# HYDROMAGNETIC STABILITY THEORY

## A BRIEF SURVEY

by

R. J. TAYLER

## ABSTRACT

This is a modified and slightly extended version of a C.T.R. Colloquium given on Thursday April 20th, 1961. Recent progress in hydromagnetic stability theory is described briefly and a detailed list of references is given. No account is given of plasma instabilities of other types.

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Culham Laboratory,
Culham, Nr. Abingdon,
Berks.

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# Contents

## 1. Introduction

In this lecture a brief survey is given of recent progress in the theory of hydromagnetic stability. By hydromagnetic instabilities are understood those which are described by closed hydrodynamic type equations and which do not depend on the detailed structure of the particle distribution functions in the equilibrium state. This means that all high frequency behaviour of the system is being neglected.

For most of the lecture the system will be supposed to be described by the idealised hydromagnetic equations:

$$\rho \frac{dv}{dt} = - \text{grad } p + \frac{j \times B}{c} , \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \text{div } \rho \underline{v} = 0, \tag{2}$$

$$\frac{1}{p} \frac{dp}{dt} = \frac{\gamma}{\rho} \frac{d\rho}{dt} , \tag{3}$$

$$E + \frac{v \times B}{c} = 0, \tag{4}$$

$$\text{curl } \underline{B} = \frac{4\pi j}{c} , \tag{5}$$

$$\text{div } \underline{B} = 0 \tag{6}$$

$$\text{and} \quad \text{curl } \underline{E} = - \frac{1}{c} \frac{\partial B}{\partial t} . \tag{7}$$

The principal approximations in these equations are the substitution of a scalar pressure for the pressure tensor in equation (1), the neglect of several terms including those due to resistivity and the Hall effect in equation (4) and the assumption of the adiabatic law equation (3). Displacement currents are also neglected but this approximation is normally justified and in any case the introduction of these terms does not essentially complicate the problem.

In the next section the stability of a system obeying the idealised hydromagnetic equations is discussed. In the final section the influence of some of the terms omitted from these equations is described. Only in one case, where low frequency behaviour is affected, are instabilities involving the detailed structure of the velocity distribution functions mentioned.

1.

## 2. Idealised hydromagnetics

### (a) Static equilibria

Before 1958 relatively few hydromagnetic stability problems had been solved exactly by a normal mode analysis; such a method is only simple when the system has a high degree of symmetry and the resulting differential equations can only be solved for very simple equilibria. One such equilibrium is the so-called stabilized pinch (Rosenbluth (1), Shafranov (2), Tayler (3)). In cases when the normal mode equations could not be solved exactly, information about the sign or reality of the disturbance growth rate could sometimes be obtained by forming volume integrals of the perturbed equations (see for example Chandrasekhar (4)).

A more powerful technique for the study of the stability of static equilibria was introduced by Bernstein, Frieman, Kruskal and Kulsrud (5) and Hain, Lust and Schluter (6). (This followed earlier work by Lundquist (7)). The hydromagnetic equations have an energy integral

$$\mathcal{E} = T + W = \int \rho \, \underline{v}^2 \, d\tau + \int \left( \frac{p}{\gamma-1} + \frac{B^2}{8\pi} \right) d\tau.$$

Because the energy remains constant, a static equilibrium can only become unstable if there exists a neighbouring state of lower potential energy. Bernstein et al. gave an expression for this change of potential energy which was suitable for calculation and they showed that the displacements considered need not satisfy all of the normal boundary conditions.

This energy principle was first applied to systems with a high degree of symmetry. For problems with cylindrical symmetry, two of the Euler-Lagrange equations of the energy integral are algebraic and minimising and only the minimisation with respect to the radial component of the perturbation presents any difficulty. In terms of this radial perturbation $(\xi_r)$, the perturbation in the energy $(\delta W)$ takes the form

$$\delta W = \int \left\{ A \, \xi_r^2 + B \left( \frac{d\xi_r}{dr} \right)^2 \right\} dr,$$

where $A$ and $B$ depend on the equilibrium conditions and the shape of perturbation chosen. The simple form of this expression for $\delta W$ enables general stability conditions to be obtained.

