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Abstract. Incompressible magnetoconvection and reduced magnetohydrodynamics are generally regarded as two separate models of very distinct physical phenomena. However, in 2-D Cartesian, 2-D cylindrical and single helicity cylindrical geometries, the two approximations yield the same equations with gravity playing the same role in the former as magnetic curvature in the latter. This equivalence does not seem to have been fully appreciated before, and important implications for both fields are explored here for the first time. Among the many possible consequences for interdisciplinary work, it is shown in particular that techniques used in magnetoconvection give new insights into the dynamics of magnetohydrodynamic double-tearing modes.

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1. Introduction

Incompressible magnetoconvection (hereinafter abbreviated as IMC) is the study of how Rayleigh-Benard convection (RBC) of an electrically conducting fluid is modified in the presence of an applied magnetic field. The governing equations were originally posed by Chandrasekhar [1, § 4]. Its major applications are to the Solar convection zone [2, 3], to industrial processes involving liquid metals such as the cooling of the Lithium blanket for a fusion reactor [4] and to flow in electrolytic cells, eg. ref [5]. High-β reduced magnetohydrodynamics (RMHD) is a model developed by Strauss [6] and independently by Kadomtsev and Pogutse [7], for modelling the behaviour of plasma in tokamak fusion experiments [8], primarily to treat global phenomena. The parameter β measures the value of plasma pressure relative to magnetic pressure. The low-β variant of RMHD is also used in nonlinear studies of fundamental phenomena in magnetohydrodynamics (MHD), such as tearing modes and coalescence instability [9, § 6], which are believed to have widespread application in astrophysics and other laboratory plasmas such as reversed-field pinches.

The equivalence of RBC and a model for tokamak micro-turbulence (resistive pressure gradient plasma turbulence in the absence of magnetic shear) was first pointed out by Horton et al [10], see the recent review by Garcia et al [11]. IMC with a sheared applied magnetic field has recently been put forward as a model for laboratory plasmas [12]. However, this is believed to be the first time that identity between a convection model, namely IMC, and a global model for tokamaks such as RMHD, has been pointed out, and this equivalence is highly significant because of the explicit appearance of the magnetic field $\mathbf{B}$.

The equivalence has potentially many significant implications from both the theoretical and computational view-points, a selection of which is discussed in more detail below. Firstly, however the next Section 2 makes the equivalence explicit, then Section 3 discusses the most important implications of IMC results for RMHD, of which Section 4 is an explicit example for the double-tearing problem. Section 5 discusses other implications, notably the relation between laboratory work and astrophysical applications.

2. Equations

Both IMC and RMHD start from a full set of 3-D, compressible MHD equations. RBC makes the Boussinesq approximation to ensure that the fluid velocity field $\mathbf{u}$ is incompressible, whereas in RMHD, the presence of a strong, uni-directional magnetic field of strength B_0 is crucial for incompressibility of the flow vector in the plane normal to the field. References to original sources for derivation of IMC and RMHD equations have already been given, and moreover there are excellent textbook references for the derivations of both RBC [13, § 14A] and RMHD [14, § 7.4], so these analyses will not be reproduced. The two models are equivalent when there is dependence only on two co-

ordinates, and it will be assumed, following the conventions of IMC, that the ignorable third coordinate is y.

Since both $\mathbf{B}$ and $\mathbf{u}$ are solenoidal, it follows that in Cartesian co-ordinates

$$\mathbf{B} = \nabla \times (0, A(x,z), 0); \qquad \mathbf{u} = \nabla \times (0, \psi(x,z), 0) \tag{1}$$

hence the electric current density j and vorticity ω have only y-components with

$$j = -\mu_0^{-1}\nabla^2 A; \qquad \omega = -\nabla^2\psi \tag{2}$$

where μ_0 is the magnetic permeability. Introducing T as the (potential) temperature (ie. the point difference between the actual temperature and the temperature of the adiabatic temperature gradient), and using standard IMC notation (see Annex A), the IMC equations may then be written as advection-diffusion equations for respectively, magnetic flux, temperature and vorticity:

$$\begin{aligned}
\frac{\partial A}{\partial t} + \frac{\partial(\psi, A)}{\partial(x,z)} &= \eta\nabla^2 A, \\
\frac{\partial T}{\partial t} + \frac{\partial(\psi, T)}{\partial(x,z)} &= \kappa\nabla^2 T, \\
\frac{\partial \omega}{\partial t} + \frac{\partial(\psi, \omega)}{\partial(x,z)} &= \nu\nabla^2\omega + \frac{1}{\rho_0}\frac{\partial(A,j)}{\partial(x,z)} - g\alpha_0\frac{\partial T}{\partial x}
\end{aligned} \tag{3}$$

The high-β RMHD equations are given by Strauss [15]. Somewhat confusingly, the absolute size of β in this model still satisfies $\beta \ll 1$, since in ref [15] β is only 'high' relative to the value assumed for the low-β variant of RMHD, wherein the effective value of $\beta = 0$, corresponding to a complete absence of pressure terms. In Strauss's notation, the poloidal magnetic flux is AR_0, where R_0 is the major radius of the torus and P is his pressure variable. The electric current density J and scalar vorticity W satisfy

$$J = -\nabla^2 A; \qquad W = \rho_0\nabla^2 U \tag{4}$$

It follows that

$$\begin{aligned}
\frac{\partial A}{\partial t} &= [U, A] + \frac{B_0}{R_0}\frac{\partial U}{\partial \varsigma}, \\
\frac{\partial P}{\partial t} &= [U, P] \\
\frac{\partial W}{\partial t} &= [U, W] - [J, A] - \frac{B_0}{R_0}\frac{\partial J}{\partial \varsigma} + \frac{2}{R_0}[R, P]
\end{aligned} \tag{5}$$

where the Jacobian $[f_1, f_2] = \nabla f_1 \times \nabla f_2 \cdot \hat{\varsigma}$ and $\hat{\varsigma}$ is the unit vector in the direction of the toroidal coordinate ς. The Jacobian bracket $[f_1, f_2]$ used in Eq. (5) is like the ∇^2 operator in that it may be defined in a coordinate-free way.

