AEA FUS 264

AEA Technology
Fusion
(UKAEA/Euratom Fusion Association)

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April 1994

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Abstract

A self contained model of the complete sawtooth cycle is developed and described. The model incorporates a trigger condition related to $m = 1$ internal kink stability and a description of the crash phase in which current relaxation in the plasma core is calculated to $0(\epsilon^2)$ accuracy ($\epsilon = a/R \ll 1$). The growth and subsequent healing of an $m = 1$, $n = 1$ magnetic island is simulated by a simultaneous relaxation event in an annular region encompassing the $q = 1$ surface. The complete model contains no arbitrary parameters and when implemented in a transport/stability code such as the LARS code will give definite predictions of eg sawtooth period, island size at the instant of fast collapse, range of variation of axial safety factor q_o, behaviour of the radius of the $q = 1$ surface.

1 Introduction

In an earlier note [1] a simple model for the sawtooth crash was developed and investigated numerically. In this model it was assumed that the initial growth of an $m = 1$ instability triggered resistive-interchange instability in the plasma core where the average curvature, as measured by D_R[2], is unfavourable. In the 'unstable' zone, which typically stretches from the magnetic axis to a radial position close to, but inside, the $q = 1$ surface, it was assumed that Taylor relaxation takes place, and the post-crash state was calculated in terms of the pre-crash state by making use of four conditions (three conserved quantities: total energy, core helicity and core toroidal flux, together with pressure balance at the mixing radius r_m).

This model has several attractive features, and several weaknesses. On the one hand, since the mixing radius r_m is defined by

$$D_R(r_m) = 0 \quad , \tag{1.1}$$

the mixing radius lies just within r_1 (where $q(r_1) \equiv 1$) and hence the inversion radius r_i (defined by $p_{\text{initial}}(r_i) = p_{\text{final}}(r_i)$) also lies inside r_1. This feature is in agreement with careful observations on eg TEXTOR[3]. In addition, when steady sawtoothing is simulated with the LARS code by initiating regular sawtooth events with a steady repetition period τ_s (taken from experimental observations) the value of axial q ($q_o \equiv q(o)$) was observed to oscillate with small amplitude $\Delta q \simeq 0.07$ around a mean value $q_o \simeq 0.77$, in broad agreement with experimental measurements on TEXTOR[4].

On the other hand the model lacks a prescription for initiating a sawtooth event, and therefore has no direct link with the $m = 1$ stability properties of the evolving equilibrium. Since r_m is typically smaller than r_1, no reconnection or pressure relaxation can occur at and beyond the $q = 1$ surface, so $m = 1$ island growth cannot be simulated. Finally, since $D_R(r) > 0$ (unstable) for $0 < r < r_m$ throughout most of the sawtooth ramp time it was unclear why core relaxations should only take place at specific regular intervals.

In this note we address these weaknesses of the original model and describe a more sophisticated, and completely self contained model of a sawtoothing Tokamak. This model contains no free parameters.

2 Sawtooth Crash Model: A Qualitative Description

Since all sawtooth events involve the growth (sometimes very rapid growth) of an $m = 1$, $n = 1$ internal mode, we start with the criterion for linear stability of this mode in the collisional diamagnetic regime[5]. This can be expressed in the form of the following condition for stability:-

$$| s\frac{\omega_A}{\omega_{*e}} | \lesssim \Delta'_{1/1} \lesssim | \omega_{*e}\tau_\eta |^{1/2} \tag{2.1}$$

where $\omega_A = v_A/\sqrt{3}R$, $\tau_\eta = 4\pi r_1^2/\eta c^2$, $s \equiv s(r_1) = r_1 q'(r_1)$, and $\Delta'_{1/1}$ is the outer region stability index for the $m = 1$, $n = 1$ mode, which can be calculated by the T3[6] or T7[7] stability codes, or estimated analytically. A second stable zone exists for $\Delta'_{1/1} < 0$, but this requires that the current profile be maintained as a shoulder profile throughout the ramp phase, a condition which was impossible to satisfy in simulations of the TEXTOR experiment.

When either inequality in equation (2.1) is violated we assume that an $m = 1$, $n = 1$ magnetic island begins to grow (the commencement of the $m = 1$ precursor oscillation), but that this triggers a sawtooth collapse only when the steep pressure gradient which develops at the separatrix reaches the unfavourable curvature ($D_R > 0$) in the core of the Tokamak. During the ramp phase, although D_R is positive in this zone it is not of sufficient magnitude to drive the resonant, tearing-parity g-modes unstable, since these modes are, of necessity, of rather high mode number ($m, n \simeq 9, 10;\ 10, 11;$ etc). However, the sharp increase of $p'(r)$ just outside the island separatrix[8] (see Section 5 for an estimate of this) may drive these rather short wavelength g-modes unstable in the region where $D_R > 0$. The equivalent pressure front advancing outwards from the $q = 1$ surface, as the $m = 1$ island grows, finds itself in a $D_R < 0$ (stable) plasma and merely diffuses away (the heat pulse propagation). A detailed discussion of the stability of tearing and twisting parity g-modes in the plasma core is presented in Appendix A.

The sawtooth collapse is itself represented by:-

(i) Taylor relaxation in the plasma core, defined by $0 \le r \le r_m$ (or by $r_{m1} \le r \le r_{m2}$ in the case where D_R changes sign between $r = 0$ and r_m; this latter circumstance occurs for hollow pressure profiles and results in predictions of partial sawteeth).

(ii) Taylor relaxation across the annulus $r_m < r < \hat{r}_m$ where the outer mixing radius, $\hat{r}_m$, is now defined as the outer separatrix position of the $m/n = 1/1$ magnetic island at the instant when the inner separatrix is located at r_m, ie

$$\psi^*(\hat{r}_m) = \psi^*(r_m), \tag{2.2}$$

where

$$\psi^*(r) = \int_o^r r dr \left(\frac{1}{q} - 1\right). \tag{2.3}$$

This outer mixing radius must, of course, be greater than r_1 but will always be less than the Kadomtsev mixing radius [9], r_{mk}, which corresponds to the total reconnection condition

$$\psi^*(r_{mk}) = \psi^*(o). \qquad (2.4)$$

Consequently both the mixing radius and the inversion radius take smaller values than in the Kadomtsev reconnection model. Reference 10 has an earlier discussion of Kadomtsev reconnection and Taylor relaxation.