By use of well chosen trial functions for the radial perturbation, Rosenbluth (8) was able to show that there exist configurations arbitrarily close to the stabilized pinch configurations which are unstable. From the above form of the energy integral several authors including Laing (9) and Suydam (10) were able to obtain sufficient conditions for plasma stability. Thus obviously stable configurations were

2.

discovered. One sufficient condition is obtained very simply; since B is positive, it is sufficient for stability that A is positive everywhere for all perturbations. One such condition can be written

$$B_\theta \frac{d}{dr} (r B_\theta) \leqslant 0 .$$

This can only be satisfied in the hard-core or inverse pinch in which a solid rod carrying a current passes through the centre of the plasma and the plasma current flows in the opposite direction to that in the core.

The third Euler-Lagrange equation is not obviously a minimising equation but, by considering perturbations which were solutions of this equation, Suydam (11) was able to obtain a necessary condition for the stability of a cylindrical plasma. The condition which must be satisfied at all points has the form

$$\left(\frac{r\mu'}{\mu}\right)^2 + \left(\frac{32\pi \, rp'}{B_z^2}\right) > 0.$$

where $\mu = B_\theta /rB_z$ and the prime denotes differentiation with respect to r. More recently Newcomb (12) and Suydam (10) have found necessary and sufficient conditions for stability. These conditions depend on the existence or non-existence of zeros of the third Euler-Lagrange equation within the plasma. K. Schwartz (52) has extended Newcomb's necessary and sufficient condition to the case in which the plasma has anisotropic pressure by using the energy principle of Kruskal and Oberman (36). With the discovery of the necessary and sufficient conditions, the study of the stability of a cylindrical system is reduced to computation and there is no difficulty in principle. The computation is quite laborious but results using the criterion have recently been obtained by Whiteman and Copley (13).

Any contained hydromagnetic equilibrium is topologically toroidal and some of the results obtained for cylindrical systems have now been extended to toroidal systems. One difficulty is that few explicit forms for toroidal equilibria are known as they can only be obtained by the solution of non-linear partial differential equations. The general properties of toroidal equilibria have been discussed by Kruskal and Kulsrud (14) and Grad and Rubin (15). Special toroidal equilibria have been derived by Shafranov (16), Laing, Roberts and Whipple (17), Whipple (18) and Whiteman (19) amongst others.**

Kadomtsev (20) has shown how to obtain an analogue of Suydam's necessary condition for stability which is applicable to general toroidal configurations.

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**See also Kadomtsev (49).

He considered a particular perturbation which was confined to the neighbourhood of one magnetic surface and which followed the magnetic field lines on that surface; on intuitive grounds Suydam (11) had suggested that these perturbations which move magnetic field lines without stretching them would be amongst the least stable. Mercier (21) has obtained another analogue of Suydam's criterion which is a necessary condition for the stability of an axisymmetric torus. This criterion is more stringent than Kadomtsev's which can of course also be applied in the special case of axial symmetry.

It seems likely that a toroidal analogue of Newcomb's necessary and sufficient conditions also exists; this would depend on whether there exist solutions of the Euler—Lagrange equation which leave an entire magnetic surface unperturbed. Such a result in the special case of axial symmetry has been given by Bineau (unpublished as yet). Sufficient conditions for the stability of toroidal configurations have been given by Mercier (21) and Suydam (22).

Few detailed applications of toroidal stability criteria have been given because of the lack of simple expressions for the equilibria. Mercier has, however, shown that one particular equilibrium (a hydromagnetic analogue of the Hill's vortex) is unstable and he has shown that criteria can be simplified in the neighbourhood of the magnetic axis. Bernstein et al. (5) have shown how to find necessary and sufficient conditions when the plasma does not contain an axial magnetic field and Tayler (unpublished as yet) has shown that all configurations without an axial field, except possibly those of an inverse pinch type, are unstable. Many calculations on rather complicated equilibria have been made by the Princeton group (see for example (23)). Their results have been obtained by treating many complicating factors as small perturbations and by expanding about a simple equilibrium. The stability of a particular toroidal analogue of the Stabilized Pinch has been studied by Lüst, Suydam, Richtmeyer, Rotenberg and Levy (physics of fluids, to be published). They show that the toroidal results are similar to the cylindrical results if the aspect ratio is not too small.

(b) Stationary Equilibria

If a hydromagnetic configuration is considered in which the fluid has a steady velocity, the stability of the system cannot be studied by means of the energy principle. As the system possesses kinetic energy in its undisturbed state, there is no necessity for the potential energy to decrease for the system to become unstable; the instability can be fed by the steady kinetic energy. Several simple problems have been solved by means of a normal mode analysis [Trehan (24), Gerjuoy and Rosenbluth (25) Pytte (26), Zabusky (27)]. More recently Frieman and Rotenberg (28) have shown that a variational principle does exist for this problem. They have

been able to show that, provided the fluid velocities are small compared to the sound velocity and the Alfven velocity, steady motions cannot make a previously stable system unstable.