The equivalence of the systems Eqs (3) and (5) begins to become apparent when it is realised that, in slab geometry, $\partial(f_1, f_2)/\partial(x,z)$ is the Jacobian of f_1 and f_2, hence taking into account the difference in signs in Eqs (2) and (4), the nonlinear terms in the two systems are the same. Moreover, because of the above-mentioned abstract property of the Jacobian, equality holds in any 2-D planar co-ordinate system. As to the linear terms, Eq. (5) lacks diffusion, which may be added just as it is included by

Chandrasekhar [1]. The critical linear term is that due to magnetic field curvature, namely $2[R, P]/R_0$, where R is cylindrical polar radius from the major axis, to be compared with $-g\alpha_0 \partial T/\partial x$ in Eq. (3). The latter may be written as

$$\alpha_0 \frac{\partial(gz, T)}{\partial(x, z)} \propto \frac{\partial(\phi_G, T)}{\partial(x, z)} \tag{6}$$

where ϕ_G is the gravitational potential. Hence if $\phi_G \propto R$, the two linear terms are equivalent to within a constant of proportionality, if P is identified with T.

In other words, apart from a constant multiplier, the magnetic curvature and gravity terms are identical. This equivalence of the effects was first pointed out in the classic FKR paper on MHD [16], although only in the context of linear theory in slab geometry. Further, the source quoted by FKR studies the physically quite distinct problem of particle orbits in magnetic fields, i.e. where collective or fluid effects are absent.

The remaining, untreated linear terms in the comparison of the two systems are those in $\partial/\partial\varsigma$. Obviously Eqs (3) and (5) may be brought into equivalence by assuming that there is no dependence on ς. However, as might be anticipated, the fundamental property making for equivalence is the presence of an ignorable coordinate. This is made explicit in the paper by Park et al [17], for the low-β RMHD equations in the case of single helicity m/n, when the transformation

$$A_h = A + \eta c_1 t, \quad J_h = J - c_1, \quad \text{where } c_1 = \frac{2n}{m}\frac{B_0}{R_0} \tag{7}$$

is shown to eliminate the terms in $\partial/\partial\varsigma$. Hence there is also an equivalence between the RMHD and IMC equations in the single helicity case, as well as in the planar co-ordinate system when the IMC y co-ordinate is ignorable. However, the effective gravity is generally position dependent as illustrated below.

The equivalence of Eqs (3) and (5) even in slab geometry has become obscured due to the choice of coordinate system in the RMHD model and the change in notation from A to ψ for the magnetic field. Moreover, as pointed out e.g. in the textbook [18, § 12.1.2], tokamak MHD and astrophysical MHD are expected to be very different because of the fact that generally $\beta \gg 1$ in the latter. However, as shown below, the reference magnetic field used in IMC may be analogous to the poloidal field in RMHD, whence the relevant β is poloidal-β, usually written β_p, and the value of β_p frequently exceeds unity in present day tokamaks.

The RMHD notation has changed since ref [6] appeared, and the equivalence between the notation in current use [14, § 7.4] is given in the Table. Regarding dimensionless groups, see Annex A, considerable care is needed in their comparison. A potentially confusing issue in Annex A is that Prandtl numbers in IMC are always defined with the thermal diffusivity as denominator, whereas in RMHD they may be defined with respect to the magnetic diffusivity. That said, important points to be drawn from Annex A are that the Chandrasekhar number q which measures magnetic field in IMC is related to the RMHD Lundquist number S, and the Rayleigh number r which measures temperature difference corresponds to a magnetic-curvature weighted β.

Table 1. IMC and present RMHD Notation Compared

Quantity	IMC	RMHD
Stream function	ψ	ϕ or φ
Vorticity	ω	$-U$
Magnetic flux potential	A	ψ or $A_{\parallel}$
Current density	j	$\pm J/\mu_0$
Thermodynamic field	T or θ	P or p or n
Horizontal/poloidal coordinate	x	y
Ignorable/toroidal coordinate	y	z
Vertical/radial coordinate	z	x
Reference length	d or d_0	a or L
Magnetic diffusivity	η	η
Thermal diffusivity	κ	D or χ_i
Kinematic viscosity	ν	μ or χ_ϕ

Note that whereas IMC always employs SI units, RMHD may use Gaussian units (where permeability $\mu_0 = 1$).

It is of course also necessary to discuss boundary conditions. The first application of the high-β RMHD equations [6] was to a tokamak of square cross-section so that there was complete equivalence between Eqs (3) and (5), even extending to a position-independent gravity, provided $\partial/\partial y = 0$. Looking at Figure 1((a) and (b)), the constancy of g is easily demonstrated by imagining cylindrical coordinates (R, Z) to be introduced into the cross-section, then, in Eq. (6), identifying IMC z coordinate with R and IMC x with Z. Unfortunately, since for tokamaks, interest attaches to a problem with closed flux-lines in planes of constant azimuthal angle, the IMC problem with its directed net flux in the corresponding plane is very different from the physical point-of-view. Similar remarks apply to the thermodynamic field. (The dynamo problem which might employ zero net flux boundary conditions is not studied in such a 2-D geometry.) However, the $\psi = 0$ boundary conditions on the flow are equivalent to the free-slip boundary conditions employed in IMC.