The two relaxation phenomena are separated because the annular relaxation in $[r_m, \hat{r}_m]$ is introduced as a device to simulate the partial Kadomtsev reconnection process and its reversal, and it is not envisaged that the $m = 1$ magnetic island is invaded by the high-n micro-turbulence associated with the $n \gtrsim 10$ g-modes which are responsible for relaxation of the core plasma $[0, r_m]$. However, the constant pressure demanded in each of these relaxation regions is assumed to have the same value, $\bar{p}$, since no thermal barrier exists between them.

In the next sections we calculate the post-crash equilibrium and derive the equations which relate the pre-crash to the post-crash equilibrium and enable the latter to be calculated explicitly.

3 The Post-crash Equilibrium State

In the post-crash state the core plasma ($0 \leq r \leq r_m$) is assumed to be in a Taylor relaxed state defined by $\boldsymbol{J} = \mu \boldsymbol{B}, p(r) = \bar{p}$ with $\bar{p}$ and μ constant. Representing the magnetic field in the form

$$\boldsymbol{B} = R_o B_o [g(r)\nabla\phi + f(r)\nabla\phi \times \nabla r], \qquad (3.1)$$

where r, θ, ϕ are field line straightened coordinates with Jacobian $J \equiv r^2 R/R_o$ and r a magnetic surface variable, it follows that

$$\mu f = -g'. \qquad (3.2)$$

As in Ref 1 we consider a large aspect ratio equilibrium with $\beta \sim 0(\epsilon^2)$ $[\epsilon = a/R_o]$, and expand

$$\begin{aligned} f &= \epsilon f_1 + \epsilon^3 f_3 + \dots, \\ g &= 1 + \epsilon^2 g_2 + \dots, \\ p &= \epsilon^2 p_2 + \dots, \\ \mu &= \epsilon \mu_1 + \epsilon^3 \mu_3 + \dots, \end{aligned} \qquad (3.3)$$

and utilise equilibrium relations derived in eg Connor & Hastie[11].

Then following Ref 1 we obtain

$$\begin{aligned} f_1(r) &= \frac{1}{2}\mu_1 r, \\ g_2(r) &= -\frac{1}{4}\mu_1^2 r^2 + \bar{g}_2. \end{aligned} \qquad (3.4)$$

An expression for $f_3(r)$ can also be obtained. This involves the Shafranov shift $\Delta(r)$ of the magnetic surfaces, as well as their shapes. Details are given in Appendix B.

To simulate the growth, saturation and decay of the $m = 1$, $n = 1$ magnetic island it is assumed that the annular region $r_m \leq r \leq \hat{r}_m$ (with $\hat{r}_m$ defined by equation (2.8)) is also in a relaxed state defined by $\boldsymbol{J} = \hat{\mu}\boldsymbol{B}, p = \bar{p}$. However, in this annular zone solution of equation (3.2) along with the radial pressure balance equation

$$\frac{p_2'}{B_o^2} + g_2' + \frac{f_1}{r}(rf_1)' = 0, \tag{3.5}$$

results in

$$f_1(r) = \frac{1}{2}\hat{\mu}_1 r + \frac{\hat{c}_1}{r}, \tag{3.6}$$

$$g_2(r) = -\frac{1}{4}\hat{\mu}_1^2 r^2 - \hat{\mu}_1\hat{c}_1 \ell n\left(\frac{r}{r_m}\right) + \hat{g}_2, \tag{3.7}$$

where the additional constant of integration $\hat{c}_1$ is related to the total excess current flowing within the core plasma [1]. An expression for $f_3(r)$ is obtained in Appendix B. Thus far the solution for the post-crash state contains six undetermined constants $(\bar{p}, \mu_1, \bar{g}_2, \hat{\mu}_1, \hat{c}_1, \hat{g}_2)$. These constants can all be calculated in terms of properties of the pre-crash equilibrium and this forms the content of Section 4 of this report. However, we first note that if this were done using the six available conservation equations relating the pre-crash to the post-crash state, the final configuration would not satisfy cylindrical pressure balance (ie continuity of $p(r) + \frac{1}{2}B^2$) at the surfaces r_m and $\hat{r}_m$. To adjust this non-equilibrium back to global equilibrium we allow for small $(0(\epsilon^2))$ radial expansion or compression of the three zones $[0, r_m]$, $[r_m, \hat{r}_m]$ and $[\hat{r}_m, a]$. These involve expansion/compression of ideally conducting plasma and do not modify the flux and helicity conservation properties. They are introduced by assuming that, in the final state, the three plasma zones are defined by $[0, r_{mf}]$, $[r_{mf}, \hat{r}_{mf}]$, $[\hat{r}_{mf}, a]$ where

$$r_{mf} = r_m + \epsilon^2\xi_2, \tag{3.8}$$

$$\hat{r}_{mf} = \hat{r}_m + \epsilon^2\hat{\xi}_2. \tag{3.9}$$

4 Sawtooth Crash Model: Derivation of Equations

The full sawtooth crash model has defined the post-crash equilibrium in terms of eight constant parameters $(\bar{p}_2, \mu_1, \bar{g}_2, \hat{\mu}_1, \hat{c}_1, \hat{g}_2, \xi_2, \hat{\xi}_2)$. The suffices refer to the order (in ϵ) of each quantity, while a circumflex is used to denote quantities associated with the annular region.

[1] For simplicity of notation and to avoid confusion with $\mu = \boldsymbol{J}.\boldsymbol{B}/B^2$ we take $\mu_0 \equiv 1$ in eqn.(3.5) and subsequently.

