AEA FUS 256

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November 1993

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Abstract

In recent papers it has been argued by Montgomery and Shan that in calculating resistive MHD instability thresholds, both resistivity and viscosity play an equally important role and may significantly modify conventional views of resistive MHD. The purpose of this paper is to discuss these arguments and put them in perspective in the context of tokamak physics. The crucial point is the following: while it is indeed true that for a *given* q–profile, the marginal stability thresholds of linear visco-resistive MHD equations depend in principle upon both resistivity and viscosity jointly through the Hartmann number, physical considerations of tokamak experiments suggest that this is an insignificant modification of well-known results in tearing mode theory as far as its applications to tokamaks are concerned. Furthermore, the uniform q (equivalently resistivity) model used by Montgomery and Shan to motivate many of their arguments and to some extent their computational techniques is non generic in a sense to be explained and is therefore not a helpful starting point for discussions of MHD stability in tokamaks. Apart from this general observation, we find that the actual results obtained by Montgomery and co-workers regarding linear instability are all contained in the 'standard' approach used in the Fusion community: specific examples are provided using the linear version of the CUTIE code developed at Culham to solve the relevant MHD equations. The CUTIE results show explicitly that JET–like tokamaks operate in a different parameter regime and involve qualitatively and quantitatively different physics to those studied by Montgomery and Shan.

1. Introduction:

In a recent set of papers, Montgomery and Shan [1-3] (see also other works cited in these papers) have considered the visco-resistive MHD equations in a periodic cylinder geometry and have argued that the linear instability thresholds of these equations for a given $q(r)$ and aspect ratio a/R, depend only on the Hartmann number, $Ha \equiv \frac{(\tau_\eta \tau_\nu)^{1/2}}{\tau_A}$.

Here, $\tau_A = a/V_A$, the Alfvén 'transit time', and τ_η, τ_ν are the resistive 'field diffusion time' $(= 4\pi a^2/c^2\eta)$ and viscous 'momentum diffusion time' $(= a^2/\nu)$ respectively. As usual, $V_A = B_0(4\pi\rho)^{-1/2}$, η is the resistivity (assumed scalar and possibly a function of r) and ν is the kinematic viscosity (assumed uniform in r).

In actual fact, Montgomery and Shan treat only the spatial profile of q as fixed and take the 'pinch ratio', $\frac{B_\theta(a)}{B_0}$ (equivalent to $q(a)$ since a/R is fixed) and Ha as the variable parameters in their stability studies. Montgomery [1] shows directly from the MHD equations by standard dimensional arguments that a given (m, n) Fourier mode is marginally stable only along a definite 'stability boundary' (ie curve) in the Ha–$\frac{B_\theta(a)}{B_0}$ plane, where the Hartmann number is evaluated at $r = 0$, say.

To reach this conclusion, it is necessary to assume (as Montgomery does) that in the unperturbed state there is no mean flow, although there is a uniform electric field in the 'toroidal' direction leading to a longitudinal current consistent with the the profiles assumed for $q(r)$ and $\eta(r)$. Furthermore, B_0 is taken uniform (no poloidal current) and the system assumed incompressible with constant density ρ. Under these circumstances, a result due to Furth *et al* [4] shows that the 'principle of exchange of stabilities' applies: ie, the real and imaginary parts of the frequency vanish together at marginality. Thus at marginal conditions, the equations reduce to a linear eigenvalue problem for the critical Hartmann number. A solution of this problem yields a visco-resistive 'neighbouring equilibrium' bifurcating from the symmetric unperturbed state with the critical Hartmann number as the 'eigenvalue' for a specified pinch ratio (or, equivalently, q_a) and current profile.

Shan and Montgomery [2,3] also consider a more special case of uniform η which results in a uniform q—profile and show how the marginal stability boundaries for various Fourier modes can be analytically obtained. They use the spatial eigenfunctions (the so–called 'Chandrasekhar-Kendall functions') of this special problem to construct a spectral method for solving the general problem with variable q and report linear and non linear calculations for various cases.

The purpose of this paper is to consider the results and conclusions reached by Shan and Montgomery and relate their work to the more traditional approaches to resistive MHD. It is useful at this point to spell out the common ground between their approach and the one adopted here.

The present analysis, based on conventional visco-resistive MHD is in agreement with Montgomery's [1] conclusion that the marginal stability thresholds for given $q(r)$ depend only upon the Hartmann number. It is also accepted generally that the non ideal MHD modes should be studied taking account of both resistivity and possible viscous effects. The actual numerical results of the calculations of Shan and Montgomery [2,3] using the 'exactly soluble' uniform-q model and those deriving from subsequent linear and non linear simulations are also accepted as correct within their stated terms of reference.

The real difficulty with the results of Montgomery and Shan [1-3] is that they have little or no relevance to present day tokamak physics. Thus, it is our purpose to show that no *new* conclusions can be drawn from their results appropriate to linear or non linear MHD instabilities in present-day experiments which are not already contained in the vast literature on tearing modes. This claim of irrelevance is a strong one and the rest of this paper is devoted to justifying it.

To this end, we re-examine Montgomery's arguments and actual examples in the light of the more conventional approaches. The results obtained by him and Shan are shown to be recovered by a straightforward time-evolution of the visco-resistive MHD equations. This has been done with a linear version of code developed at Culham called CUTIE. This examination shows that the values of resistive and viscous diffusivities pertinent to Montgomery's papers are far from those typical of tokamak conditions. Furthermore, the uniform-q model is non generic in a specific sense to be explained and is in fact unsuitable as a starting point for more realistic tokamak simulations. Linear CUTIE simulations show that under these conditions, the linear theory of the equations is essentially that due to FKR [4] and is qualitatively and quantitatively quite different to the situation prevailing near transitional values of the Hartmann number. In particular, under these conditions, in contrast to the results obtained by Montgomery and Shan [1-3], the standard stability condition determined by Δ' is only insignificantly modified by the Hartmann number. Furthermore, the importance or otherwise of viscosity on linear tearing modes is examined. It will be seen that the proposals made by Shan and Montgomery regarding the 'correct' form of viscosity must be viewed with caution. The same caveat applies to the relevance or otherwise of their non linear simulations to actual tokamak phenomena.

2. Hartmann number and standard tearing theory:

The standard MHD equations for an incompressible plasma with isotropic resistivity and viscosity take the following well-known forms [4]:

$$\nabla . \mathrm{v} = 0 \tag{1}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times \left[(\frac{c^2 \eta}{4\pi}) \nabla \times \mathbf{B} \right] \tag{2}$$

$$\nabla \times (\rho \frac{d\mathbf{v}}{dt}) = \nabla \times [(1/4\pi)(\nabla \times \mathbf{B}) \times \mathbf{B}] + \nabla \times (\rho \nu \nabla^2 \mathbf{v}) \tag{3}$$

In writing the above equations, we assume periodic cylinder tokamak ordering and treat $\eta(r)$ and ν (fixed constant) as specified constitutive properties. Furthermore we assume that the constant density ρ and the magnetic field in the z (ie 'toroidal') direction are specified. The knowledge of η can be used together with $\mathbf{v} = 0$ in the equilibrium state to calculate the toroidal current and q profiles for a given uniform toroidal field. Alternatively, given $q(a)$ we can determine the value of E_z and the q profile using the equation, $E_z = \eta(r).j_z(r)$. The Eqs.(1-3) then determine the stability of this equilibrium to small amplitude perturbations of $\mathbf{B}$ and $\mathbf{v}$, subject to the usual boundary conditions of regularity at the magnetic axis and conducting wall at $r = a$.

Since all equilibrium quantities are functions only of r, we may assume solutions of the Fourier form $f_{m,n}(r,\omega) \exp[im\theta - inz/R - i\omega t]$. If the resulting linear eigenvalue problem for ω (in general a complex number) is solved and if it turns out that for given (m,n) the imaginary part of ω (we denote it by γ) is positive, the mode is said to be unstable. If γ is zero, we say that the mode is marginal. Negative γ implies linear stability of the profile to the Fourier mode in question. Nondimensionalising the equations reveals the parameters upon which the growth rate γ can depend. If we choose the minor radius a to scale all lengths, we arrive at the aspect ratio, a/R, as one of the relevant parameters. Since the Alfvén velocity, V_A, provides a typical velocity, the resistivity and the kinematic viscosity can be rescaled by introducing the 'magnetic Reynolds number' $S = \tau_\eta/\tau_A$ and the analogous ratio for viscous effects, $M = \tau_\nu/\tau_A$ respectively.