Physically useful correspondence between IMC and RMHD is achievable by moving to problems localised in minor radius, ie. which occupy all or part of an annular region in a plane defined by constant azimuthal angle, see Figure 1((a) and (c)). If minor radius, or better a coordinate in the direction of increasing poloidal magnetic flux, is taken to correspond to the vertical coordinate of IMC, then there is a net flux in the poloidal direction in the tokamak corresponding to an imposed horizontal magnetic field in IMC. Now, in the tokamak [17], the poloidal flux is produced by induction, so the wall boundary condition is specified as

$$\frac{\partial A_h}{\partial t} = E \tag{8}$$

for electric field E. However, as the timescale for changes induced by E is usually much

slower than other MHD timescales, it is frequently assumed [14, § 6.4] that the total poloidal flux is invariant, in agreement with the assumption made in IMC.

Turning to temperature/pressure, there is an obvious correspondence between the boundary conditions of constant temperature difference and constant pressure on a flux surface for outboard regions close to the horizontal mid-plane of the tokamak. However, note from Figure 1(b) that the effective direction of gravity in the localised IMC model geometry rotates as a flux surface is followed around the minor axis. There may be a beneficial side to this, in that the resulting explicit dependence of IMC gravity on the coordinate x caused by the mapping between coordinate systems, may be mimicked by the use of boundary conditions corresponding to non-propagating disturbances in constant gravity IMC. Instabilities in RMHD are often locked in place poloidally, and any tokamak magnetic field ripple may reinforce the effect.

3. Implications of the Equivalence

One of the key, positive points to emerge from studies of IMC is the predictive power of low order modal truncation, to be discussed further in Section 4. Although formally valid only in the limit where the amplitude of motion as measured by Reynolds number $Re = |\mathbf{u}|d_0/\nu$ is small, such truncated systems describe qualitatively well the results of numerical simulations where $Re = \mathcal{O}(10)$, see Figs 18 and 20 of [21]. There are both mathematical and physical reasons for this. Mathematical rigour is provided by bifurcation theory, which rules out qualitatively different effects for sufficiently small amplitude. The physical reason is that the truncated model's second order spatial modes capture in an approximate sense the formation of boundary layers as Reynolds number increases. In convection problems generally, further increase in Reynolds number leads to a boundary layer instability, which cannot occur in this model. However, computations with a higher order model may mis-represent this instability, and actually give more inaccurate results (consistent with the non-monotone convergence mentioned below) unless the number of modes is large enough to resolve details of the boundary layer.

As illustrated by Knobloch [22], equivariant bifurcation theory provides a very powerful guide to the nonlinear behaviour of IMC once the linear stability and symmetry properties of the model are known. This fact has already been used to devise models for tokamak behaviour [23, 24].

A negative point is that IMC can be quantitatively misleading. Full IMC calculations may systematically over-estimate the heat flux if the boundary layers are not resolved on the computational mesh [25] (for RBC the convergence is generally non-monotonic [26]), and widening the layers by increasing the diffusivities also increases the heat flux, typically by $(\kappa^2/\nu)^{1/3}$ in RBC [11]. As in RBC [27] the IMC numerical solution may appear satisfactory when in fact it is eg. grossly overpredicting the heat flux.