These constants will now be determined in terms of the pre-crash equilibrium by using the following conservation and equilibrium relations: -

(i) Conservation of core helicity

$$K = 2\int_0^{r_m} f(r)dr \int_0^r r'g(r')dr'. \tag{4.1}$$

(ii) Conservation of core toroidal flux

$$\Phi = \int_0^{r_m} rgdr. \tag{4.2}$$

(iii) Conservation of helicity in the annulus $[r_m, \hat{r}_m]$

$$\hat{K} = 2\int_{r_m}^{\hat{r}_m} fdr \int_{r_m}^{r} g(r')r'dr'. \tag{4.3}$$

(iv) Conservation of toroidal flux in the annulus

$$\hat{\Phi} = \int_{r_m}^{\hat{r}_m} rgdr. \tag{4.4}$$

(v) Conservation of poloidal flux in the annulus

$$\hat{\Psi} = \int_{r_m}^{\hat{r}_m} fdr. \tag{4.5}$$

(vi) Conservation of global energy

$$W = \int_0^a rdr\left[\frac{3}{2}p + \frac{1}{2}\left\langle\frac{B^2R^2}{R_0^2}\right\rangle\right]. \tag{4.6}$$

(vii) Cylindrical pressure balance at $r_m + \epsilon^2\xi_2$

$$\left[p_2 + \frac{1}{2}B_0^2\left(f_1^2 + 2g_2\right)\right]_{r_m+\epsilon^2\xi_2} = 0. \tag{4.7}$$

(viii) Cylindrical pressure balance at $\hat{r}_m + \epsilon^2\hat{\xi}_2$

$$\left[p_2 + \frac{1}{2}B_0^2\left(f_1^2 + 2g_2\right)\right]_{\hat{r}_m+\epsilon^2\hat{\xi}_2} = 0, \tag{4.8}$$

where we have used the notation (in (4.7) and (4.8))

$$[X]_{r_i} = \lim_{\delta\to 0}\left\{X(r_i+\delta) - X(r_i-\delta)\right\}, \tag{4.9}$$

and $< X >= \frac{1}{2\pi}\oint X d\theta$.

In applying global energy conservation, (vi), we have assumed, in effect, that the post-crash equilibrium is being calculated after sufficient time has elapsed that any turbulent motion associated with the growth and decay of the $m = 1$, $n = 1$ kink mode or the g-modes has been damped by viscosity and appears as thermal plasma energy. The asymmetry in the poloidal flux constraints (annular poloidal flux is conserved, while core poloidal flux is not) arises because the annulus is bounded on both sides by perfect conductors (by assumption) while the core plasma is contained by a conductor on its outermost boundary only. Thus an annular relaxation does not reduce to a core relaxation in the limit $r_m \to 0$ (ie inner radius of the annulus becoming vanishingly small). In this limit a singular axial current is present in the annular relaxation, but absent in a core relaxation. It may seem contradictory that the core plasma is regarded as ideal when considering the relaxation of the annular region $[r_m, \hat{r}_m]$ and vice versa, but we stress that the use of a relaxation model for the annulus is only a device to simulate the effect of growth and decay of an $m = 1$, $n = 1$ island. Both relaxation and island 'healing' should leave behind a plasma in which $j_{\|} \simeq$ constant and $p \simeq$ constant.

Returning to equations (4.1) - (4.8) we now obtain:-

$$K_1 = \frac{1}{8}\mu_1 r_m^4, \tag{4.10}$$

$$\Phi_2 = r_m\xi_2 - \frac{1}{16}\mu_1^2 r_m^4 + \frac{1}{2}\bar{g}_2 r_m^2. \tag{4.11}$$

$$\hat{K}_1 = \frac{1}{2}(\hat{r}_m^2 - r_m^2)\left\{\hat{c}_1 + \frac{1}{4}\hat{\mu}_1(\hat{r}_m^2 + r_m^2) - \frac{1}{2}\hat{\mu}_1 r_m^2 - \frac{2\hat{c}_1 r_m^2}{(\hat{r}_m^2 - r_m^2)}\ln(\hat{r}_m/r_m)\right\}, \tag{4.12}$$

$$\begin{aligned}\hat{\Phi}_2 &= \hat{r}_m\hat{\xi}_2 - r_m\xi_2 + \frac{1}{2}(\hat{r}_m^2 - r_m^2)\left[\hat{g}_2 + \frac{1}{2}\hat{\mu}_1\hat{c}_1 - \frac{1}{8}\hat{\mu}_1^2(\hat{r}_m^2 + r_m^2)\right] \\ &- \frac{\hat{\mu}_1\hat{c}_1}{2}\hat{r}_m^2\ln(\hat{r}_m/r_m),\end{aligned} \tag{4.13}$$

$$\hat{\Psi}_1 = \frac{1}{4}\hat{\mu}_1(\hat{r}_m^2 - r_m^2) + \hat{c}_1\ln(\hat{r}_m/r_m), \tag{4.14}$$

$$\begin{aligned}W_2 &= \frac{3}{4}\frac{\bar{p}_2}{B_0^2}\hat{r}_m^2 + \frac{1}{32}\mu_1^2 r_m^4 + \frac{1}{32}\hat{\mu}_1^2(\hat{r}_m^4 - r_m^4) + \frac{1}{4}\hat{\mu}_1\hat{c}_1(\hat{r}_m^2 - r_m^2) \\ &+ \frac{1}{2}\hat{c}_1^2\ln(\hat{r}_m/r_m),\end{aligned} \tag{4.15}$$

$$0 = 2(\hat{g}_2 - \bar{g}_2) + \frac{1}{4}(\mu_1^2 r_m^2 - \hat{\mu}_1^2\hat{r}_m^2) + \hat{\mu}_1\hat{c}_1 + \frac{\hat{c}_1^2}{r_m^2}. \tag{4.16}$$

$$
\begin{aligned}
P_2 &= 2\left(\hat{g}_2 + \frac{\bar{p}_2}{B_0^2}\right) - 4\frac{\hat{r}_m \hat{\xi}}{(a^2 - \hat{r}_m^2)} \\
&- \frac{1}{4}\hat{\mu}_1^2 \hat{r}_m^2 - 2\hat{\mu}_1 \hat{c}_1 \ln(\hat{r}_m / r_m) + \hat{\mu}_1 \hat{c}_1 + \frac{\hat{c}_1^2}{\hat{r}_m^2},
\end{aligned} \tag{4.17}
$$

where the quantites $K_1, \hat{K}_1, \Phi_2, \hat{\Phi}_2, \hat{\Psi}_1$, W_2 and P_2 are defined as follows:-