(c) Large Amplitude Disturbances

If a system is unstable against small perturbations, it is nevertheless possible that the amplitude of the instabilities might be limited by non-linear effects; conversely a system executing large stable oscillations might become unstable against further perturbations.

Friedrichs (29) has investigated the first possibility for the case of the pinched discharge with surface currents. He has used a classical bifurcation analysis, similar to that used in the study of the stability of rotating liquid masses, in the hope that he would be able to show that a series of stable distorted equilibria comes into existence when the cylindrical equilibria become unstable. However what he has shown is that distorted equilibria only exist in the neighbourhood of undistorted equilibria when the latter are stable; in this case the distorted equilibria are unstable. This suggests that the initially stable configurations may be unstable at large amplitudes.

If a plasma is oscillating about a steady state, there are times when it is acted on by accelerating forces which, to a first approximation, make the problem of stability against further small perturbations resemble the Rayleigh-Taylor stability problem of a plasma supported against gravity by a magnetic field. Because of this there has been much interest in the Rayleigh-Taylor stability problem. Early results were obtained by Kruskal and Schwarzschild (30) and Meyer (31). General stability criteria for plane and cylindrical systems have been given by Newcomb (32)* and Tayler (33). One simple dynamical problem, in which a small perturbation is superimposed upon a steady motion, has been discussed by Tayler (34). He has shown that the small perturbation can grow but it is not clear that this is a true instability rather than a transfer of energy from one steady oscillation to another.

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*Criteria for plane systems have also been obtained by Cowley (51).

## 3. Non-idealised hydromagnetics

### (a) Double adiabatic hydromagnetics

If the collision frequency in a plasma is not sufficiently high, motions along and across the magnetic field lines are not closely coupled. If the flow of heat along the field lines is neglected, closed hydromagnetic equations can be obtained in which the single adiabatic law relating pressure and density variations is replaced by two governing the parallel and perpendicular components of the pressure tensor. These equations are

$$\frac{d}{dt}(p_{\shortparallel} \, B^2/\rho^3) = 0$$

and
$$\frac{d}{dt}(p_{\perp}/B\rho) = 0 \; .$$

These equations were derived by Chew, Goldberger and Low (35). It was subsequently shown by Kruskal and Oberman (36) and Rosenbluth and Rostoker (37) that a modified energy principle applies in this case. Furthermore, if the equilibrium has isotropic pressure, the double adiabatic equations predict greater stability than the idealised hydromagnetic equations.

### (b) Small Larmor radius

In deriving their equations, Chew, Goldberger and Low assumed that the particle Larmor radii were small. Chandrasekhar, Kaufman and Watson (38, 39) considered the form of these equations when heat flow along the field lines was not neglected. Kruskal and Oberman (36) and Rosenbluth and Rostoker (37) also obtained energy principles for this problem and showed that, for isotropic equilibrium, the stability predicted by these equations is intermediate between that predicted by the double adiabatic and idealised hydromagnetic equations. Thus

$$\delta W_{DA} \geqslant \delta W \geqslant \delta W_H \; .$$

### (c) Anisotropic instabilities

If the plasma equilibrium has anisotropic pressure, both the Chew, Goldberger and Low and the Chandrasekhar, Kaufman and Watson equations predict the occurrence of new types of instabilities. Thus, for example, Lüst (40) and Chandrasekhar, Kaufman and Watson (41) have shown that plane waves in an infinite homogeneous medium with a uniform magnetic field become unstable if either the parallel or the perpendicular component of pressure becomes too large. The latter instabilities are commonly called mirror instabilities because they are to be expected in a

magnetic mirror machine.    The criteria for instability obtained from the Chew,
Goldberger, Low equations are

$$p_{\shortparallel} > p_{\perp} + B^2/4\pi$$

and
$$p_{\perp}^2/p_{\shortparallel} > 6(p_{\perp} + B^2/8\pi).$$

The first criterion is also obtained from the treatment of Chandrasekhar, Kaufman
and Watson but the second criterion depends on the precise form of the equilibrium
distribution functions.    If electrons and ions both have Gaussian distributions
in both components of velocity and the same ratio of parallel and perpendicular
pressures, it becomes

$$p_{\perp}^2/p_{\shortparallel} > (p_{\perp} + B^2/8\pi).$$

Chandrasekhar, Kaufman and Watson (41) have also considered the problem of
the stabilized pinch with anisotropic pressure and have shown how new instabilities
arise.*