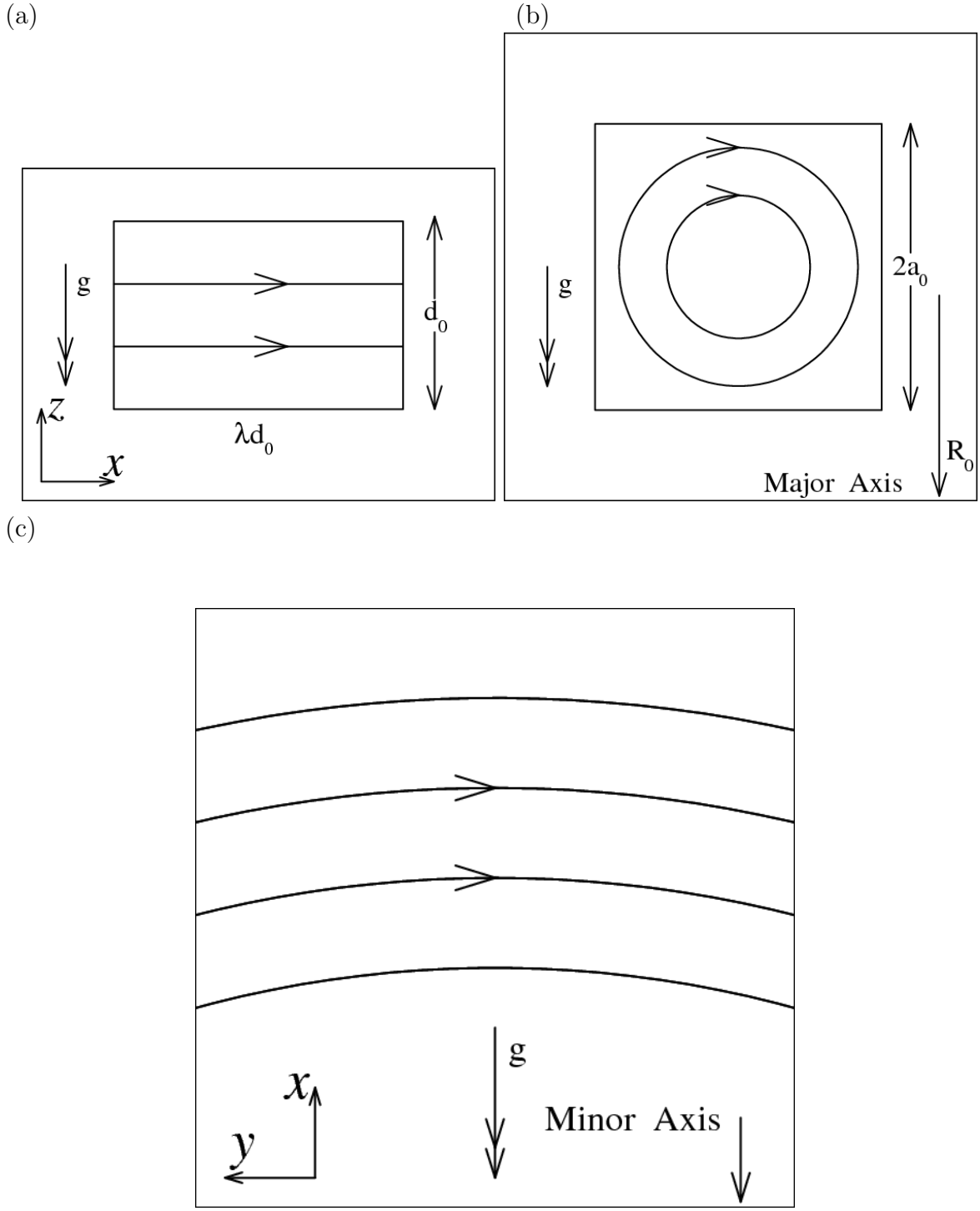

Figure 1. Illustration of the equivalence between the configuration of magnetic field, drawn with single arrows, and gravity used in IMC (*a*, top left) and in tokamak RHMD (*b*, top right and *c*, bottom). The subfigure (b) shows the poloidal component of magnetic field only in a whole cross-section of the torus in minor radius, while the diagram (c) represents an expanded version of a region close to the top of (b). Note that for the purpose of comparison, whereas normally the tokamak is drawn with the major axis vertical, here the major axis is shown running horizontally along the bottom of subfigure (b).

4. Double Tearing

Double tearing is a topic presently of high interest because of its possible role in advanced tokamak scenarios which have a minimum in the radial distribution of magnetic field twist [28], see also Annex B. As with any instability in the context of controlled magnetic fusion, it would be better if it could be avoided, and if this is not always possible, then to try to establish circumstances under which it saturates at small amplitude. The double tearing instability is of particular concern, because of the indications in ref [29] that it becomes a relatively fast instability compared to ordinary tearing instability as Lundquist number increases. 'Single' tearing is itself implicated in a variety of unwanted behaviours, some of which are associated with even apparently saturated, 'magnetic island' states. (Although it is fair to point out that more complicated models of the double tearing instability indicate that asymmetries caused by diamagnetic effects [30] or radially varying toroidally-directed flow [31] may make at least the initial growth resemble single tearing more.)

The equivalence between IMC and RMHD enables a rapid analysis of the possible qualitative behaviour of double-tearing using truncated models. In the interests of clarity and readability, the detailed linear and nonlinear analysis of these systems has been placed in Annex A, the notation from which will be used below. The physics of Annex A differs from refs [25, 32] where an electric field $\propto (1-\phi)\sin(2\pi z)$ is applied in the y-direction to produce the current configuration which is relevant to the double-tearing instability. (Note that $1-\phi$ is an arbitrary electric field strength parameter.)

Compared to the derivation of double-tearing by Pritchett *et al* [29], see also ref [33, § 4.2], there is an implicit assumption in IMC that magnetic field perturbations and accompanying flows vanish completely outside $0 < z < 1$, rather than only at infinity. The two jumps in the radial structure of the unstable mode expected at high S which lead to the 'isolated plateau' profile of ref [29, Fig. 2(c)], are replaced by the smooth cosine structure corresponding to $\partial\psi/\partial z$ in Annex A. The IMC model is slightly more complicated than Pritchett *et al*'s, in that it partially accounts for the effect of a net toroidal current by having a fixed net (poloidal) flux, absent from the RMHD paper. Nonetheless, the IMC model is unlikely to be quantitatively accurate for RMHD because it is derived in limit where diffusive processes are dominant, whereas theoretical estimates of diffusion in tokamaks yield small values unless turbulence is important.

The truncated fifth order time dependent system including an applied electric field of the above special form, may be produced by inspection from that in Annex A on substituting $e \to e + 1 - \phi$, viz.

$$\begin{aligned}
\dot{a} &= \sigma[-a + rb - \zeta qd(\varpi + (\varpi - 1)(e - \phi)] \\
\dot{b} &= -b + a(1 - c) \\
\dot{c} &= -\varpi(c - ab) \\
\dot{d} &= -\zeta d + a(\phi - e) \\
\dot{e} &= -\varpi(\zeta e - ad)
\end{aligned} \tag{9}$$

In the above system, $a(t)$ is the time dependent amplitude of the velocity perturbation which is accompanied by thermal modes of amplitudes $b(t)$ and $c(t)$ and magnetic modes of amplitudes $d(t)$ and $e(t)$. Chandrasekhar parameter q is here a measure of absolute magnetic field strength in contrast to its normal meaning as measure of field twist in tokamak physics. Parameters σ and ζ are the Prandtl number and the magnetic (or third) Prandtl number respectively, and ϖ is a measure of the mode geometry, see Annex A for a full definition of quantities in Eq. (9). It follows that the IMC analysis presented in ref [25] may be applied with only minor modification to RMHD double tearing. Indeed, writing $\phi_1 = \varpi + \phi(1-\varpi)$, it follows that the linear analysis of ref [25] applies provided $q \to \tilde{q} = \phi\phi_1 q$. For example, if $\tilde{q} > 0$ and $\zeta > 1$ then growth is direct.