It then follows that the non dimensional equations only involve $a/R, S, M$ and the spatial profile, $q(r)$. The latter can be used to represent the radial profile of $\eta(r)$, assuming that S is evaluated at some definite reference point (which may be taken to be $r = 0$ without loss of generality). In turn this implies that the non dimensional growth rate $\gamma\tau_A$ (for a given (m,n) mode), is a function of the above parameters and a functional (in general non local) of $q(r)$. If we fix a/R and $q(r)$, $\gamma\tau_A$ must be a function only of S and M. Provided there are no equilibrium flows/diamagnetic effects, it turns out that $\gamma = 0$ (ie the 'marginally stable state') corresponds to $\omega = 0$. In this case, we can consider the non dimensional equations at marginality, ie when $\omega = \gamma = 0$. The linearized marginal equations then take the dimensionless forms:

$$\nabla^*.\mathbf{v}^* = 0 \quad (4)$$

$$\nabla^* \times (\mathbf{v}^* \times \mathbf{b}) = \nabla^* \times \left[(\frac{1}{h(r)S}) \nabla^* \times \mathbf{B}^* \right] \quad (5)$$

$$\nabla^* \times [(1/4\pi)(\nabla^* \times \mathbf{B}^*) \times \mathbf{b}] = -\nabla^* \times [(1/4\pi B)(\nabla^* \times \mathbf{B}) \times \mathbf{B}^*] - \nabla^* \times (\frac{1}{M}) \nabla^{*2} \mathbf{v}^* \quad (6)$$

In these equations, $\mathbf{b}$ represents the unit vector along the unperturbed magnetic field $\mathbf{B}$ and the velocity is scaled relative to a/τ_A. Starred quantities are the dimensionless perturbed fields whilst ∇^* is the gradient with respect to the non dimensional coordinate variables; the function $h(r) = \frac{\eta(r)}{\eta(0)}$ is the prescribed spatial profile of the resistivity.

We note that although these equations contain two *distinct* dimensionless parameters (ie S, M), we can rescale $\mathbf{v}^* \equiv (\frac{M}{S})^{1/2}\mathbf{w}$ and obtain the same set of equations with $\mathbf{v}^*$ replaced by $\mathbf{w}$ and S, M replaced by a *single* non dimensional number, $Ha \equiv (SM)^{1/2}$. Note also that it is not in general possible to make such a reduction in the case of the full (ie non marginal) time evolution equations, Eqs.(1-3). It is now plain that for given $\mathbf{B}(r)$ (ie $q(r), h(r)$), boundary conditions, and Fourier mode (ie m, n), these equations do not in general have non trivial solutions. Indeed, we must solve them regarding the *critical* Hartmann number, Ha_{crit} as an eigenvalue. Thus we recover Montgomery's [1] statement, starting from the standard approach.

It should be carefully noted however that growth or decay rates depend upon S and M *individually* in general and not merely on the combination Ha. It therefore makes physical sense to consider (for example) the limit when S is large but *finite*, letting $M \to \infty$ and calculate growth rates, as is done by Furth *et al* [4] in the main part of their paper. The marginality condition in this asymptotic limit when M, Ha have been taken to infinity is now a condition on the *q profile* and has nothing to do with the Hartmann number (and as a matter of fact does not involve S either).

Let us now consider the relevance or otherwise of the above result to tokamak experiments. In most tokamaks, the conditions are usually such that S is of the order $10^6 - 10^8$. The kinematic viscosity from classical theory is typically of the order of resistive diffusivity. Thus, $c^2\eta/4\pi\nu \simeq (\frac{c^2 m_e}{4\pi n_e e^2 \rho_i^2})(\tau_i/\tau_e)$. This ratio is of order unity under standard, medium to large-scale tokamak conditions. This implies that the Hartmann number is of the same order as S. Taking neoclassical effects into account does not significantly change the ratio. If ν is assumed *anomalous* and of the order of the anomalous thermal diffusivity (ie $\simeq 1 - 10 m^2/s$) whilst treating η classical/neoclassical, we get M to be somewhat smaller than S but the Hartmann number is still $\geq O(10^6)$. In all these cases we must take ν to be the *radial* momentum diffusivity for reasons which will be discussed in detail later.

As we shall show by explicit numerical examples and also from the results of Montgomery and Shan, it follows that for most 'reasonable' q profiles, the critical Hartmann number is many orders of magnitude smaller than the values appropriate to experiment (typically, $Ha_{crit} \simeq 10^3$ for the profiles considered). The physical reason for this is very simply explained: the magnetic free energy available to drive the mode is far larger than the viscous stress available to damp the mode except with unrealistically large viscosity (which of course corresponds to 'small' Hartmann numbers, for given resistivity). To overcome the effects of the Lorentz couple in the vorticity diffusion equation, the viscous forces must be large, and this can happen only at rather modest values of the Hartmann number.

The *relevant* condition for visco-resistive stability of Fourier modes when S, M are 'large' is actually related to the well-known Δ' of the FKR theory [4]. This quantity is uniquely determined (for given m, n and q profile) by the solution of the 'marginal' equilibrium equation, $\nabla \times [(\nabla \times \mathbf{B}) \times \mathbf{B}] = 0$, and is *independent* of S, M for asymptotically large values. It has been shown to be related to the free magnetic energy available in the equilibrium configuration to drive the instability (see [5]). If $M \gg S \gg 1$, the instability condition is the well-known $\Delta' > 0$ criterion, the mode being marginal if $\Delta' = 0$.

Note that the emphasis here is decidedly different from that of the work of Montgomery and Shan. The parameters S, M are in fact regarded as *fixed, determined by the experimental conditions* and we ask for the conditions to be satisfied by the q profile in order that a given m, n Fourier mode may be marginally stable. Under these circumstances, the effect of a small but non zero viscosity (ie a large but finite Hartmann number) can be assessed physically by considering the equation of motion: the viscous stress term is only a small perturbation on the Lorentz force term which leads to the $\Delta' = 0$ criterion; thus, in this case, the marginal stability criterion will take the form, $\Delta' - F(Ha) > 0$ where $F(Ha)$ is a function of the Hartmann number (it will in general involve q and a/R of course) which must vanish as $Ha \rightarrow \infty$.

This is very different from the structure of the stability condition [2,3] when the Hartmann numbers are 'small' (ie $O(1000)$). It should not be forgotten that in tokamak physics, one is not merely interested in marginal stability conditions (which as we have seen, depend only on the Hartmann number for given q) but also in the linear *growth* rates of the Fourier modes. These growth rates are functions not only of S and M, but actually depend on local and global profile properties such as the magnetic shear (ie q' and Δ'). Indeed, it has been shown [6-9] that when the linear growth rates are weak the modes quickly evolve into a 'Rutherford' regime and often saturate to form quasi stationary (rotating) 'islands'. Many active methods of profile control have been suggested to weaken the growth rates to render the modes 'benign'. The Hartmann number is scarcely relevant to such studies except in so far as viscous layers may dominate over resistive ones in some circumstances.

It is well-established since the work of Furth *et al* [4] that the spatial structure of the modes involves viscoresistive *critical layers* at radii r_s where $q(r_s) = m/n$ with thicknesses determined by the combined effects of S and M. The theory of Shan and Montgomery applies only when S, M and Ha are so small (relative to experimental conditions) that there is no 'singular' layer structure at all. This fundamental fact will also be illustrated by explicit calculations.

Shan and Montgomery consider a 'uniform resistivity model' to illustrate some of their points. In their work, they have essentially rederived many earlier results by several authors [10-13] This model was actually considered by Shafranov [13] within the context of ideal MHD and turns out to be soluble exactly in terms of Bessel functions, even with the inclusion of non ideal effects such as the Hall effect in Ohm's law [12] and viscosity in the equation of motion. In the context of tokamak physics, it must be said that the model has little or no relevance, since experimental measurements of q by Soltwisch [14] and many others show conclusively that the current density is not spatially uniform as required by this model, in general. Very flat q profiles over a substantial portion of the tokamak would imply that the strongly stabilizing 'line bending' term is absent suggesting *ideal* instability. Viscosity and resistivity (at high S, M) would not normally be expected to significantly affect the stability in this case. Of course the 'layer' would extend everywhere and the mode would not have the characteristic 'tearing' structure. In short, this is a non generic profile which is atypical of most tokamak discharges. For this reason, we believe that results relating to such specialized profiles are of little (if any) relevance to tokamak applications. While Shan and Montgomery do carry out calculations for more general sheared profiles, the eigenfunctions obtained from the uniform case are not an appropriate set to expand the solutions in the general case at physically interesting values of S, M.

3. Numerical results:

In this section we present results from the linear CUTIE code developed at Culham. This code essentially solves the linearized equations of motion, Eqs.(1-3) given m, n, the equilibrium $q(r)$ profile and and the parameters S, M. Thus starting from arbitrarily prescribed initial data, the time evolution of the linear mode is calculated by solving two coupled partial differential equations satisfied by the magnetic flux function Ψ and the velocity stream function, Φ. In the tokamak limit (which is the main concern in this work), it is easily seen that (Hazeltine and Meiss [15]) the equations of motion reduce to,

$$\frac{\partial\Psi}{\partial t} + \nabla_{\parallel} c\Phi = \frac{c^2\eta(r)}{4\pi}\nabla_{\perp}^2\Psi \qquad (7)$$

$$\frac{dc\nabla_{\perp}^2\Phi}{dt} = -V_A^2\nabla_{\parallel}(\nabla_{\perp}^2\Psi) + \nu\nabla_{\perp}^4 c\Phi \qquad (8)$$

where, $\nabla_{\|} = \frac{\mathbf{B}}{B}.\nabla$ and $j_{\|} = -\frac{c}{4\pi}\nabla_{\perp}^2\Psi$. The parallel component of the vorticity is given by, $\Omega_{\|} = \frac{c}{B}\nabla_{\perp}^2\Phi$.