(d)  Finite transport processes

If the plasma has high density or low temperature, the normal transport effects
described by Chapman and Cowling (42) become important.    This is also true in a
conducting liquid.    Hare (43) has shown that viscosity can never increase the
instability of a static equilibrium of an incompressible fluid.    Tayler (44) has
also shown that the same thing is true for one particular problem involving a
compressible fluid and he has shown how growth rates at short wavelengths are
reduced by viscosity.

Tayler (45) has also considered a special problem involving a viscous fluid
of finite conductivity.    He has shown that the presence of a finite conductivity
can lead to greater instability.    This can be explained because the conductivity
plays a dual role in stability problems:   although Joule heating may be expected
to damp instabilities, the finite conductivity encourages field diffusion and
allows plasma to move across field lines.    Similarly finite viscosity may lead
to greater instability of a stationary equilibrium.    This is well known from
hydrodynamics.

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*Mercier and Cotsaftis (50) have recently obtained a stability criterion for a
 toroidal plasma with anisotropic pressure.
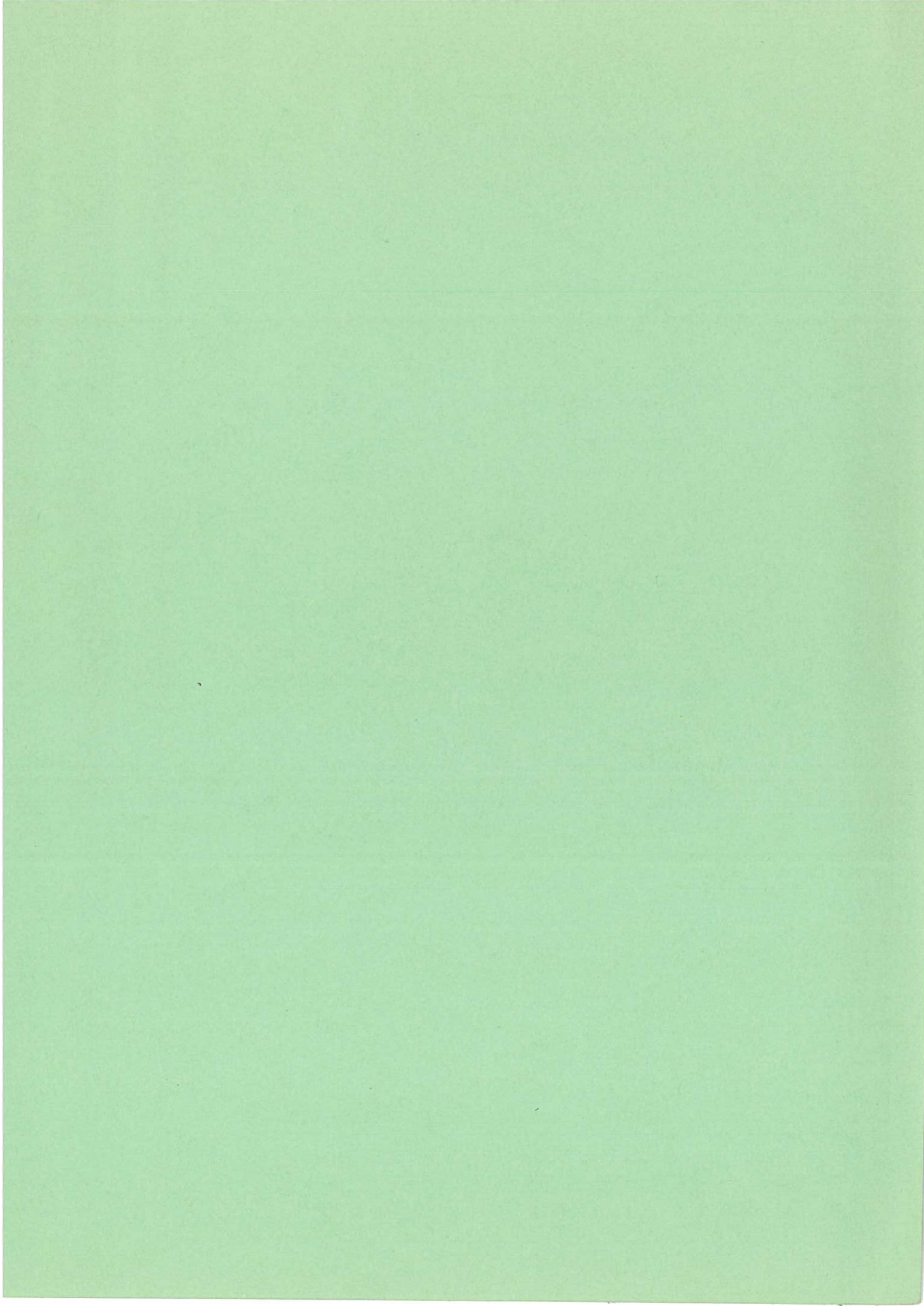
## (e) Hall effect