In relation to tokamaks, where a key question is whether or not magnetic 'islands' are formed by double-tearing instability [34, § 8.2.4], nonlinear stability of the zero amplitude IMC solution implies that complete reconnection occurs, i.e. that islands are not found. This requires not only linear stability, but the absence of subcritical behaviour, which can be checked by finding the steady branch of nonlinear solutions. Eliminating the other steady amplitudes in terms of a in Eq. (9) gives the branch implicitly as

$$r = 1 + \tilde{q} + r_2 a^2 + r_4(a^2) \tag{10}$$

where

$$r_2 = 1 + \left(\zeta^2 - 2 + \frac{\varpi}{\phi_1}\right)\frac{\tilde{q}}{\zeta^2} \tag{11}$$

and r_4 is given in Annex A. From Eq. (10) the linear stability criterion follows as $\tilde{q} + 1 - r > 0$, but in the absence of overstability or equivalently $\zeta > 1$, there is the possibility of subcritical solutions, since r_2 cannot be guaranteed to be positive as described in Annex A. Subcriticality is potentially very important because it implies that an un-reconnected mode may suddenly appear before linear theory predicts it should.

One point worth emphasising is that the analytic model indicates a potentially very complicated nonlinear dependence of the instability on the various parameters. Such an effect might easily be overlooked by numerical simulations which, since they are usually restricted to one particular set of parameter values per computation, tend to examine relatively small regions of parameter space.

5. Summary and Further Implications

The paper has demonstrated the equivalence between equations, and to some extent boundary conditions, originally derived for two quite different problems in MHD. It has focused on two of the most important implications of this equivalence for RMHD, namely (i) the value of bifurcation theory in predicting qualitative nonlinear behaviour, yet (ii) the difficulty of obtaining quantitatively accurate, full numerical solutions of the system. Double-tearing has provided a specific, and significant example of how the former (i) can be exploited. The 5-mode LOMT can also be applied to the equations

for plasma pressure driven turbulence of Itoh & Itoh [20], and this will be pursued in a separate publication. As to the latter (ii), one implication for the numerical limitations of astrophysical MHD simulation is discussed in ref [35].

Furthermore, full simulation of IMC [25] for horizontally elongated domains has produced rolls which stably fill the domain. This strongly supports the notion that the coalescence instability, whereby small wavelength features merge to give larger ones [9, § 6.5.3], operates at high-β. Although this instability cannot be completely modelled by low order models which assume fixed wavelength, truncated systems might be used to investigate coalescence at high β, and also tearing instability, for which it will be helpful that there are similar models of applied vertical fields [36] that might be easily adapted.

There are other potentially very important, but more speculative implications. The existence of subcritical solutions in the truncated models is consistent with a rapid growth of double-tearing instability. 'Single'-tearing instability is likely to share this property, since hysteresis has previously been reported for low-β RMHD [37], and subcriticality is a common feature of double convection problems. Note that IMC is an example of double convection because two advecting and diffusing scalar fields, in this case the temperature and the magnetic field, produce competing forces.

In normal application of IMC, gravity has always been taken to be destabilising, because in the 2-D configuration, the uniform applied magnetic field is always stabilising. This regime corresponds to $r > 0$ and $\tilde{q} > 0$. However in a tokamak, the analogue of gravity is proportional to curvature which may have either sign and similarly, the sheared magnetic field may be destabilising corresponding to $\tilde{q} < 0$. Hence tokamak RMHD may have much in common with thermosolutal double convection, where a solvent such as salt is the second convection field. Thermosolutal experiments yield at least three different convection regimes, one of which occurs when the salinity gradient is destabilising and is descriptively known as 'salt fingering' where the formation of a 'thermohaline staircase' may be observed [13, § 23]. Conceivably, the fingers correspond to the streamers observed in some tokamak simulations [38]. If sustained, the analogy with double convection would provide much physical insight into plasma behaviour, because of the direct, detailed measurements of salt fingering that are available, cf. [39].

The RMHD equations in toroidal geometry have also been proposed as a model for ELMs (Edge Localised Modes [40]) in tokamaks [41]. Understanding ELMs, together with Outer Modes or Edge Harmonic Oscillations [42], is important for the successful operation of large tokamaks including ITER. On the basis that IMC was originally proposed as a model for the astrophysical phenomena, there may be a relation between ELM filaments and both (1) the emergence of small but intense magnetic flux features called plages/faculae through the visible layers of the Sun [43, § 7], and (2) the sunspot penumbra [43, § 5]. Unfortunately, neither Solar phenomenon has a definitive explanation. Yet, again the solar observations and laboratory experiments may be usefully complementary, for example, since the corresponding region of the tokamak is optically thin, ELMs may be insightful for Solar subsurface MHD.

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Annex A: Details of the Derivation of Double Tearing Model

Derivation of Model System

This section explains how the 2-D model system described in the main text follows from the original 3-D model proposed by Chandrasekhar. The following sections summarise respectively the derivation of a low order truncated model, analysis of its linear stability, and lastly its nonlinear properties.