Setting $\Psi = Z_{m,n}(r,t)\exp(i(m\theta - nz/R))$, $c\Phi = iV_A\Phi_{m,n}(r,t)\exp(i(m\theta - nz/R))$ and introducing the radial Laplacian operator, $D_{m,n}^2 \simeq \frac{1}{r}\frac{d}{dr}(r\frac{d}{dr}) - (\frac{m}{r})^2$, the coupled equations for a given Fourier mode (m,n) become,

$$\frac{\partial Z_{m,n}}{\partial t} - V_A\frac{(m-nq)}{qR}\Phi_{m,n} = \frac{c^2\eta(r)}{4\pi}D_{m,n}^2 Z_{m,n} \tag{9}$$

$$\frac{\partial D_{m,n}^2\Phi_{m,n}}{\partial t} + V_A\frac{(m-nq)}{qR}D_{m,n}^2 Z_{m,n} = m.Z_{m,n}.\frac{V_A}{cqR}.\frac{4\pi}{B_{0\theta}}\frac{dj_{0z}}{dr} + \nu.D_{m,n}^2(D_{m,n}^2\Phi_{m,n}) \tag{10}$$

We do not describe the numerical methods used to solve these equations here. Suffices to state that standard (radial and temporal) unconditionally stable schemes are employed to time advance the equations and solve the Poisson-type equations to calculate the fields $Z_{m,n}, \Phi_{m,n}$.

We start with simulations demonstrating that the stability of 2/1 and 3/2 modes calculated by using the above scheme is the same as that obtained by Shan and Montgomery. Thus, following these authors, we adopt the resistivity profile, $\eta(r) = \eta(0)\left[1 + (r/r_0)^{2\Lambda}\right]^{1+1/\Lambda}$ with $r_0 = 0.6a, \Lambda = 4$. This can be shown to result in the q profile, $q(r) = q(0)\left[1 + (r/r_0)^{2\Lambda}\right]^{1/\Lambda}$. Fig. 1 shows the equilibrium profiles of q, and j_z used in the simulations (identical with those of Fig. 1 in Ref. [2]). We take the minor radius, $a = 100$ cms, whilst $R = 2.5 \times a = 250$ cms. This leads to the same value of $R/a = 2.5$ as that used in Ref. [2] . We set $q(0) = 1.38$ as in Ref. [2], leading to $q_a = 3.85$. Choosing $B = 2$ Tesla, we obtain the plasma current to be 1.03 MA, a value typical of JET conditions. The Alfvén velocity is chosen to be 2.2×10^8 cm/sec, corresponding to a nominal central ion density of 4.0×10^{14} protons/cc. Runs 1-3 in Ref. [2] are simulated by taking $S = 2 \times 10^4$ and varying ν so that the Hartmann number Ha is varied through 707, 2828 and 14142. These values correspond to the following values of $\nu/\frac{c^2\eta(0)}{4\pi}$: 800, 50, 2, respectively.

In complete agreement with Shan and Montgomery, we find that for $Ha = 707$ ('Run 1' in Table 1 of Ref. [2]), both the 2/1 and 3/2 are stable. When $Ha = 2828$ ('Run 2' of Ref. [2]), the 2/1 mode is unstable with growth rate, $\gamma = 6.7 \times 10^{-3}/\tau_A$; however, the 3/2 mode is stable. Finally, when Ha is increased to 14142 ('Run 3'), both the 2/1 and the 3/2 modes are unstable, with growth rates, $\gamma_{2,1} = 1.8 \times 10^{-2}/\tau_A$ and $\gamma_{3,2} = 5 \times 10^{-4}/\tau_A$ respectively.

Several comments are in order. The values of S and Ha used in these simulations correspond to (in the case of Run 1 with $Ha = 707$ for example) a resistive diffusivity of 1.1×10^6 cm^2/sec and $\nu = 8.7 \times 10^8$ cm^2/sec respectively! This is far larger (by three to four orders of magnitude) than the anomalous thermal diffusivity in tokamaks. If the plasma resistance at these levels of resistivity is estimated, a loop voltage of nearly 2000 volts would be needed to drive the 1 MA current. Thus these numbers show clearly that Shan and Montgomery's simulations have little relevance to JET-like tokamak conditions.

Fig. 2 shows the typical 2/1 eigenfunctions for the 'Run 3' case (ie $Ha = 14142$). The $\Phi_{m,n}$ plot (normalized) shows that there is no real 'layer' structure and the mode is 'global' in character in these highly resistive and viscous conditions.

The purpose of the above results was to demonstrate that the present numerical techniques are indeed capable of simulating the *linear* stability behaviour of profiles considered by Shan and Montgomery. We next demonstrate that the code can also recover the critical Hartmann number and (in principle) the stability thresholds calculated by these authors. For this purpose, we fix the 'pinch ratio', $B_{0\theta}(a)/B = 0.103$ (it corresponds to the profiles considered above) and study the growth rate of the 2/1 mode as a function of the Hartmann number, keeping S fixed. We find that for $Ha = 943$, the mode is weakly growing but for $Ha = 894$, it is weakly damped. By a straightforward process of 'interval bisection' we find that the critical Hartmann number for this profile and pinch ratio is 910, to adequate accuracy. This is in excellent agreement with Fig. 1 of Ref. [3], obtained for the identical conditions using their method based on spectral expansion in terms of the eigenfunctions of the constant shear problem. It is also easy to show by similar calculations that the marginal condition is unchanged when both S and M are varied, keeping Ha fixed, in agreement with the general theory and the arguments of Montgomery [1]. For reasons of space we do not show the explicit results for this as well as grid refinement studies demonstrating convergence of the present scheme.

We conclude this section presenting three runs representing more realistic cases. The point is that under these conditions the physics is of the 'singular layer' type and the methods of Shan and Montgomery would involve prohibitive spectral calculations. Our finite difference technique is able to resolve the layer physics properly (with relatively few grid points within the layer, usually 10 are found to give adequate accuracy on both growth rates and eigenfunctions) and obtain well-converged solutions and growth rates. The results also show that the width of the layer depends upon the values of S, M prevalent *at the resonant radius*, as expected from the standard analysis of FKR [4].

Taking the same q profile as above, we consider the case when $S = 1.0 \times 10^7$. The nominal loop voltage to drive the plasma current then turns out to be 4 Volts, showing that this case is much closer to experimental conditions than the simulations of Shan and Montgomery.

First consider $\nu = \frac{c^2\eta(0)}{4\pi} = 2.2 \times 10^3$ cms^2/sec, corresponding to $S = M = Ha$. The radial grid resolution in this case is, $\Delta r = a/1800$, corresponding to about 10 mesh points within a 'resistive layer width', $\delta \simeq a/S^{2/5}$. Fig. 3 shows the normalized eigenfunctions after 1 msec of 'real time' evolution of the mode.

The layer structure and the characteristic 'kink' in the Ψ function are brought out clearly by the plots of $\Phi_{m,n}(r)$ and $Z_{m,n}(r)$ respectively. The growth rate is calculated to be, $\gamma_{2,1} = 8.2 \times 10^{-5}/\tau_A \simeq 1.8 \times 10^3/\text{sec}$. In this case, the resistive and viscous diffusivities are equal.

Next consider what happens if we take the same S but decrease ν such that $\nu = 0.01 \times \frac{c^2\eta(0)}{4\pi}$. The Hartmann number in this case is, $Ha = 10 \times S = 1.0 \times 10^8$. We expect the mode to be resistivity dominated and indeed, as Fig. 4 shows, the layer width is lower and corresponds to the 'traditional' resistive tearing theory à la FKR. The growth rate is, $\gamma_{2,1} = 1 \times 10^{-4}/\tau_A$, somewhat (ie 25 %) faster than in the previous case. Finally, to demonstrate the effects of 'viscosity dominance' we show the results in Fig. 5 of a run with $Ha = S/10 = 1.0 \times 10^6$. In this case, the resistive diffusivity is 2.2×10^3 cm^2/sec whereas the kinematic viscosity is 100 times larger (indeed, much larger than the typical tokamak perpendicular thermal diffusivity due to turbulence, even in L-modes). The $\Phi_{2,1}$ plot clearly shows the much 'thickened' (compare with the II case above) viscous layer. The growth rate is reduced due to viscous damping and is found to be, $\gamma_{2,1} = 4 \times 10^{-5}/\tau_A$.

It is now clear that the 'critical' Hartmann number calculated for the profile is essentially irrelevant at high S since it corresponds to impossibly large kinematic viscosity. The real marginal stability depends upon the magnetic free energy to drive the mode contained in the equilibrium poloidal field and is measured by the Δ' of the mode in question. Under realistic conditions, linear visco-resistive modes can be made stable only by profile control or suitable 'dynamic' [16] or 'feed-back' stabilization.