$$K_1 = \int_0^{r_m} f_{1i}(r) r^2 dr, \tag{4.18}$$

$$\hat{K}_1 = \int_{r_m}^{\hat{r}_m} f_{1i}(r)(r^2 - r_m^2) dr, \tag{4.19}$$

$$\Phi_2 = \int_0^{r_m} g_{2i}(r) r dr, \tag{4.20}$$

$$\hat{\Phi}_2 = \int_{r_m}^{\hat{r}_m} g_{2i}(r) r dr, \tag{4.21}$$

$$\hat{\Psi}_1 = \int_{r_m}^{\hat{r}_m} f_{1i}(r) dr, \tag{4.22}$$

$$W_2 = \int_0^{\hat{r}_m} r dr \left[\frac{3}{2}\frac{p_{2i}(r)}{B_0^2} + \frac{1}{2} f_{1i}^2(r)\right], \tag{4.23}$$

$$P_2 = \left[2\frac{p_{2i}(\hat{r}_m)}{B_0^2} + 2g_{2i}(\hat{r}_m) + f_{1i}^2(\hat{r}_m)\right], \tag{4.24}$$

and the subscript i in equations (4.18)-(4.24) indicates that the initial, or pre-crash, equilibrium profiles are used to calculate these quantities.

Solution of the eight algebraic equations (4.10)-(4.17) now determines the parameters defining the post-crash equilibrium state. However, at this level of approximation in the ϵ expansion $q(r) = rg/R_0 f$ is only determined to leading order since we have not yet determined $f_3(r)$. Because stability of the $m = 1, n = 1$ resistive kink mode, which is responsible for triggering the sawtooth event, is sensitively dependent on the radial profile of $q(r)$ it is important to determine $q(r)$ more accurately, and to evolve it accurately (to order q^2) throughout the subsequent ramp phase of the sawtooth cycle. To this end we calculate $f_3(r)$ after the crash, by equating the third order helicities $K_3, \hat{K}_3$ to their post-crash values. This requires some care because of the $0(\epsilon^2)$ displacement of the two mixing radii, r_m and $\hat{r}_m$. This calculation is presented in appendix C.

5 Summary and Discussion

The model described above is completely self contained, requiring no additional input or intervention if it is implemented in a transport code linked to a calculation of $\Delta'_{1/1}$.

To summarise:- The sawtooth event is assumed to be initiated by the destabilisation of the resistive kink mode (determined in a regime where diamagnetic effects are important). At the instant of $m = 1, n = 1$ destabilisation it is assumed that the $m = 1, n = 1$ precursor activity commences. The rapid thermal collapse is assumed to occur when the $m = 1, n = 1$ island separatrix, advancing inwards towards the axis reaches the region of unfavourable curvature ($D_R > 0$). By this time a large pressure gradient will have developed in a narrow layer close to the advancing separatrices. The magnitude of this pressure gradient has been estimated in Ref 8. It depends on the magnitude of the thermal diffusivity $\chi_\perp$ (which attempts to smooth out the steep gradient) and the rate of growth of the island, dW/dt, responsible for eroding the temperature profile and generating a pressure 'cliff' at the separatrix. In Ref 8 we find an estimate of the width, δ, of this layer to be of order

$$\delta \sim \chi_\perp \left(\frac{dW}{dt}\right)^{-1} . \qquad (5.1)$$

Taking $\chi_\perp \sim 1m^2/sec, W \simeq W_0 e^{\gamma t}$ we estimate the pressure gradient magnification factor,

$$\frac{W}{\delta} \approx \frac{W^2\gamma}{\chi_\perp} \qquad (5.2)$$

to be ≈ 10 for a 10cm island width and $\gamma^{-1} \simeq 1$ msec or a 3cm island with $\gamma^{-1} \sim 100\ \mu$ sec.

This enhanced pressure gradient is assumed to destabilise resistive g-modes in the $D_R > 0$ region, which are then progressively destabilised inwards towards the magnetic axis leading to a current and pressure relaxation to a Taylor state ($j_\parallel/B = \mu, p = \bar{p}$ with $\mu, \bar{p}$ constant) in this region. The mechanism for this propagating collapse should be similar to that which initiates the first g-mode growth, via rapid island growth generating ever larger local pressure gradients as the original pressure profile is eroded.

As this relaxation event takes place it is unclear how the original $m = 1, n = 1$ island should evolve. However, since experimental observations indicate that continued island growth and complete Kadomtsev reconnection does *not* occur, we assume that the $m = 1$ island now heals leaving in its wake an annular region in which both pressure and $j_\parallel/B$ have been flattened. To simulate this process the values of flattened current and pressure in this region are calculated by representing this process also as a (separate) relaxation event to a Taylor state.

All these events are treated as instantaneous on the time scale of the sawtooth ramp so that this model of the sawtooth collapse can be implemented, as a single event, within a 1-1/2 D transport/stability code such as LARS.

An interesting prediction of the model concerns the magnitude of the $m = 1$ island at the instant of thermal collapse. This is given by

$$\frac{W}{2} \approx (r_1 - r_m), \tag{5.3}$$

where $D_R(r_m) = 0$ defines r_D .

For a circular cross section plasma $r_1 - r_m$ is readily estimated by expanding $D_R(r)$ around r_1; thus from

$$q^2(r_m) - 1 - q^2\left(\frac{R}{r}s\Delta_s'\right)_{r_m} = 0, \tag{5.4}$$

with Δ_s the Shafranov shift, and $s = rq'/q$, we obtain

$$(r_1 - r_m) \sim \frac{r_1}{2}\left(\frac{1}{4} + \beta_p\right), \tag{5.5}$$

where we have estimated $\Delta_s'(r)$ for the case of low shear, and

$$\beta_p = -\frac{2}{B_\theta^2(r_1)}\int_0^{r_1}\frac{dp}{dr}\left(\frac{r}{r_1}\right)^2 dr. \tag{5.6}$$

Thus, at the end of the phase of island growth (the precursor oscillations) the island width predicted for a circular cross-section Tokamak is

$$\frac{W}{r_1} \approx \frac{1}{4} + \beta_p, \tag{5.7}$$

in broad agreement with observations on eg TFR(12).