Ware (46) has recently stressed that, although the ideal hydromagnetic equations assume that the fluid and magnetic field move together, the truth is that when the Hall current is included the magnetic field is more closely tied to the motion of the electrons.  This suggests that instability will be replaced by instability waves (overstability).  Tayler (unpublished as yet) has solved one problem including the Hall effect exactly;  instability waves do occur but stability conditions are virtually unaltered unless the ion gyration frequency is small compared to typical hydromagnetic frequencies and the growth rates are, if anything, reduced.  Other stability problems including the Hall effect have been studied by Tserkovnikov, (47) and Kadomtsev (48).

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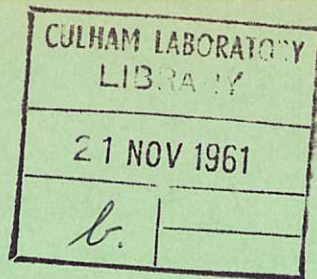48. B.B. Kadomtsev, Soviet Physics, J.E.T.P. 10, 780, 1960.

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United Kingdom Atomic Energy Authority

RESEARCH GROUP

Lecture

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## A Brief Survey

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Culham, Oxfordshire.

1961

U.D.C.

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(b) Stationary Equilibria

If a hydromagnetic configuration is considered in which the fluid has a steady velocity, the stability of the system cannot be studied by means of the energy principle. As the system possesses kinetic energy in its undisturbed state, there is no necessity for the potential energy to decrease for the system to become unstable; the instability can be fed by the steady kinetic energy. Several simple problems have been solved by means of a normal mode analysis [Trehan (24), Gerjuoy and Rosenbluth (25) Pytte (26), Zabusky (27)]. More recently Frieman and Rotenberg (28) have shown that a variational principle does exist for this problem. They have

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been able to show that, provided the fluid velocities are small compared to the sound velocity and the Alfven velocity, steady motions cannot make a previously stable system unstable.

(c) Large Amplitude Disturbances

If a system is unstable against small perturbations, it is nevertheless possible that the amplitude of the instabilities might be limited by non-linear effects; conversely a system executing large stable oscillations might become unstable against further perturbations.

Friedrichs (29) has investigated the first possibility for the case of the pinched discharge with surface currents. He has used a classical bifurcation analysis, similar to that used in the study of the stability of rotating liquid masses, in the hope that he would be able to show that a series of stable distorted equilibria comes into existence when the cylindrical equilibria become unstable. However what he has shown is that distorted equilibria only exist in the neighbourhood of undistorted equilibria when the latter are stable; in this case the distorted equilibria are unstable. This suggests that the initially stable configurations may be unstable at large amplitudes.

If a plasma is oscillating about a steady state, there are times when it is acted on by accelerating forces which, to a first approximation, make the problem of stability against further small perturbations resemble the Rayleigh-Taylor stability problem of a plasma supported against gravity by a magnetic field. Because of this there has been much interest in the Rayleigh-Taylor stability problem. Early results were obtained by Kruskal and Schwarzschild (30) and Meyer (31). General stability criteria for plane and cylindrical systems have been given by Newcomb (32)* and Tayler (33). One simple dynamical problem, in which a small perturbation is superimposed upon a steady motion, has been discussed by Tayler (34). He has shown that the small perturbation can grow but it is not clear that this is a true instability rather than a transfer of energy from one steady oscillation to another.

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*Criteria for plane systems have also been obtained by Cowley (51).

