PLASMA CONTAINMENT AND STABILITY THEORY

by

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A B S T R A C T

Models of the behaviour of high temperature plasma are applied to the problem of plasma confinement in magnetic traps. A wide variety of possible instabilities is disclosed. In magnetic mirror traps the low frequency instabilities can be overcome by design of the magnetic field. The high frequency instabilities, particularly those associated with the loss-cone character of the equilibrium distribution function, are more persistent and appear to impose severe restrictions on the dimensions of the plasma. Consequently toroidal traps seem to offer a better prospect for long term containment but at present they are subject to low frequency instabilities which persist even when conditions for hydromagnetic stability have been met. These instabilities may be due to small resistive effects or to an unstable drift wave. The resistive instabilities should disappear at high temperature and the drift-wave instability should be overcome by increased shear in the magnetic field.

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C O N T E N T S

INTRODUCTION

There are two main aspects to the theory of plasma confinement in a magnetic field. One is the introduction of suitable models, the other is the application of these models to determine the properties of specific magnetic 'traps'. In the latter problem the question of stability is of paramount importance.

In this review some of the conventional plasma models are briefly described and are then used to discuss the stability properties of confinement systems. These systems are of two main types, one is the adiabatic trap which relies on magnetic mirrors to restrict the escape of particles along the lines of force; the other is the closed toroidal trap in which plasma flows freely along lines of force but these remain within a closed toroidal region.

Theory suggests that there is a hierarchy of instabilities which may afflict both systems. The most important are those of low frequency and long wavelength which are described by the fluid or the guiding centre models. Using these models the problem of confinement and stability can be completely solved even for realistic, and therefore complex, geometrical situations. The results show that gross stability depends principally upon the magnetic field configuration and that grossly stable configurations of both adiabatic and closed types can be constructed. Stable versions of the mirror trap are based upon the 'magnetic well' - a system in which the field gradient is everywhere of the correct sign for stability. The stable closed traps depend either upon 'magnetic shear' - a differential twisting of the lines of force about a magnetic axis, or are based on the 'average magnetic well' in which the <u>average</u> field gradient is of the correct sign for stability.

Once gross stability has been achieved then small scale instabilities become important and can be investigated. To discuss these one must usually resort to the Vlasov model which describes the microscopic behaviour of plasma through the one particle distribution function. Unfortunately this model is intractable except in rather idealized configurations; nevertheless these are sufficient to illustrate the general character of the instabilities to be expected in more realistic situations.

In mirror traps the most important micro-instabilities are of higher frequency (i.e., comparable to ion cyclotron frequency). They depend on the velocity distribution and can be alleviated by choosing this suitably. However, the inevitable loss-cone in a mirror confined plasma leads to some seemingly inescapable instabilities which can be controlled only by imposing restrictions on the overall size of the plasma.

In closed systems the high frequency instabilities are not likely to be serious, instead they are plagued by additional low-frequency instabilities which are not described by the fluid model and which persist even when hydromagnetic stability has been achieved. Some of these are due to finite resistance or electron inertia and can be treated by suitably modifying the fluid model, others are short wavelength instabilities associated with the drift wave and again require the Vlasov model for their discussion. The resistive instabilities should disappear at high temperatures and the drift wave instability should be stabilized if the shear is increased substantially beyond that necessary for hydromagnetic stability, or if restrictions are placed on the size of the system.

1. PLASMA MODELS

(a) The Hydromagnetic (Fluid) Model

The simplest model of plasma is that of an ordinary conducting fluid with density ρ, scalar pressure p and velocity $\underset{\sim}{v}$ obeying the equations

$$\rho \frac{d\underset{\sim}{v}}{dt} = - \underset{\sim}{\nabla} p + \frac{1}{c} (\underset{\sim}{j} \times \underset{\sim}{B}) \qquad \ldots (1.1)$$

$$\frac{1}{p} \frac{dp}{dt} = \frac{\Upsilon}{\rho} \frac{d\rho}{dt} \qquad \frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla} \cdot (\rho \underset{\sim}{v}) = 0$$

where $\underset{\sim}{j}$ and $\underset{\sim}{B}$ are the current and magnetic field. The electromagnetic field is governed by the Maxwell equations

$$\underset{\sim}{\nabla} \times \underset{\sim}{B} = \frac{4\pi \underset{\sim}{j}}{c} \qquad \underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{E} = \frac{1}{c} \frac{\partial \underset{\sim}{B}}{\partial t} \qquad \ldots (1.2)$$

and is coupled to the fluid motion by an elementary form of "Ohm's law"

$$\underset{\sim}{E} + \frac{\underset{\sim}{v} \times \underset{\sim}{B}}{c} = 0 \qquad \ldots (1.3)$$