Viscosity certainly affects layer properties and growth rates, but the essential theory of this was already given by FKR in Ref. [4] and can in any given case be calculated readily by codes such as CUTIE. Shan and Montgomery have argued that the 'effective' viscosity could be much higher in tokamaks than implied in the standard approaches. In particular they suggest that *parallel* ion viscosity be used in the stability analysis [2]. This is physically unlikely to be important in tearing stability for two reasons: firstly, parallel viscosity essentially involves the *parallel* gradient operator, $\mathbf{b}.\nabla$ and as such is negligible within the resonant layer, at least in the cylinder. In a torus, it can be as important as the perpendicular viscosity effects. In addition, it can never be a crucial term in the momentum balance anyway, since the Lorentz force is by far the dominant term in the parallel vorticity equation and leads directly to the usual 'outer solution' of standard tearing theory. Furthermore, within the layer, it is the *radial* derivatives of velocity which can be expected to play a role, and these are controlled by the *perpendicular* viscosity [4], which may be classical (ie $\simeq \frac{\rho_i^2}{\tau_i}$, possibly neoclassically 'enhanced' by geometric and trapped particle effects) or 'anomalous' like the ion thermal diffusivity. None of these effects can produce a low Hartmann number, although they can certainly affect growth rates and layer widths like many *other* effects such as thermal diffusivity, Landau damping, Hall effect, electron inertia, etc. Secondly, in the low collisionality regime typical of tokamaks, the parallel diffusivity of momentum and energy are decidedly *not* given by classical expressions like $\simeq V_{thi}^2 \tau_i$ but have to be 'Knudsen-corrected' for long mean-free-path [17,18]. The combined effect of such corrections and the size of $k_{||}/k_{radial}$ is to reduce the 'effective' kinematic viscosity to values not much above those implied by anomalous perpendicular transport. Thus although such 'Knudsen-corrected' diffusivities are available, as pointed out above, they can have little direct effect on mode stability as apparently envisaged by Montgomery.

4. Summary and conclusions:

In this paper, the recent work of Shan and Montgomery on viscoresistive instabilities in tokamaks is considered. Analytical and numerical considerations indicate that whilst their results can indeed be recovered by more standard approaches, they are largely irrelevant to realistic present day tokamak conditions. Furthermore, the numerical methods and the exact solutions found by these workers have little or no application to Fusion plasma physics (except possibly in highly collisional edge conditions or in small machines) although they may be more useful in other types of MHD applications involving larger resistivity and viscosity. The relative roles of viscosity and resistivity in allowing the equilibrium free-energy to drive low mode number instabilities have been understood since the classic paper of Furth *et al* [4] and the more recent studies of Montgomery and co-workers do not significantly change the situation. It has also been shown that a numerical code developed at Culham (called CUTIE) can reproduce both the results (in the linear regime) of Shan and Montgomery and the well-known tearing theory under appropriate conditions.

Acknowledgements: The author thanks M.K. Bevir, J.W. Connor, R. Fitzpatrick, C.G. Gimblett for useful discussions. This work was funded jointly by the UK Dept. of Trade and Industry and Euratom.

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FIGURE CAPTIONS

Fig. 1a: Equilibrium $q(r/a)$. Fig. 1b: Equilibrium $j_z(r/a)$. See text for functional forms.

Fig. 2a: Typical 2/1 eigenfunction of the perturbed $Z_{m,n}(r/a)$ (ie normalized radial magnetic field perturbation profile). 'Run 3' conditions with $Ha = 14142; S = 2 \times 10^4$. Fig. 2b: Plot of associated $\Phi_{m,n}(r/a)$ eigenfunction (ie normalized radial velocity perturbation profile).

Fig. 3a: Typical 2/1 eigenfunction of the perturbed $Z_{m,n}(r/a)$ (ie normalized radial magnetic field perturbation profile). 'Realistic' conditions: $S = 1.0 \times 10^7; Ha = S; \nu = \frac{c^2\eta(0)}{4\pi} = 2.2 \times 10^3$ cms^2/sec. 1800 radial mesh points. Eigenfunction after 1 msec time-evolution of growing mode. Fig. 3b: Plot of associated $\Phi_{m,n}(r/a)$ eigenfunction (ie normalized radial velocity perturbation profile). Note strong localisation near the resonant point.

Fig. 4a: Typical 2/1 eigenfunction of the perturbed $Z_{m,n}(r/a)$ (ie normalized radial magnetic field perturbation profile). 'Realistic' conditions, 'resistivity dominated': $S = 1.0 \times 10^7; Ha = 10 \times S; \nu = 0.01 \times \frac{c^2\eta(0)}{4\pi} = 2.2 \times 10^1$ cms^2/sec. 1800 radial mesh points. Eigenfunction after 1 msec time-evolution of growing mode. Fig. 4b: Plot of associated $\Phi_{m,n}(r/a)$ eigenfunction (ie normalized radial velocity perturbation profile). Note greater localisation near the resonant point as compared with Fig. 3b.

Fig. 5a: Typical 2/1 eigenfunction of the perturbed $Z_{m,n}(r/a)$ (ie normalized radial magnetic field perturbation profile). 'Realistic' conditions, 'viscosity dominated': $S = 1.0 \times 10^7; Ha = S/10; \nu = 100 \times \frac{c^2\eta(0)}{4\pi} = 2.2 \times 10^5$ cms^2/sec. 1800 radial mesh points. Eigenfunction after 1 msec time-evolution of growing mode. Fig. 5b: Plot of associated $\Phi_{m,n}(r/a)$ eigenfunction (ie normalized radial velocity perturbation profile). Note much broader 'layer' as compared with Figs. 3b, 4b.