Extending this prediction to include Tokamaks of shaped cross-section we note that(13)

$$D_R \simeq -\frac{2rp'q^2}{s^2B_0^2}\left\{1 - \frac{1}{q^2} + s\frac{R}{r}\Delta_s' - \frac{3}{4}\left(E' + \frac{E}{r}\right)\right\}, \tag{5.8}$$

where $E/r = (\kappa - 1)/(\kappa + 1)$ is a measure of the surface ellipticity, so that r_m is greater than for a circular cross-section, and can even exceed r_1. However, during the early phase of a Tokamak discharge while the plasma current is increasing (the current ramp phase) the shape of the magnetic surfaces in the plasma core is only weakly elliptical so that $r_m \approx r_{mcircular}$, and sawteeth might be expected to display $m = 1$ precursor behaviour. Later, during current flat top, when the magnetic surfaces within the $q = 1$ region are quite strongly elliptical, much shorter precursor trains are to be expected, or indeed none at all if $r_m \gtrsim r_1$. These qualitative predictions could account for the presence of precursor oscillations in JET sawteeth early in the discharge history, and their disappearance in the current flat-top phase.

6 Acknowledgements

The authors thank Dr J W Connor for discussions. This work was funded jointly by the UK Department of Trade and Industry and Euratom.

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Appendix A Tearing parity g-modes for $q < 1$

The dispersion relation for tearing parity modes in the presence of unfavourable curvature at the singular surface, was originally derived, in the absence of diamagnetic effects, by Glasser, Greene and Johnson[2]. It takes the form

$$\Delta' = \frac{2.1}{s^{1/2}}\left(\frac{S}{n^2}\right)^{3/4}\hat{\gamma}^{5/4}\left[1 - \frac{\pi}{4}\frac{D_R s}{\hat{\gamma}^{3/2}}\left(\frac{n^2}{S}\right)^{1/2}\right], \tag{A.1}$$

where $s = rq'/q$ is the shear, n the toroidal mode number, $S = (\tau_\eta/\tau_A)$ is the magnetic Reynold's number with $\tau_A = Rq\sqrt{1+2q^2}/v_A$ and $\tau_\eta = r_s^2/\eta_{||}c^2$.

The eigenvalue $\hat{\gamma} = (\gamma\tau_A)$ and the quantity D_R whose sign determines whether the layer physics has a stabilizing or destabilizing influence on the tearing-parity mode resonant at $r = r_s$, is given in the circular cross-section, $\beta \sim O(\epsilon^2)$, large aspect ratio approximation, by

$$D_R = +\frac{2rp'q^2}{B_0^2 s^2}\left[1 - \frac{1}{q^2} + \frac{R}{r}\Delta_s' s\right], \tag{A.2}$$

where $\Delta_s(r)$ is the Shafranov shift of the equilibrium magnetic surfaces.

In the original paper[2] Glasser, Greene and Johnson were particularly interested in the additional stabilization introduced by the favourable curvature ($D_R < 0$) for tearing modes at surfaces with $q > 1$ (as for the $m = 2, n = 1$ mode). For such modes, they demonstrated that (A.1) generates a marginal stability condition only when Δ' is positive and sufficiently large, the critical value, Δ_c', being given by

$$\Delta_c' = \frac{1.52}{s^{1/2}}\left[-D_R s\right]^{5/6}\left(\frac{S}{n^2}\right)^{1/3}. \tag{A.3}$$

A similar situation arises for tearing modes which are resonant at surfaces where $q(r_s)$ lies in the range $q_0 < q < 1$. In this case, however, m and n must, of necessity, be rather large ($m = 9, n = 10$ is resonant at $q = 0.9$, and similar mode numbers are required for $q = \frac{10}{11} \simeq 0.91, q = \frac{11}{12} = 0.917$ etc.) so that $\Delta_{m/n}'$ is certainly negative, and well approximated by

$$\Delta_{m/n}' \approx -2m \approx -2n. \tag{A.4}$$

Now, setting $\gamma = -i\omega$ and equating real and imaginary parts of (A.1) to zero, marginal stability is found to occur when

$$\left(-\Delta_{m/n}'\right) = \frac{2.74}{s^{1/2}}\left[D_R s\right]^{5/6}\left(\frac{S}{n^2}\right)^{1/3}. \tag{A.5}$$

Interpreted as a criterion for the mode number n, (A.5) states that modes with $n > n_c$ (a critical value) are stable where n_c is given by

$$n_c = 1.2[D_R]^{1/2}(sS)^{1/5}, \tag{A.6}$$

which indicates that all possible modes resonant inside the $q = 1$ surface should be stable if $q_0 > \frac{3}{4}$ and dp/dr is not abnormally large at any resonant surface.

A more relevant form of the dispersion relation (A.1), incorporating electron and ion diamagnetic drift effects, was obtained by Bussac et al[14] and Ara et al[15]. This takes the form

$$\Delta' = \frac{2.1}{s^{1/2}}\left(\frac{S}{n^2}\right)^{3/4}\left[\gamma(\gamma + i\omega_R)(\gamma + i\omega_R + i\omega_{*e})^3\right]^{1/4}$$
$$\times\left\{1 - \frac{\pi}{4}D_R s\left(\frac{n^2}{S}\right)^{1/2}[\gamma(\gamma + i\omega_R + i\omega_{*e})]^{-1/2}\right\}, \tag{A.7}$$

where ω_{*e} is the electron diamagnetic frequency and ω_R is the plasma rotation frequency in the laboratory frame.