These equations represent only the simplest form of hydromagnetic model corresponding to an ideal fluid in which resistance and viscosity are absent. In some cases these non-ideal properties must be included, as must certain specifically plasma properties such as the so-called "finite Larmor radius corrections". (Rosenbluth, Krall and Rostoker 1962; Roberts and Taylor 1962) The origin of these corrections is found in the derivation of fluid models of plasma, which are based, not on the small mean free path, (which indeed is frequently very long compared to the dimensions of the apparatus) but on the small Larmor radius of the charged particles. Fluid-like equations can be derived formally (Chew, Goldberger and Low 1956) by an asymptotic expansion in the ratio of the Larmor radius to the scale length and a finite value for this ratio leads to additional terms (Roberts and Taylor 1962; Rosenbluth and Simon 1965) in the fluid equations, just as a finite mean free path leads to viscosity in an ordinary fluid. Indeed, the finite Larmor radius terms are mathematically similar to those describing viscosity but differ in that they lead to no dissipation of energy.

(b) The Guiding Centre Model

Another model of plasma behaviour, the guiding centre model, concentrates on the motion of individual particles. This consists of a rapid gyration about a slowly drifting "guiding centre", and for the description of long wavelength, low frequency phenomena only the motion of this guiding centre is important. In a uniform magnetic field alone, the guiding centre does not move across the magnetic field, but if a transverse electric field is also present it drifts perpendicular to both the electric and magnetic field with a velocity

$$\underset{\sim}{V}_E = \frac{\underset{\sim}{E} \times \underset{\sim}{B}}{B^2} c \ , \qquad \ldots (1.4)$$

which is just the velocity of the reference frame in which the transverse electric field vanishes. If the magnetic field itself is non-uniform there is an additional guiding centre drift across the field, (Alfven 1950; Northrop 1963) known as the "gradient-B drift", given approximately by

$$\underset{\sim}{V}_B = -\frac{m}{e}\frac{\nabla B \times \underset{\sim}{B}}{B^3}\left(\tfrac{1}{2}v_\perp^2 + v_\parallel^2\right) \qquad \ldots (1.5)$$

where $v_\perp$ and $v_\parallel$ are velocity components perpendicular and parallel to the field. It is important to note that the electric drift is the same for all particles but the gradient-B drift is in opposite directions for ions and electrons and depends on the particle energy.

(c) The Vlasov Model

Both the fluid model and the guiding centre model are adapted to the investigation of low frequency long wavelength phenomena; in order to study high frequency or short-wave phenomena one must resort to the so-called Vlasov model in which the one-particle distribution function for the plasma $f(\underset{\sim}{r},\underset{\sim}{v})$ is governed by the equation

$$\frac{\partial f}{\partial t} + \underset{\sim}{v}\cdot\underset{\sim}{\nabla} f + \frac{e}{m}(\underset{\sim}{E} + \underset{\sim}{v}\times\underset{\sim}{B})\cdot\frac{\partial f}{\partial \underset{\sim}{v}} = 0 \ . \qquad \ldots (1.6)$$

In this paper the Vlasov equation will be used only to study electrostatic modes of oscillation, for which the magnetic field is regarded as fixed and the electric field determined by

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{E} = \int f\, d^3v \ . \qquad \ldots (1.7)$$

If δ-function distributions were allowed in f then equation (1.6) would be exact since the only forces acting are those due to the field of the particles. However the Vlasov model implies that these fields are averaged over regions containing many particles so that the short range effects of neighbouring particles are smoothed out. Alternatively, therefore, one may describe this model as one in which short range collisions have been neglected and for this reason equation (1.6) is also referred to as the "collisionless Boltzmann equation".

The following sections illustrate how these several models have been used to study containment and stability and describe some of the more important conclusions which have been reached.