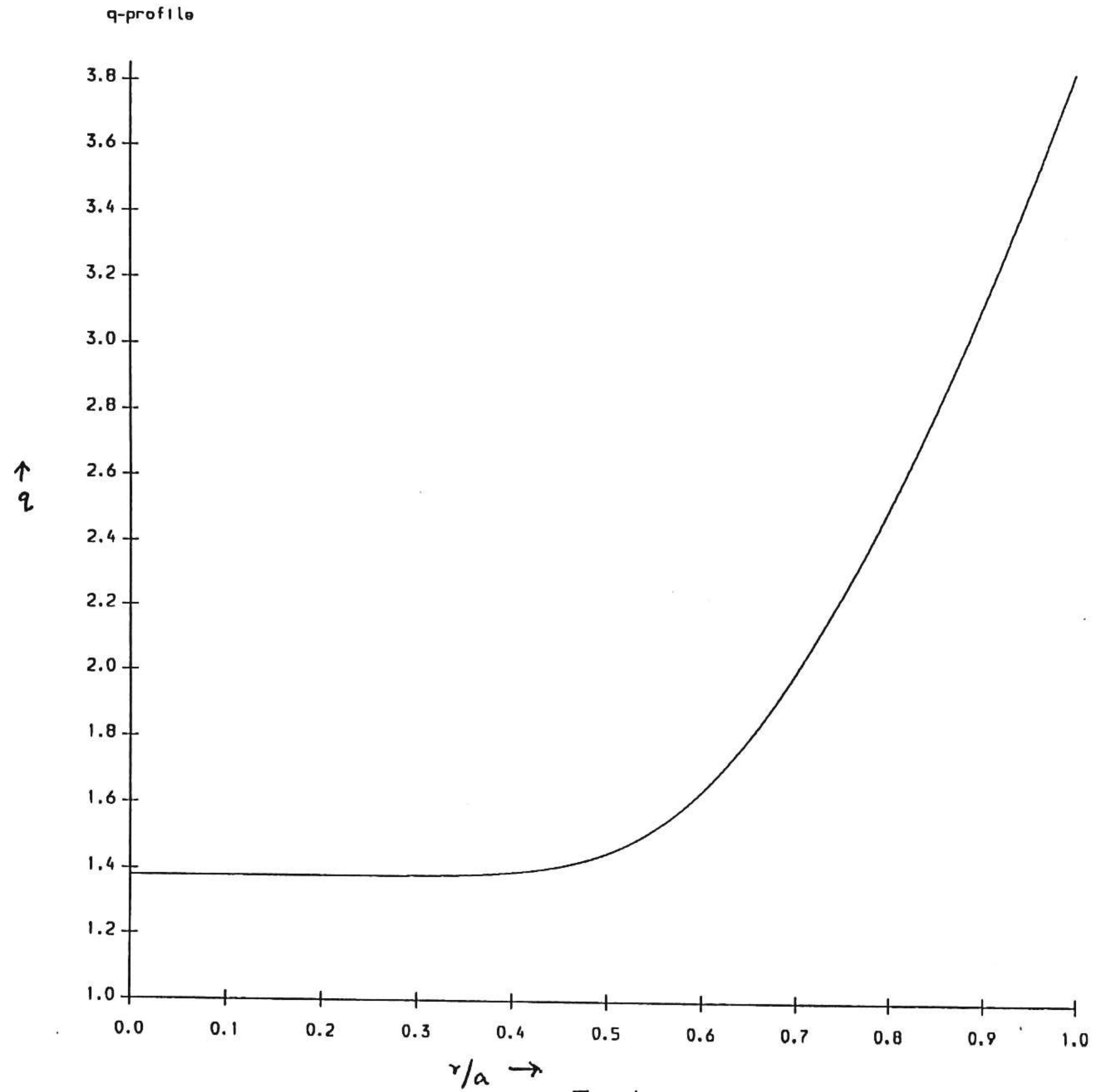
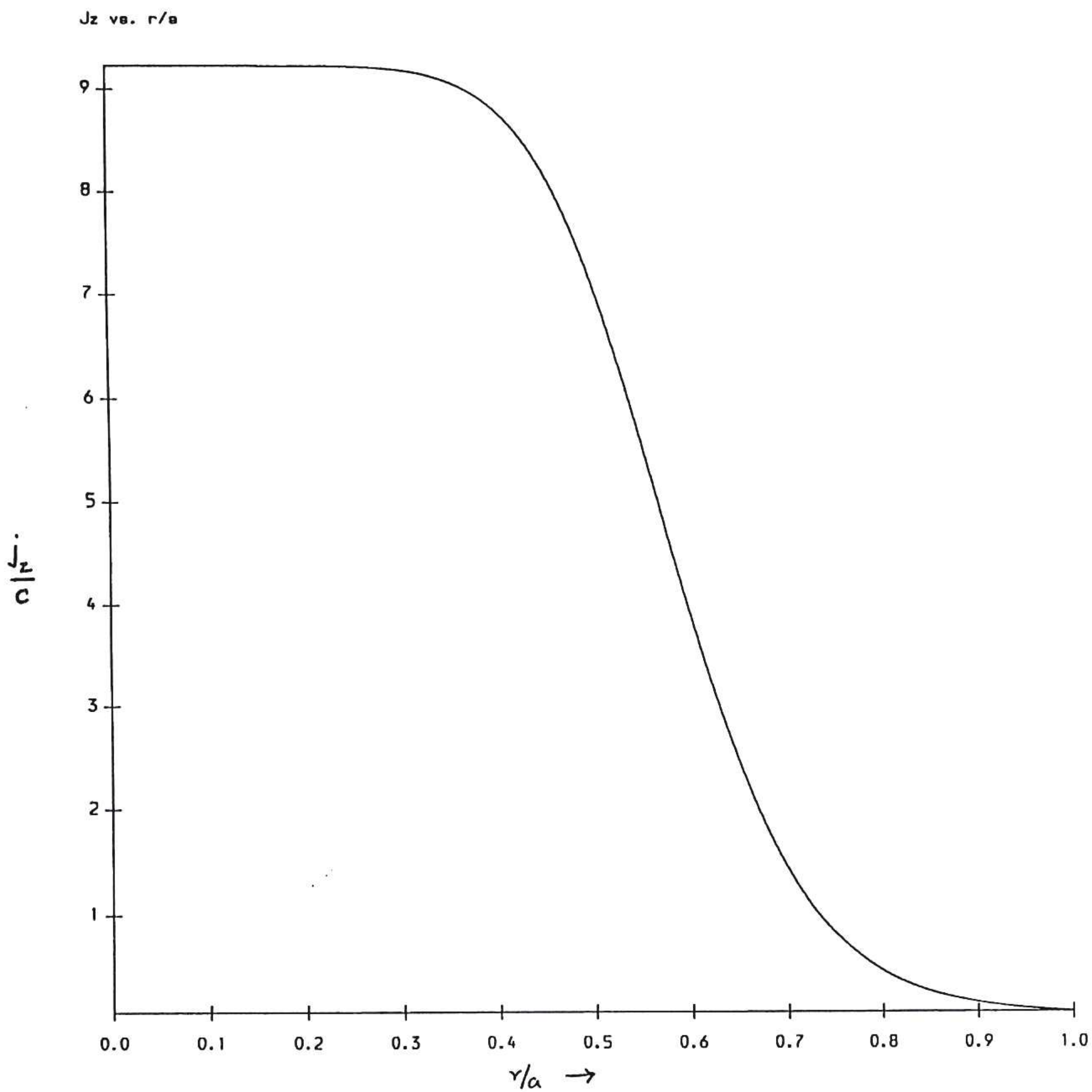

q-profile
3.8
3.6
3.4
3.2
3.0
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
↑ q
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
r/a →

Fig. 1a.


Jz vs. r/a
9
8
7
6
5
4
3
2
1
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
r/a →
$\frac{j_z}{c}$

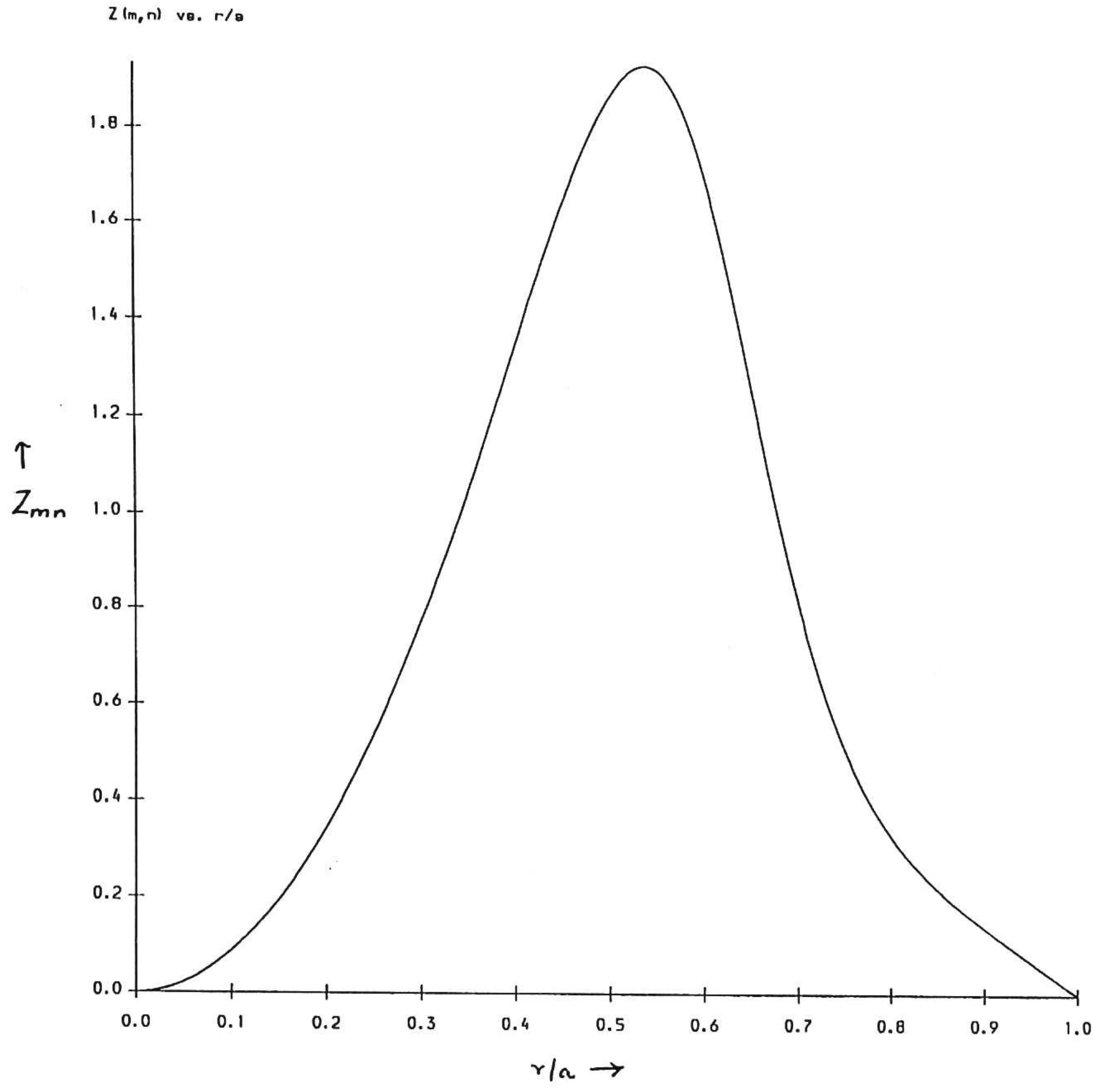
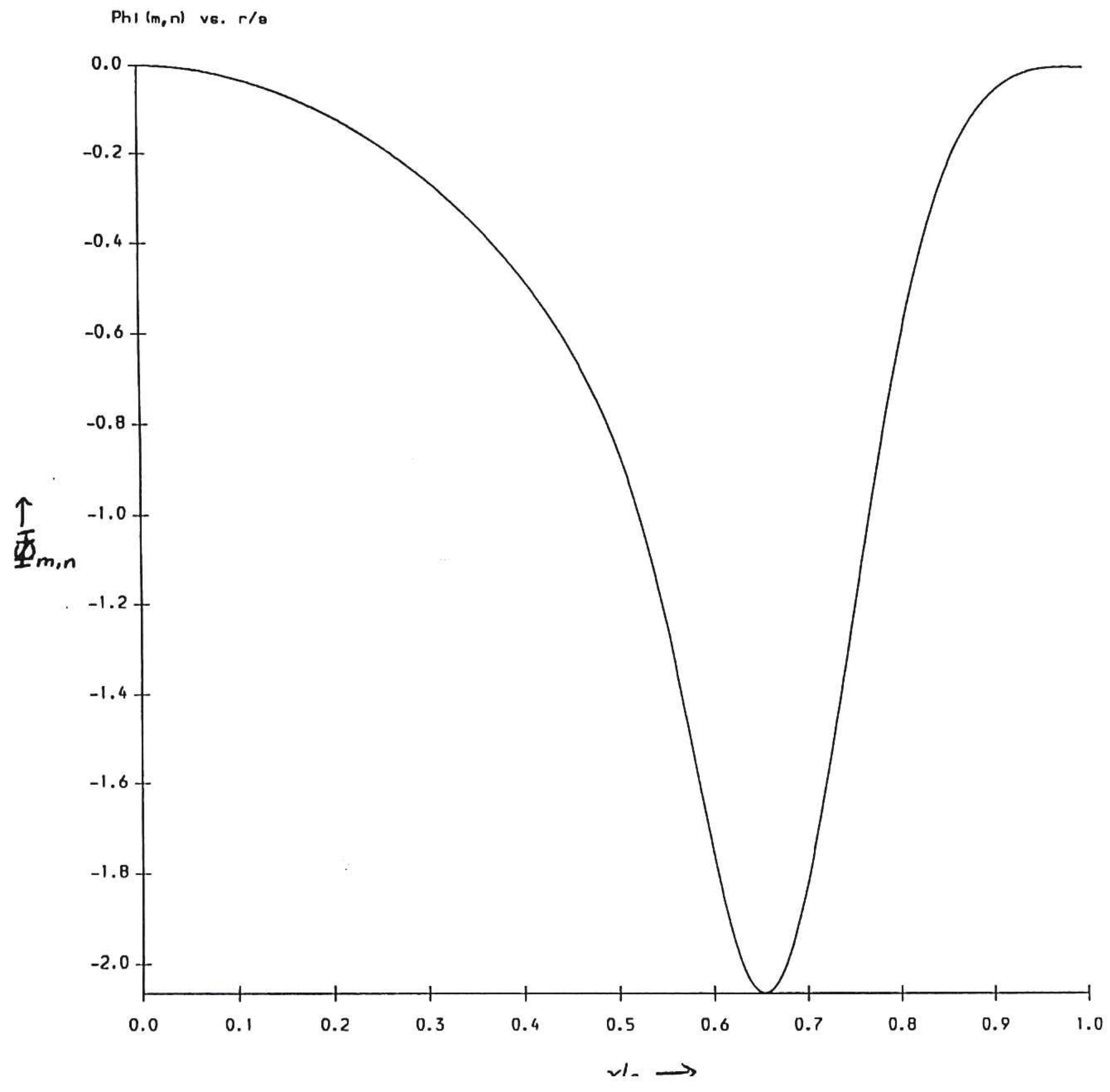