In one sense, Chandrasekhar's incompressible magnetoconvection (IMC) model is a confinement problem [32], in that it is designed to investigate how magnetic field can suppress the natural tendency of a layer of electrically and thermally conducting fluid heated from below to overturn convectively and thereby cool more rapidly. The dimensional parameters for Chandrasekhar's problem [1] are η, κ and ν which are the magnetic, thermal and viscous diffusivities respectively. Magnetic diffusivity $\eta = 1/(\mu\sigma_E)$, where μ is the permeability and σ_E is the electrical conductivity. The definition of the fluid is completed by specifying its density ρ_0 and coefficient of thermal expansion α_0. Gravity g acts in the vertical z direction, where there is a temperature difference ΔT across the fluid layer which has depth d_0. The horizontal magnetic field is initially applied as a uniform flux $\mathbf{B_0}$

The MHD equations in the Boussinesq (slightly compressible) approximation are then:

$$\rho_0\left(\frac{\partial}{\partial t}+\mathbf{u}\cdot\nabla\right)\mathbf{u} = -\nabla p+\rho' g\hat{\mathbf{z}}+\frac{1}{\mu}(\nabla\times\mathbf{B})\times\mathbf{B}+\rho_0\nu\nabla^2\mathbf{u} \tag{12}$$

$$\left(\frac{\partial}{\partial t}+\mathbf{u}\cdot\nabla\right)T = \kappa\nabla^2 T \tag{13}$$

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{u}\times\mathbf{B})-\nabla\times(\eta\nabla\times\mathbf{B}) \tag{14}$$

where $\mathbf{B}$ is magnetic field, $\mathbf{u}$ is flow field, p is a pressure field, T is temperature and where the spatially variable part of the density field which gives rise to the buoyancy force is $\rho' = \rho_0\alpha_0 T$. (Note the generalisation of Eq. (12) whereby $g\hat{\mathbf{z}}$ is replaced by $-\nabla\phi_G$, where ϕ_G is the gravitational potential.) Pressure p is customarily eliminated from consideration by forming the curl of the momentum Eq. (12).

The 2-D MHD system is derived by taking the y-component of the vector equations for vorticity and magnetic field together with the scalar T equation. Quantities $A(x,z), \psi(x,z)$ and $\omega(x,z)$ are introduced such that $\mathbf{B} = \nabla\times(A\hat{\mathbf{y}})$, $\mathbf{u} = \nabla\times(\psi\hat{\mathbf{y}})$ and ω is the y-component of $\nabla\times\mathbf{u}$. The MHD equations are made dimensionless by scaling time with respect to the thermal timescale ${d_0}^2/\kappa$ and distance with respect to the layer depth d_0 (hence velocity is scaled by κ/d_0). This introduces the following dimensionless groups

$$Ra = g\alpha_0\Delta T {d_0}^3/(\kappa\nu) \tag{15}$$

$$Q = \mathbf{B_0}^2 {d_0}^2/(\mu\rho_0\eta\nu) \tag{16}$$

$$\sigma = \nu/\kappa \tag{17}$$

$$\zeta \quad = \eta/\kappa \tag{18}$$

where Ra is the Rayleigh number and Q is the Chandrasekhar number. The Rayleigh number Ra is a measure of the driving, thermal term and the Chandrasekhar number, which is equal to the square of the Hartmann number, similarly measures the magnetic field. Parameters σ and ζ are respectively the Prandtl number and the magnetic (or third) Prandtl number.

The IMC equations may then be written

$$\begin{aligned}
\frac{\partial \omega}{\partial t} + \frac{\partial(\psi,\omega)}{\partial(x,z)} &= \sigma\nabla^2\omega + \sigma\zeta Q\frac{\partial(A,j)}{\partial(x,z)} - \sigma Ra\frac{\partial T}{\partial x} \\
\frac{\partial T}{\partial t} + \frac{\partial(\psi,T)}{\partial(x,z)} &= \nabla^2 T, \\
\frac{\partial A}{\partial t} + \frac{\partial(\psi,A)}{\partial(x,z)} &= \zeta\nabla^2 A + E_y,
\end{aligned} \tag{19}$$

where E_y is normalised, applied electric field and the non-linear terms may be written more explicitly as for example

$$\frac{\partial(\psi,A)}{\partial(x,z)} = \frac{\partial\psi}{\partial x}\frac{\partial A}{\partial z} - \frac{\partial\psi}{\partial z}\frac{\partial A}{\partial x} \tag{20}$$

The IMC plasma or liquid metal is confined by stress-free boundaries at top and bottom. These boundaries have perfect electrical conductivity and fixed uniform temperatures. Laterally, the boundary conditions are consistent with relatively passive, periodic boundary conditions, except that the zero gradient conditions prevent global travelling wave instabilities. These properties equate to boundary conditions on the dimensionless variables of

$$\psi = \omega = 0 \qquad \text{on all four sides} \tag{21}$$

$$T = 1 \;\; (z=0), \quad T = 0 \;\; (z=1), \quad \frac{\partial T}{\partial x} = 0 \;\; (x = 0, \lambda), \tag{22}$$

$$A = 1 \;\; (z=0), \quad A = 0 \;\; (z=1), \quad \frac{\partial A}{\partial x} = 0 \;\; (x = 0, \lambda), \tag{23}$$

Fully periodic lateral boundaries have been modelled bu Knobloch [22].