When a similar marginal stability analysis is performed on this dispersion relation the marginal stability condition takes the form

$$\left(-\Delta'_{m/n}\right) = \frac{4.2}{s^{1/2}}[D_R s]^{3/2}\left|\frac{\omega_A}{\omega_{*e}}\right|, \tag{A.8}$$

where we have dropped ω_R for simplicity. Again, translating this into a stability boundary ($n > n_c$ stable) we find the critical value of n given by

$$n_c = 1.4 s^{1/2} D_R^{3/4}\left|\frac{\omega_A}{\omega_{*1}}\right|^{1/2}, \tag{A.9}$$

where ω_{*1} is the electron diamagnetic frequency with $n \equiv 1$. This result again predicts that all possible resonant modes within the $q = 1$ surface should be stable when typical equilibrium pressure profiles $p(r)$ are used to evalue D_R. The twisting parity g-mode[16] is also stable (stabilized by compressible effects) when

$$D_R \frac{B_0^2}{\gamma p} < 1. \tag{A.10}$$

These calculations suggest that there is no difficulty in explaining the absence of local g-mode activity in the core of a Tokamak under normal (sawtooth ramp) conditions.

However, the increased pressure gradient which forms in front of the advancing separatrix of a growing $m = 1, n = 1$ island could, if it is large enough, precipitate the growth of either twisting or tearing parity g-modes within the $q = 1$ surface. This enhanced pressure gradient effect has been invoked before by several authors[8,17], but usually to argue for the instability of ideal or resistive ballooning modes as the secondary, fine scale, mechanism of the sawtooth collapse. Such modes would appear to be possible both on the outer ($q > 1$) separatrix and the inner ($q < 1$) one. The attraction of the tearing parity g-mode as an explanation of the sawtooth thermal collapse is that it is naturally one-sided and could cause Taylor relaxation as well as rapid thermal transport only within the $D_R > 0$ region.

Where large positive values of D_R are generated in the steep pressure gradients near the inner separatrix the growth rate of unstable g-modes may be enhanced by electron inertial effects in much the same way as Porcelli[18] has described for the linear phase of the $m = 1, n = 1$ resistive kink. Consequently timescales of order $100\mu s$ should be possible if the $m \approx n \simeq 10$ g-modes become unstable.

Finally, another predictive feature of this model of the sawtooth crash is that, where the growth of the $m = 1, n = 1$ island remains small (for whatever reason) it is possible that the pressure gradient generated in front of the advancing separatrix never exceeds the critical value necessary to precipitate g-mode instability. In this case one might expect the $m = n = 1$ reconnection to continue to completion as originally envisaged by Kadomtsev, and as appears to happen in some 'slow sawtooth collapses' on TFTR[19].

Appendix B Third Order Equilibrium

The functional dependence of $f_3(r)$ in the post-crash state is calculated in this Appendix. Making use of equation (3.2), we find in third order

$$\mu_1 f_3(r) + \mu_3 f_1 = -g_4'(r), \tag{B.1}$$

and

$$\hat{\mu}_1 \hat{f}_3 + \hat{\mu}_3 \hat{f}_1 = -\hat{g}_4'(r), \tag{B.2}$$

where equation (B.1) refers to the core $[0, r_m]$, and (B.2) to the annulus $[r_m, \hat{r}_m]$. Considering first the core plasma, we have

$$f_1(r) = \frac{1}{2}\mu_1 r, \tag{B.3}$$

and we use the pressure balance equation in ϵ^4 order to relate f_3 to g_4'. This takes the form[13],

$$\begin{aligned} \frac{d}{dr}(r f_3) &= \left\{\mu_3 r + \mu_1 \bar{g}_2 r - \frac{1}{4}\mu_1^3 r^3\right\} \\ &+ \mu_1 \left\{\frac{3}{2}\frac{r^3}{R_0^2}\mu_1 - \Delta'^2 r + \Delta'\frac{r^2}{R_0} + \frac{r}{R_0}\Delta - r S_n'^2 + (n^2-1) S_n S_n'\right\} = 0, \end{aligned} \tag{B.4}$$

where the shaping terms S_n are to be summed over n, with $S_2(r)$ the elliptic distortion and $S_3(r)$ the triangularity of the surfaces. Making use of the analytic solutions for $\Delta(r), S_2(r)$ and $S_3(r)$ as in Ref 1 we finally obtain,

$$\begin{aligned} f_3(r) = \frac{1}{2}\mu_3 r &- \frac{1}{2}\mu_1 r\left(\frac{\Delta_0}{R_0} - \bar{g}_2\right) - \frac{1}{16}\mu_1^3 r^3 - \frac{29}{64}\frac{\mu_1 r^3}{R_0^2} \\ &- \mu_1 r \left(\left(\frac{\bar{E}}{r_m}\right)^2 + 3\left(\frac{\bar{T}}{r_m}\right)^2\left(\frac{r^2}{r_m^2}\right)\right), \end{aligned} \tag{B.5}$$

where Δ_0 is the value of the Shafranov shift on axis, and $\bar{E}$ and $\bar{T}$ are the constants

$$S_2(r) \equiv E(r) = \bar{E}\frac{r}{r_m} \ , \tag{B.6}$$

$$S_3(r) \equiv T(r) = \bar{T}\left(\frac{r}{r_m}\right)^2 \ . \tag{B.7}$$

The constant μ_3 will be determined using conservation of core helicity in Appendix C.

Next we solve for $f_3(r)$ in the annular region $[r_m, \hat{r}_m]$. In this region the analogue of equation (B.4) takes the form:-

$$
\begin{aligned}
\frac{d}{dr}\left(r\hat{f}_3(r)\right) &= \left\{\hat{\mu}_3 r - \frac{1}{4}\hat{\mu}_1 r^3 + \hat{\mu}_1^2 \hat{c}_1 r \ln\left(r/r_m\right)\right\} \\
&- \hat{c}_1 \left\{\frac{3}{2}\frac{r}{R_0^2} + \frac{2\Delta'}{R_0} + \frac{\Delta'^2}{R_0} + \frac{S_n'^2}{r} + \frac{2S_n S_n'}{r^2}(n^2-1) - (n^2-1)\frac{S_n^2}{r^3}\right\} \\
&- \hat{\mu}_1 \left\{\frac{3}{2}\frac{r^3}{R_0^2} - \hat{g}_2 r + \frac{r}{R_0}\Delta - r\Delta'^2 + \Delta'\frac{r^2}{R_0} - rS_n'^2 + (n^2-1)S_n S_n'\right\} \quad \text{(B.8)}
\end{aligned}
$$