Z(m,n) vs. r/a
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
Z_{mn} ↑
r/a →

Fig. 2a.



Phi (m,n) vs. r/a
0.0
-0.2
-0.4
-0.6
-0.8
-1.0
-1.2
-1.4
-1.6
-1.8
-2.0
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0

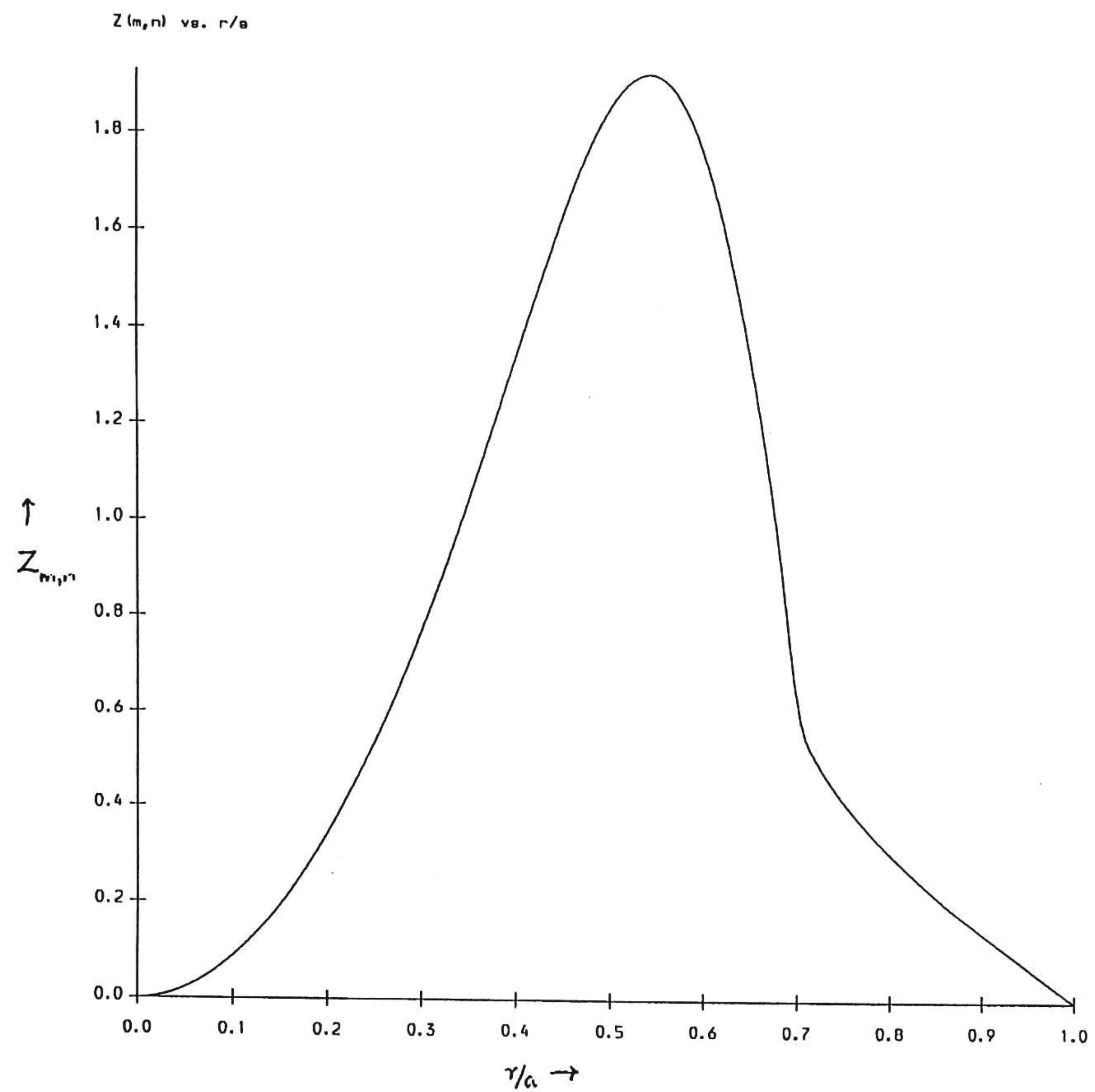
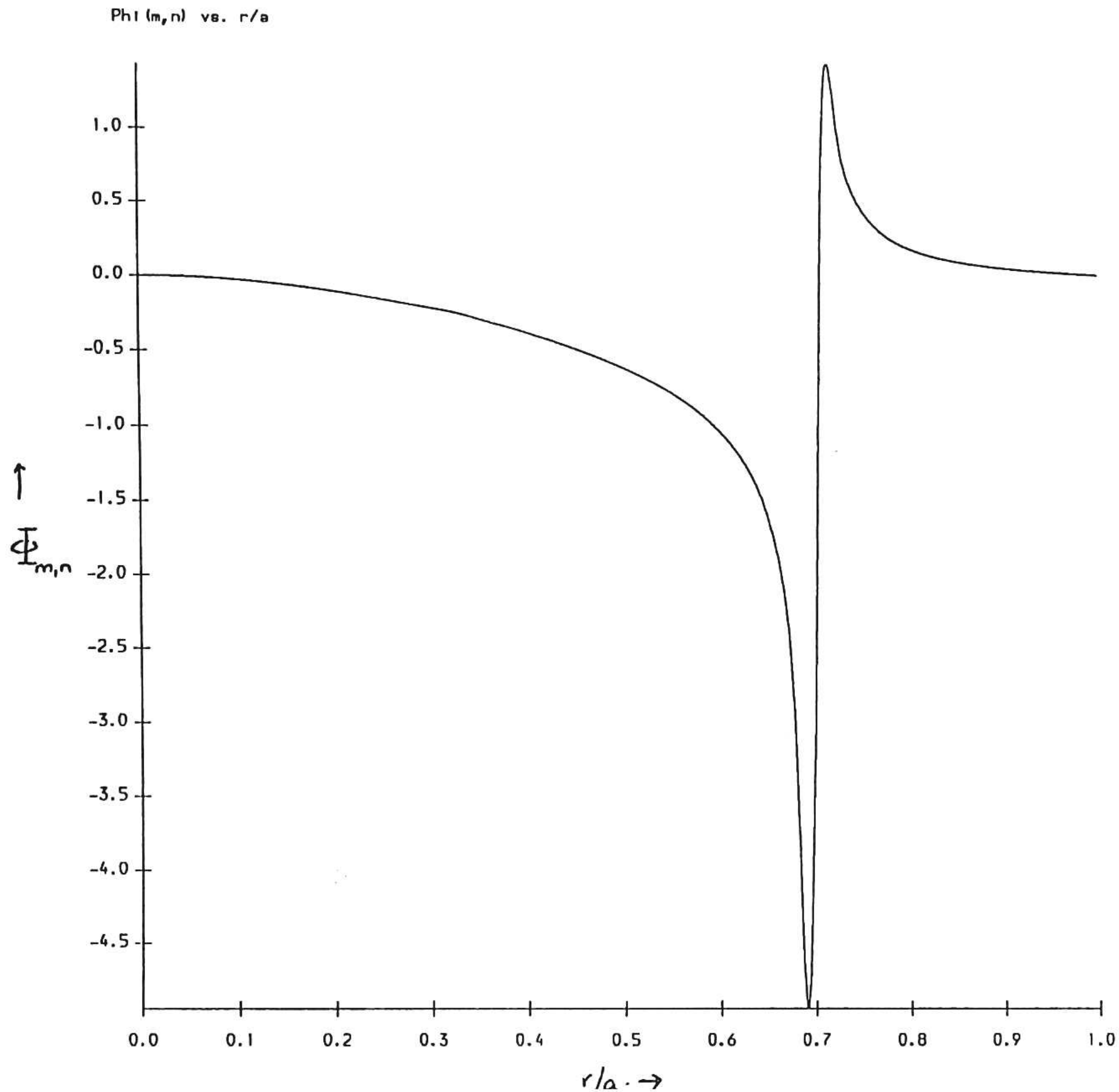