Low Order Modal Truncation

For more details of the 2-D IMC model and its replacement by a low order system of ordinary differential equations, see refs [21, 44, 25, 22], which should be consulted for a wealth of further information. The above boundary conditions allow the following truncated expansion of the fields.

$$\psi = 2^{3/2}\lambda\Lambda a(t)\sin(\pi x/\lambda)\sin(\pi z) \tag{24}$$

$$T = 1 - z + (2^{3/2}/\pi\Lambda)b(t)\cos(\pi x/\lambda)\sin(\pi z) - (1/\pi)c(t)\sin(2\pi z) \tag{25}$$

$$A = 1 - z + (2^{3/2}/\pi\Lambda)d(t)\cos(\pi x/\lambda)\sin(\pi z) - (1/\pi)e(t)\sin(2\pi z) \tag{26}$$

The most important point to note is that the modes selected in Eqs (24)–(26) are a consistently truncated Galerkin set and therefore that the above representation

will result in a model with excellent conservation properties relative to the full 2-D partial differential equation system. The chosen form Eq. (24) of the stream function corresponds to a single 2-D convective eddy of geometric aspect ratio λ, also measured by parameter Λ, where $\Lambda^2 = 1+1/\lambda^2$. It is convenient to introduce the following derived dimensionless parameters

$$r = Ra/(\pi^4\lambda^2\Lambda^6) \tag{27}$$

$$q = Q/(\pi^2\lambda^2\Lambda^4) \tag{28}$$

$$\varpi = 4/\Lambda^2 \tag{29}$$

where r is the (reduced) Rayleigh number and q is the (reduced) Chandrasekhar number. Geometry parameter $0 \leq \varpi \leq 4$.

In terms of dimensionless groups used in fusion physics, the Chandrasekhar number

$$Q = \frac{\zeta S^2}{\sigma}, \quad \text{where} \quad S = \frac{|\mathbf{B}_0|d_0}{\eta\sqrt{\mu\rho_0}} \tag{30}$$

is the Lundquist number. The Rayleigh number

$$Ra = \frac{\zeta Q\beta_{\Delta p}}{\varrho_C} = \frac{\zeta^2 S^2\beta_{\Delta p}}{\sigma\varrho_C}, \quad \text{where} \quad \beta_{\Delta p} = \frac{2\mu_0\Delta p}{|\mathbf{B}_0|^2} \tag{31}$$

defines the plasma β in terms of the pressure drop Δp across the layer depth d_0, and where ϱ_C is the radius of curvature of the magnetic field in units of d_0. Normally, to a good approximation in a tokamak, if θ_T is the poloidal angle and R_T is the major radius of the torus,

$$\frac{1}{\varrho_C d_0} = \frac{1}{R_T}\cos\theta_T \tag{32}$$

Plausible values of momentum and heat diffusivity for fusion plasmas indicate that taking the two equal makes no bigger an error than many other assumptions. However, the magnetic field diffusivity, which is inversely proportional to conductivity, is usually taken to be either significantly smaller or larger than the other two. The precise values of the diffusivities depend upon kinetic theory and whether or not turbulence is present.

Note that for the locally equivalent model indicated by Figure 1((a) and (c)), $\mathbf{B}_0$ is the effective poloidal, not toroidal, magnetic field, whence $\beta_{\Delta p}$ is of order β_p, the poloidal β. The parameter q is here a measure of absolute magnetic field strength more like S, in contrast to its normal meaning as a measure of field line twist in tokamak physics.

Substituting the representation Eq. (24)-(26) in the fluid dynamic equations Eq. (19), with applied electric field $E_0 = (4\pi\eta|\mathbf{B}|_0/d_0)(1-\phi)\sin(2\pi z)$ (where ϕ is an arbitrary constant), leads to the fifth order time dependent system

$$\dot{a} = \sigma[-a + rb - \zeta qd(1 + (\varpi - 1)e)] \tag{33}$$

$$\dot{b} = -b + a(1-c) \tag{34}$$

$$\dot{c} = -\varpi(c - ab) \tag{35}$$

$$\dot{d} = -\zeta d + a(1-e) \tag{36}$$

$$\dot{e} = -\varpi(\zeta e - ad) + \varpi\zeta(1-\phi) \tag{37}$$

The first three equations when $q = 0$ are identical to the system originally derived by Lorenz [45]. The last two equations represent magnetic field evolution and the two corresponding variables appear in a magnetic force term proportional to q in the first equation. The applied electric field in Eq. (37) is new relative to [25], and is important as it allows for the magnetic field to drive as well as suppress instability.

Linear stability analysis

The studies [44, 25], following Chandrasekhar, examine the linear stability of the model system in the case where there is no applied electric field $\phi = 1$. (It will be assumed henceforth in this section, unless explicitly stated, that $\phi = 1$.) These results also apply directly to the linear stability of the full partial differential equations in 2-D, and indeed to its leading nonlinear order interactions as indicated below. The linear system reduction of Eq. (33)-(37) may exhibit both overstability and direct growth depending on the magnetic field strength q. Linearising about the zero amplitude solution $a = b = c = d = e = 0$, the two coefficients of the second order spatial harmonics always exhibit free decay, hence there is a cubic dispersion relation for the growth (or decay) rate which determines the system's linear behaviour. It is straightforward to show that there is direct growth provided the normalised temperature difference exceeds $r^{(e)}$, ie. where

$$r > r^{(e)} = 1 + q \tag{38}$$

and overstability when

$$r > r^{(o)} = (\sigma + \zeta)\left(\frac{1+\zeta}{\sigma} + \frac{\zeta q}{1+\sigma}\right) \tag{39}$$

When

$$q = q^{(c)} = \frac{(1+\sigma)\zeta}{\sigma(1-\zeta)} \tag{40}$$

then the criteria Eq. (38) and Eq. (39) are the same and $r^{(e)} = r^{(o)}$. Note that if $\zeta \geq 1$ it follows that the instability is always direct. Which of the two is greater, magnetic diffusivity or thermal diffusivity, is very important.

Suppose $\zeta < 1$, so $q^{(c)} > 0$. The stability of the zero solution of system Eq. (33)-(37) is relatively straightforward to understand when q is not close to $q^{(c)}$. For $q < q^{(c)}$, there is direct growth of a mode and for $q > q^{(c)}$ there is overstability. (It is helpful to note that for q near $q^{(c)}$, one linear growth rate is always close to -2.)