## 3. Non-idealised hydromagnetics

### (a) Double adiabatic hydromagnetics

If the collision frequency in a plasma is not sufficiently high, motions along and across the magnetic field lines are not closely coupled. If the flow of heat along the field lines is neglected, closed hydromagnetic equations can be obtained in which the single adiabatic law relating pressure and density variations is replaced by two governing the parallel and perpendicular components of the pressure tensor. These equations are

$$\frac{d}{dt}(p_{\shortparallel} B^2/\rho^3) = 0$$

and

$$\frac{d}{dt}(p_{\perp}/B\rho) = 0 .$$

These equations were derived by Chew, Goldberger and Low (35). It was subsequently shown by Kruskal and Oberman (36) and Rosenbluth and Rostoker (37) that a modified energy principle applies in this case. Furthermore, if the equilibrium has isotropic pressure, the double adiabatic equations predict greater stability than the idealised hydromagnetic equations.

### (b) Small Larmor radius

In deriving their equations, Chew, Goldberger and Low assumed that the particle Larmor radii were small. Chandrasekhar, Kaufman and Watson (38, 39) considered the form of these equations when heat flow along the field lines was not neglected. Kruskal and Oberman (36) and Rosenbluth and Rostoker (37) also obtained energy principles for this problem and showed that, for isotropic equilibrium, the stability predicted by these equations is intermediate between that predicted by the double adiabatic and idealised hydromagnetic equations. Thus

$$\delta W_{DA} \geqslant \delta W \geqslant \delta W_{H} .$$

### (c) Anisotropic instabilities

If the plasma equilibrium has anisotropic pressure, both the Chew, Goldberger and Low and the Chandrasekhar, Kaufman and Watson equations predict the occurrence of new types of instabilities. Thus, for example, Lüst (40) and Chandrasekhar, Kaufman and Watson (41) have shown that plane waves in an infinite homogeneous medium with a uniform magnetic field become unstable if either the parallel or the perpendicular component of pressure becomes too large. The latter instabilities are commonly called mirror instabilities because they are to be expected in a

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magnetic mirror machine.    The criteria for instability obtained from the Chew,
Goldberger, Low equations are

$$p_{\shortparallel} > p_{\perp} + B^2/4\pi$$

$$\text{and} \qquad p_{\perp}^2/p_{\shortparallel} > 6(p_{\perp} + B^2/8\pi).$$

The first criterion is also obtained from the treatment of Chandrasekhar, Kaufman
and Watson but the second criterion depends on the precise form of the equilibrium
distribution functions.    If electrons and ions both have Gaussian distributions
in both components of velocity and the same ratio of parallel and perpendicular
pressures, it becomes

$$p_{\perp}^2/p_{\shortparallel} > (p_{\perp} + B^2/8\pi).$$

Chandrasekhar, Kaufman and Watson (41) have also considered the problem of
the stabilized pinch with anisotropic pressure and have shown how new instabilities
arise.*

(d)   Finite transport processes

If the plasma has high density or low temperature, the normal transport effects
described by Chapman and Cowling (42) become important.    This is also true in a
conducting liquid.    Hare (43) has shown that viscosity can never increase the
instability of a static equilibrium of an incompressible fluid.    Tayler (44) has
also shown that the same thing is true for one particular problem involving a
compressible fluid and he has shown how growth rates at short wavelengths are
reduced by viscosity.

Tayler (45) has also considered a special problem involving a viscous fluid
of finite conductivity.    He has shown that the presence of a finite conductivity
can lead to greater instability.    This can be explained because the conductivity
plays a dual role in stability problems:    although Joule heating may be expected
to damp instabilities, the finite conductivity encourages field diffusion and
allows plasma to move across field lines.    Similarly finite viscosity may lead
to greater instability of a stationary equilibrium.    This is well known from
hydrodynamics.

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*Mercier and Cotsaftis (50) have recently obtained a stability criterion for a
 toroidal plasma with anisotropic pressure.

(e) <u>Hall effect</u>

Ware (46) has recently stressed that, although the ideal hydromagnetic equations assume that the fluid and magnetic field move together, the truth is that when the Hall current is included the magnetic field is more closely tied to the motion of the electrons.   This suggests that instability will be replaced by instability waves (overstability).   Tayler (unpublished as yet) has solved one problem including the Hall effect exactly;   instability waves do occur but stability conditions are virtually unaltered unless the ion gyration frequency is small compared to typical hydromagnetic frequencies and the growth rates are, if anything, reduced.   Other stability problems including the Hall effect have been studied by Tserkovnikov, (47) and Kadomtsev (48).

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