Z(m,n) vs. r/a
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
$Z_{m,n}$ ↑
r/a →

Fig. 3a


Phi (m,n) vs. r/a
1.0
0.5
0.0
-0.5
-1.0
-1.5
-2.0
-2.5
-3.0
-3.5
-4.0
-4.5
$\Phi_{m,n}$ ↑
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
r/a →

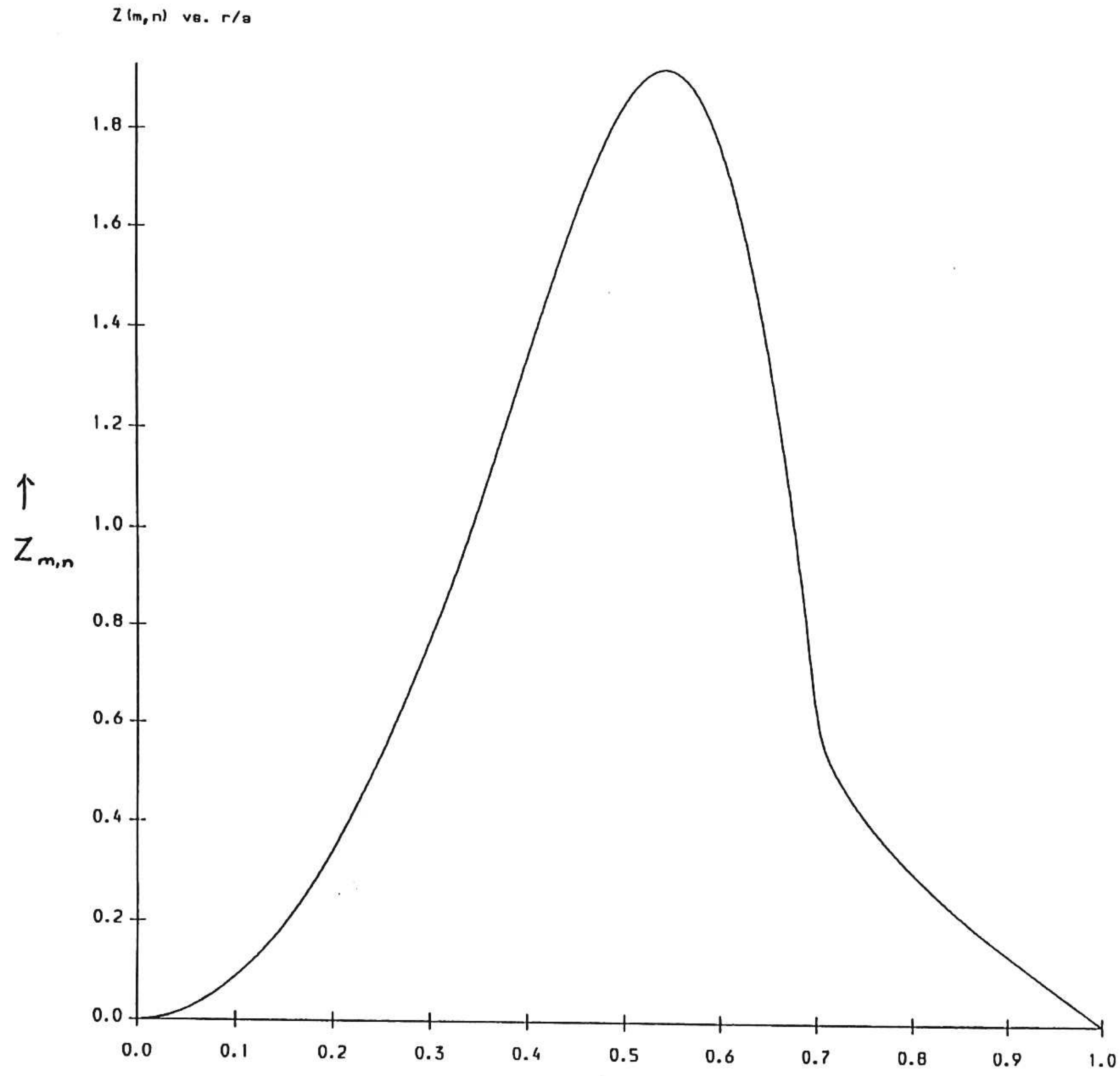
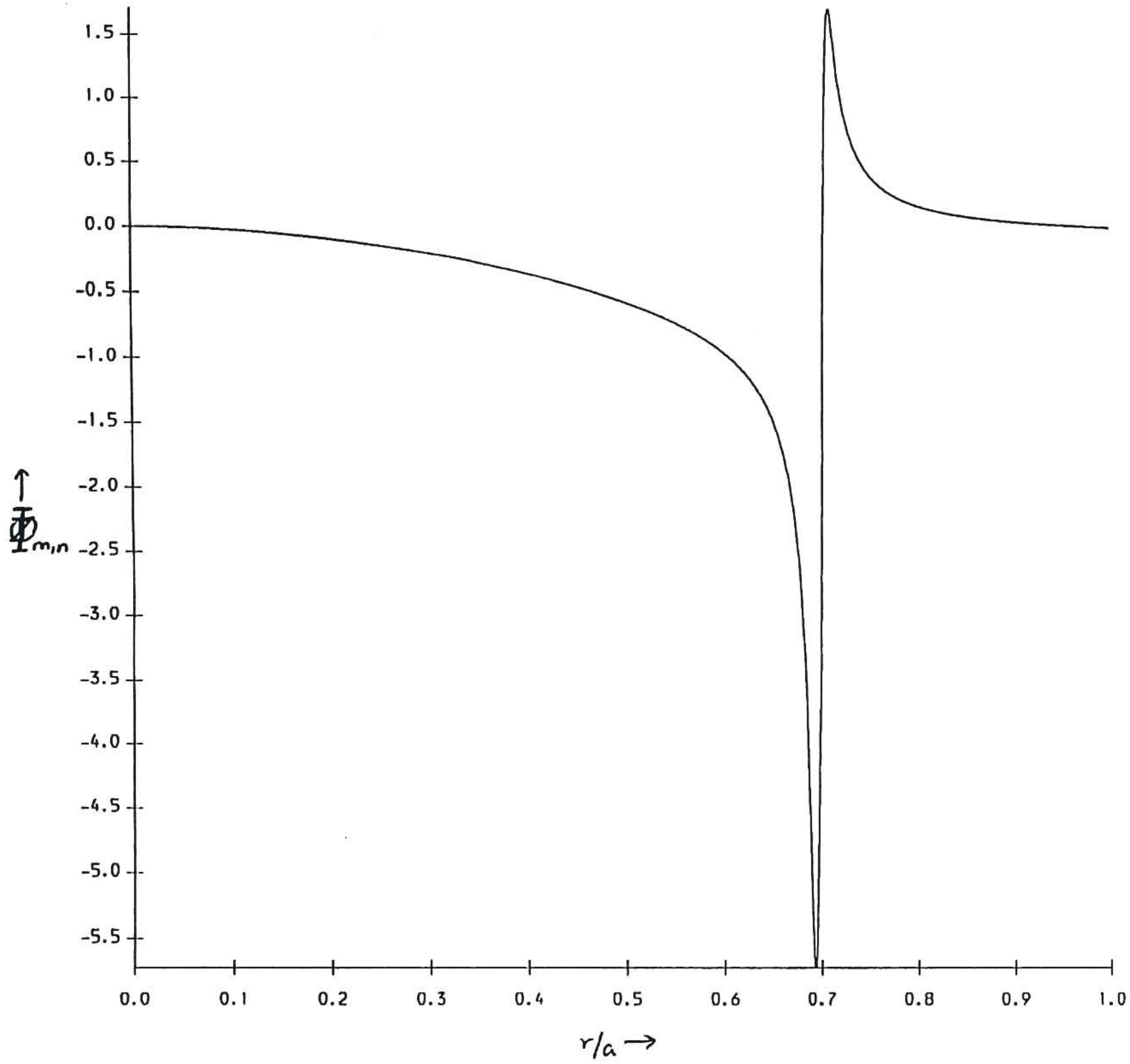
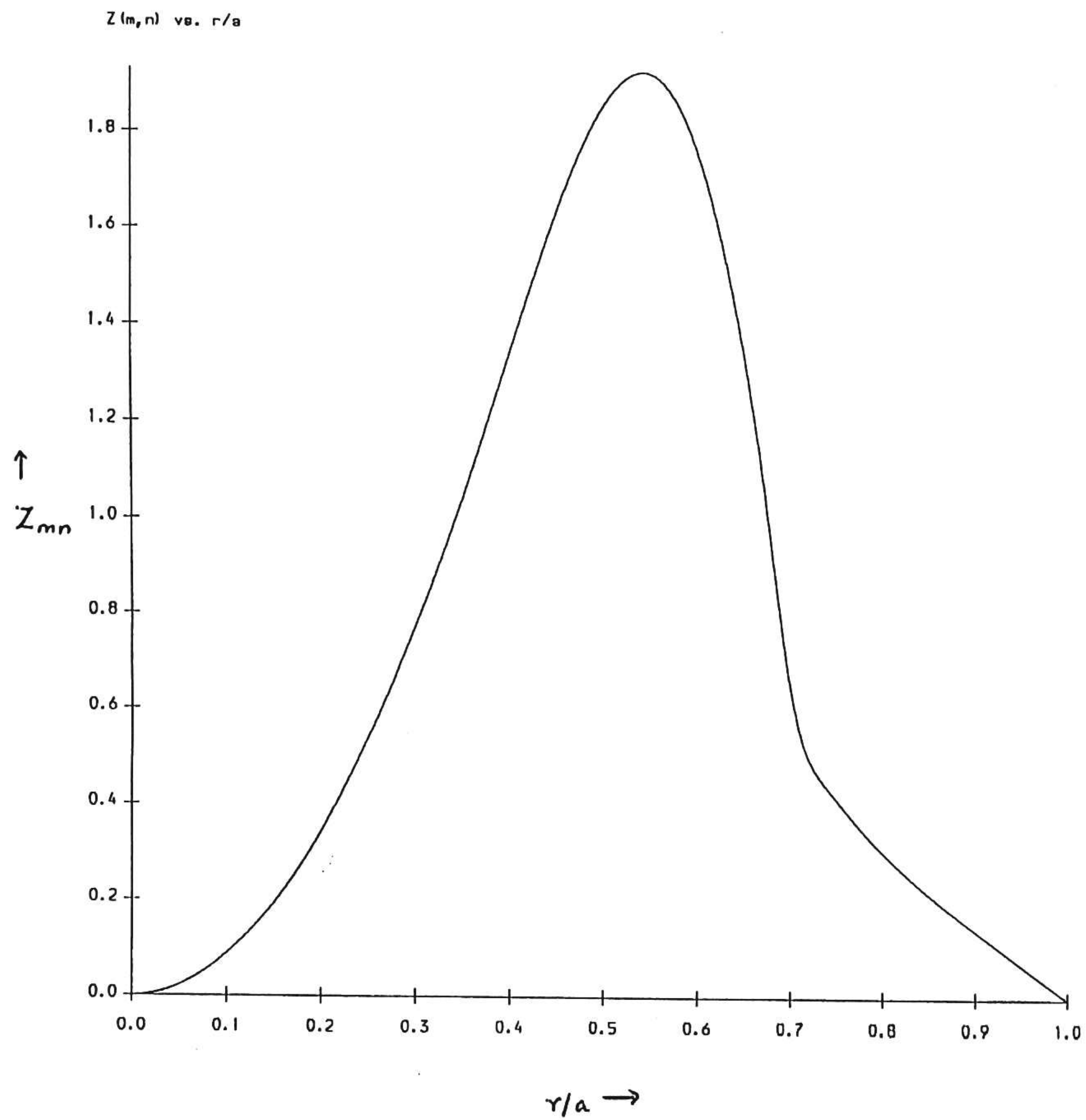
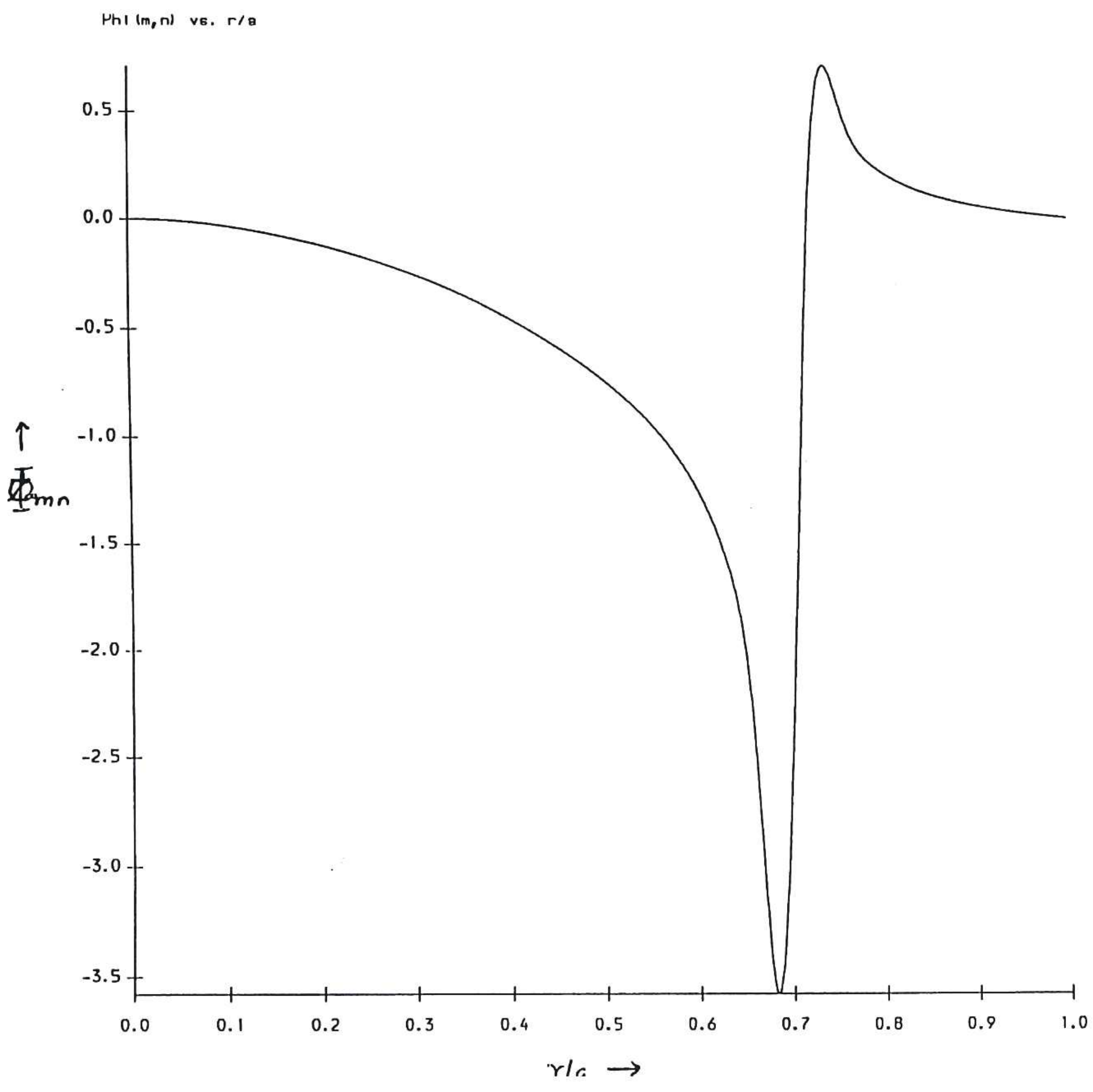

Z (m, n) vs. r/a
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
$Z_{m,n}$
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
r/a

Fig. 4a

Phi (m,n) vs. r/a


1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
-2.0
-2.5
-3.0
-3.5
-4.0
-4.5
-5.0
-5.5
$\Phi_{m,n}$
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
r/a


Z (m,n) vs. r/a
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
Z_{mn}
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
r/a



Phi (m,n) vs. r/a
0.5
0.0
-0.5
-1.0
-1.5
-2.0
-2.5
-3.0
-3.5
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
Φ_{mn} ↑
r/a →