However, the form of $f_1(r) = \frac{1}{2}\hat{\mu}_1 r + \hat{c}_1/r$ in the annular zone $r_m \le r \le \hat{r}_m$ does not permit simple analytic solutions for $\Delta(r), S_2(r)$ and $S_3(r)$ to be obtained so equation (B.8) must be integrated numerically, and the constant of integration $\hat{c}_3$ determined from poloidal flux conservation in the annulus, where

$$
r\hat{f}_3(r) = \hat{c}_3 + \int_{r_m}^{r} [R.H.S.of B.8]dr, \quad \text{(B.9)}
$$

and

$$
\begin{aligned}
\hat{\Psi}_3 = \int_{r_m}^{\hat{r}_m} f_{3i}(r) &= \xi_2 f_{1i}(r_m) - \hat{\xi}_2 f_{1i}(\hat{r}_m) + \hat{c}_3 \ln\left(\hat{r}_m/r_m\right) \\
&+ \int_{r_m}^{\hat{r}_m} dr \int_{r_m}^{r} [R.H.S.of B.8]dr'. \quad \text{(B.10)}
\end{aligned}
$$

Equation (B.10) can be regarded as determining the constant $\hat{c}_3$, while $\hat{\mu}_3$ is finally determined by helicity conservation in the annulus. The evaluations of μ_3 and $\hat{\mu}_3$ are discussed in Appendix C.

The shaping factors, and Shafranov shift $\Delta(r)$ are known analytically in the core region in terms of three unknown constants $\Delta_0, \bar{E}$ and $\bar{T}$. They take their initial functional forms $\Delta_i(r), E_i(r), T_i(r)$ in the outermost region $[\hat{r}_m, a]$, where only small, $0(\epsilon^2)$, changes have occurred in the pressure and poloidal field profile functions $p_2(r), f_1(r)$. However, in the annular zone $[r_m, \hat{r}_m]$ they are obtained by numerical solution of the equations

$$
\Delta'' + \left(\frac{2(rf_1)'}{rf_1} - \frac{1}{r}\right)\Delta' = \frac{1}{R_0}, \quad \text{(B.11)}
$$

$$
S_n'' + \left(\frac{2(rf_1)'}{(rf_1)} - \frac{1}{r}\right)S_n^2 - (n^2-1)\frac{S_n}{r^2} = 0, \quad \text{(B.12)}
$$

with $n = 2$ for $E(r)$ and $n = 3$ for $T(r)$. These solutions are matched at $\hat{r}_m$ to the external solutions and at r_m to the analytic solutions in the core.

Integrating (B.12) across r_m and $\hat{r}_m$ one obtains

$$[S'_n f_1^2]_{\hat{r}m} = 0, \tag{B.13}$$

$$[S'_n f_1^2]_{rm} = 0, \tag{B.14}$$

determining jump conditions for S'_n at these surfaces, while

$$[\Delta' f_1^2]_{rm} = 0, \tag{B.15}$$

and

$$[\Delta' f_1^2]_{\hat{r}m} = -\left[\frac{2p}{B_0^2}\right]_{\hat{r}_m} \frac{\hat{r}_m}{R_0} \tag{B.16}$$

determine jump conditions for Δ' at r_m and $\hat{r}_m$. The functions $\Delta(r)$ and $S_n(r)$ are continuous at r_m and $\hat{r}_m$.

Thus the evaluation of the post-crash equilibrium involves, firstly, the calculation of $p_2(r), f_1(r)$ and $g_2(r)$ in terms of the eight constants ($\bar{p}, \mu, \bar{g}_2, \hat{\mu}_1, \hat{c}_1, \hat{g}_2, \xi_2$ and $\hat{\xi}_2$). The shift and shape of the surfaces, $\Delta(r), S_n(r)$ can then be evaluated by numerical solution of the equilibrium equations (B.11) and (B.12) in each of the three zones $[0, r_m], [r_m, \hat{r}_m], [\hat{r}_m, a]$ with the appropriate boundary conditions (B.13) - (B.16). Finally $f_3(r)$ can be obtained in terms of the three additional constants $\mu_3, \hat{\mu}_3$ and $\hat{c}_3$, and integrals over the known $\Delta(r)$ and $S_n(r)$ functions. The constants $\mu_3, \hat{\mu}_3$ and $\hat{c}_3$ are finally determined by helicity conservation in the core and in the annulus, together with poloidal flux conservation in the annulus $[r_m, \hat{r}_m]$. At this stage the safety factor

$$q(r) = \frac{r(1+\epsilon^2 g_2)}{R_0(f_1 + \epsilon^2 f_3)}, \tag{B.17}$$

is known to $0(\epsilon^2)$ accuracy, and is evolved to this accuracy by the LARS $1\frac{1}{2}$D(20) transport code during the subsequent sawtooth ramp phase.

Appendix C The calculation of $\mu_3, \hat{\mu}_3$ and $\hat{c}_3$

The poloidal field function $f_3(r)$ in the post-crash state is calculated using helicity conservation in the core $[0, r_m]$ and in the annulus $[r_m, \hat{r}_m]$ together with poloidal flux conservation in the annulus. From the definition

$$K = 2\int_0^{r_m} f\,dr \int_0^r r'g(r')dr', \tag{C.1}$$

we have, for the initial value in the core,

$$K_i = \epsilon\int_0^{r_m} (f_1 + \epsilon^2 f_3)r^2 dr + 2\epsilon^3 \int_0^{r_m} f_1(r)dr \int_0^r r'g_2(r')dr'. \tag{C.2}$$

In the final state

$$\begin{aligned} K_f &= \epsilon\int_0^{r_m+\epsilon^2\xi_2} r^2 dr\left[\frac{1}{2}\mu_1 r + \epsilon^2 f_3(r)\right] + \epsilon^3\int_0^{r_m+\epsilon^2\xi_2} \mu_1 r dr \int_0^r r'g_2(r')dr' \\ &= \epsilon\frac{\mu_1 r_m^4}{8} + \epsilon^3\left[\frac{\mu_1}{2}r_m^3\xi + \int_0^{r_m} r^2 f_3(r)dr + \frac{1}{8}\bar{g}_2 r_m^4 - \frac{1}{96}\mu_1^3 r_m^6\right] + 0(\epsilon^5). \end{aligned} \tag{C.3}$$