Nonlinear analysis

The system Eq. (33)-(37), is a simplification of the complex system studied by [22]. The import of the simplification is well understood, namely that assuming that the modal coefficients are real instead of complex, eliminates the possibility of travelling wave solutions in Eq. (33)-(37). As discussed by Knobloch [22], there is a rigorous mathematical underpinning to the use of his complex system as a third order accurate

expansion (in amplitude a) to the full 2-D IMC problem. In fact, as is now explained, it is possible to treat all orders of a in the calculation of the steady branch for the system Eq. (33)-(37), although the applicability of fourth and higher order terms to the full 2-D IMC problem can only be justified heuristically. However, as shown in refs [21, 25], most of the time dependent behaviour of the simple system corresponds to that exhibited by direct simulations of the full 2-D partial differential equations governing magneto-convection, well beyond the domain of small amplitude perturbations where the fifth order (in time) model is rigorously accurate.

The branch of steady solutions to Eq. (33)-(37) with non-zero a^2 may be found by setting $\dot{a} = \dot{b} = \dot{c} = \dot{d} = \dot{e} = 0$. The other variables may then be expressed in terms of a, so that for example $c = a^2/(1+a^2)$. Eliminating also b, d and e in terms of a leads to a relation $r(a^2)$, given originally in [25]. Since the fifth order model system with applied electric field ($\phi \neq 1$) has different coefficients, the analysis, although completely analogous, yields significantly different results.

A full description of the double tearing problem is clearly outside the scope of this annex, however it is necessary to show that the quantity

$$r_2 = 1 + \left(\zeta^2 - 2 + \frac{\varpi}{\phi_1}\right)\frac{\tilde{q}}{\zeta^2} \tag{41}$$

may be negative when $\zeta > 1$. (Ref [25] already establishes this possibility for $\zeta < 1$.) Rewriting $\tilde{q}$ in terms of q and then rearranging Eq. (41) as an expression for q, a positive q exists such that $r_2 < 0$ if

$$\phi\left(\phi_1 + (\varpi - 2\phi_1)/\zeta^2\right) < 0 \tag{42}$$

which upon using the definition of ϕ_1 is equivalent to

$$\phi\left(1 - \phi/\varpi_2 - \mathcal{O}(1/\zeta^2)\right) < 0 \tag{43}$$

where $\varpi_2 = \varpi/(\varpi - 1)$. Hence $r_2 < 0$ when ζ is large and, for example, $\varpi_2 > 0$ and $\phi < 0$. This corresponds to a strong applied electric field and modes with wavelength $\lambda > 1/\sqrt{3}$.

The quantity r_4 containing the higher order terms in the expression for $r(a^2)$ has also to be specified, and is

$$r_4(a^2) = \frac{\left([(\zeta^2-2)\zeta^2 - a^2]\varpi - [(2\zeta^2-3)\zeta^2 + (\zeta^2-2)a^2]\phi_1\right)a^4\phi q}{\left(a^2+\zeta^2\right)^2\zeta^2} \tag{44}$$

Annex B: Tearing Mode Theory

This section has been added for readers unfamiliar with magnetic fusion MHD theory. For a more detailed, mathematical approach see the text by Biskamp [33, § 4.2].

Tearing mode theory describes MHD instabilities for which the magnetic diffusivity is important. These instabilities might be thought unlikely to occur in magnetic fusion plasmas because the electrical conductivity is so high, but because it multiplies the diffusive term with its higher order spatial derivative, it is possible for resistivity η to be

significant in narrow layers. As might be expected on the basis of this argument, steep magnetic field gradients are the most liable to lead to tearing instability. Evidently, the most dangerous situations are those where the least energy is required to produce a perturbation. It should be plausible that in a toroidal field configuration, such perturbations are those localised in radius, and indeed which follow a magnetic field line. The latter can be justified heuristically from the picture of the plasma as independent particles, which find it easier to move long distances along magnetic field lines than normal to them. It follows that the lowest energy perturbations to generate are therefore those which follow the field lines of the shortest length.

In a tokamak configuration, the effect of simultaneously applied current and magnetic field in the toroidal direction is to give rise to helical field lines. In a conventional tokamak, the current peaks in the centre of the device, so since the applied magnetic field is approximately spatially constant, the pitch of the fieldline helices increases outward from the (minor) axis. The shortest field lines are those which close after one circuit in the toroidal direction (giving rise to so-called resonant surfaces), and hence they lie at well separated locations within the plasma. At these surfaces, there is greatest risk of the standard, 'single' tearing instability, wherein field lines reconnect and so 'tear up' the initial simple field topology.

The seriousness of this instability should be evident from the fact that it may be driven purely by energy from the magnetic field that is intended to confine the plasma, hence even though it may have a small beginning, there is no obvious, guaranteed saturation mechanism. Fortunately, rigorous mathematical analysis using matched asymptotic expansions shows that tearing mode growth rates are relatively small, proportional to a power, typically $\eta^{3/5}$ of the very small resistivity, enabling the modes to be controlled or even eliminated in many circumstances.

Double tearing occurs when two resonant surfaces with the same twist are close together. Such a configuration requires a local off-axis maximum in the current, and so is not expected to appear, except transiently, in conventional tokamaks, although some advanced designs require the feature. However, since 'like currents attract', perturbations at each surface can feed back upon one another, and the instability may be more vigorous than the usual tearing instability with growth rate scaling as $\eta^{1/3} \gg \eta^{3/5}$.