Thus,

$$K_{f1} = \frac{1}{8}\mu_1 r_m^4, \tag{C.4}$$

and on making use of (B.5)

$$\begin{aligned} K_{f3} = {} & \frac{1}{2}\mu_1 r_m^3\xi - \frac{1}{48}\mu_1^3 r_m^6 + \frac{1}{8}\mu_3 r_m^4 - \frac{1}{8}\mu_1 r_m^4\left(\frac{\Delta_0}{R_0} - 2\bar{g}_2\right) \\ & -\frac{29}{384}\frac{\mu_1 r_m^6}{R_0^2} - \frac{1}{4}\mu_1 r_m^4\left[\left(\frac{\bar{E}}{r_m}\right)^2 + 2\left(\frac{\bar{T}}{r_m}\right)^2\right], \end{aligned} \tag{C.5}$$

while

$$K_{1i} = \int_0^{r_m} f_{1i} r^2 dr, \tag{C.6}$$

$$K_{3i} = \int_0^{r_m} f_{3i} r^2 dr + 2\int_0^{r_m} f_{1i}(r)dr\int_0^r r'g_{2i}(r')dr'. \tag{C.7}$$

In the annular region the two constants $\hat{c}_3, \hat{\mu}_3$ are determined by third order helicity and poloidal flux conservation. For poloidal flux $\hat{\Psi}$ the initial value is

$$\hat{\Psi}_i = \epsilon \int_{r_m}^{\hat{r}_m} (f_{1i} + \epsilon^2 f_{3i}) dr, \tag{C.8}$$

$$= \epsilon \hat{\Psi}_1 + \epsilon^3 \hat{\Psi}_3.$$

In the final state

$$\hat{\Psi}_f = \epsilon \int_{r_m+\epsilon^2 \xi_2}^{\hat{r}_m+\epsilon^2 \hat{\xi}_2} \left(\frac{1}{2}\hat{\mu}_1 r + \frac{\hat{c}_1}{r} + \epsilon^2 f_3 \right) dr, \tag{C.9}$$

$$= \epsilon \left\{ \frac{1}{4}\hat{\mu}_1 (\hat{r}_m^2 - r_m^2) + \hat{c}_1 \ln(\hat{r}_m / r_m) \right\}$$

$$+ \epsilon^3 \left\{ \frac{1}{2}\hat{\mu}_1 (\hat{r}_m^2 \hat{\xi}_2 - r_m \xi_2) + \hat{c}_1 \left(\frac{\hat{\xi}_2}{\hat{r}_m} - \frac{\xi_2}{r_m} \right) + \int_{r_m}^{\hat{r}_m} f_3 dr \right\}, \tag{C.10}$$

with

$$\hat{\Psi}_{f1} = \frac{1}{4}\hat{\mu}_1 (\hat{r}_m^2 - r_m^2) + \hat{c}_1 \ln(\hat{r}_m / r_m), \tag{C.11}$$

$$\hat{\Psi}_{f3} = \frac{1}{2}\hat{\mu}_1 (\hat{r}_m \hat{\xi}_2 - r_m \xi_2) + \hat{c}_1 \left(\frac{\hat{\xi}_2}{\hat{r}_m} - \frac{\xi_2}{r_m} \right) + \int_{r_m}^{\hat{r}_m} f_3 dr. \tag{C.12}$$

For the initial helicity we have

$$K_i = 2\epsilon^2 \int_{r_m}^{\hat{r}_m} (f_1 + \epsilon^2 f_3)[1 + \epsilon^2 g_2] r dr, \tag{C.13}$$

$$= \epsilon \int_{r_m}^{\hat{r}_m} r^2 f_1 dr + \epsilon^3 \left\{ \int_{r_m}^{\hat{r}_m} r^2 f_3 dr + 2 \int_{r_m}^{\hat{r}_m} f_1 dr \int_{r_m}^{r} r' g_2 dr' \right\},$$

$$= \epsilon K_{1i} + \epsilon^3 K_{3i}, \tag{C.14}$$

while in the final state the following expressions are obtained for $\hat{K}$, after some algebraic manipulation.

$$\hat{K}_f = \epsilon \hat{K}_{f1} + \epsilon^3 \hat{K}_{f3}, \tag{C.15}$$

with

$$\hat{K}_1 = \frac{1}{8}\hat{\mu}_1^2(\hat{r}_m^2 - r_m^2)^2 + \frac{\hat{c}_1}{2}(\hat{r}_m^2 - r_m^2) - \hat{c}_1 r_m^2 \ln(\hat{r}_m/r_m), \tag{C.16}$$

$$\begin{aligned}\hat{K}_3 = & \int_{r_m}^{\hat{r}_m}(r^2 - r_m^2) f_{3f} dr + \frac{1}{2}\hat{\mu}_1(\hat{r}_m^2 - r_m^2)(\hat{r}_m\hat{\xi}_2 - r_m\xi_2) \\ & + \hat{c}_1\left\{(\hat{r}_m^2 - r_m^2)\frac{\hat{\xi}_2}{\hat{r}_m} - r_m\xi_2\ln(\hat{r}_m/r_m)\right\} \\ & - \frac{\hat{\mu}_1^3}{12}(\hat{r}_m^2 - r_m^2)^2(\hat{r}_m^2 + 2r_m^2) \\ & + \frac{\hat{\mu}_1^2\hat{c}_1}{16}\left\{(\hat{r}_m^2 - r_m^2)^2 - 2(\hat{r}_m^4 - r_m^4)\ln(\hat{r}_m/r_m)\right\} \\ & + \frac{1}{2}\hat{\mu}_1\hat{c}_1^2\left\{(\hat{r}_m^2 - r_m^2) - (\hat{r}_m^2 + r_m^2)\ln(\hat{r}_m/r_m)\right\} \\ & + \hat{g}_2\left\{\frac{1}{8}\mu_1(\hat{r}_m^2 - r_m^2) + \frac{c_1}{2}(\hat{r}_m^2 - r_m^2) - c_1 r_m^2\ln(\hat{r}_m/r_m)\right\}.\end{aligned} \tag{C.17}$$