2. ADIABATIC TRAPS

A confinement scheme, the theory of which has been studied in detail during the past few years, is that known as a "Magnetic Well", a "Minimum-B" trap or a "Hybrid" trap. (Andreoletti 1963; Taylor 1963; Taylor 1964; Taylor and Hastie 1965) This is derived from the well-known adiabatic, or Magnetic Mirror, trap in which loss of particles along the magnetic field is inhibited by the magnetic mirror effect. This depends on the existence of an adiabatic invariant, $\mu = v_\perp^2/B$, associated with the periodic nature of the motion round the Larmor orbit. This invariant is essentially constant if the magnetic field varies smoothly, so that on entering a stronger field the transverse energy of the particle increases at the expense of its parallel energy according to

$$\frac{v_\parallel^2}{2} = \varepsilon - \mu B \qquad \ldots (2.1)$$

where ε is the energy per unit mass. In fact, μB acts as a potential for the parallel motion so that particles in a field B_o are reflected from a field B_m if $\mu B_m > \varepsilon$, i.e., if

$$\frac{v_\perp}{v} > \left(\frac{B_o}{B_m}\right)^{\frac{1}{2}} \ . \qquad \ldots (2.2)$$

Particles moving more nearly parallel to the field will be able to surmount the mirror and are described as being in the "mirror loss-cone".

A simple magnetic containment system exploiting the magnetic mirror effect is provided by an axi-symmetric magnetic field which increases towards each end so that particles are contained axially by the mirror effect (Fig. 1). There is inevitably a slight decrease of the field in the radial direction but the resulting cross field gradient-B drifts are in the azimuthal direction and result in no loss of particles.

This simple mirror system suffers from two disadvantages. One is the "loss-cone" itself which means that the system cannot contain an isotropic distribution of plasma but only a "loss-cone distribution" in which

$$f(\underset{\sim}{r},\underset{\sim}{v}) = 0 \quad \text{if} \quad v_\perp < \lambda \, v_\parallel \qquad \ldots (2.3)$$

where λ is related to the "mirror ratio" B_m/B_o. This feature will be important in later discussion. The other failing of the simple mirror system is that it is subject to a serious instability - the "flute" instability.

The origin of the flute instability is easily seen. If a deformation occurs on the plasma surface then the azimuthal drift of ions and electrons in opposite directions, due to the radial gradient in B, produces a space charge. This in turn produces an electric field and consequently an $\underset{\sim}{E} \times \underset{\sim}{B}$ drift which is in a direction such as to increase the amplitude of the original deformations, see Figs. 2a-2c. The existence of this flute instability of the mirror is fully confirmed by a detailed theoretical analysis (Rosenbluth and Longmire 1957). However, the simple picture given here already suggests a means of overcoming the instability, for if the direction of the radial field gradient could be reversed then the gradient-B drifts, and therefore also the ultimate $\underset{\sim}{E} \times \underset{\sim}{B}$ drifts, could be reversed and so would tend to reduce the amplitude of the original perturbation. One might hope, then, for stable plasma confinement if the magnetic field increased in the radial direction as well as in the axial. In fact one would like a magnetic field whose strength increased in <u>all</u> directions from the centre.

It is impossible to create a simple axi-symmetric field having this property, but quite easy if one relaxes the symmetry requirement. For example, one may superimpose on the elementary mirror a second field produced by four straight rods lying parallel to the axis of the mirror coils, (Gott, Ioffe and Telkovsky 1962) with adjacent rods carrying current in opposite directions (see Fig. 3). The field of the rods increases with distance from the axis and is uniform along it. By superimposing this on the basic mirror and adjusting its strength appropriately one can produce a field which increases both axially and radially. This is shown in Figs. 4a and 4b which show a cross-section of the B = constant contours (magnetic isobars) for various values of the current in the rods. For small rod current B increases axially and decreases radially as in the simple mirror. With larger current in the rods the topology for the B = constant curves changes and a region of closed isobars surrounding a minimum in B is created.

Minimum-B fields, or magnetic wells, of which this is an example, can thus be defined by the property that:

<u>the magnetic isobars, B = constant, from a set of closed nested surfaces surrounding a non-zero minimum of B.</u>

(A non-zero minimum value is necessary to preserve the invariance of the magnetic moment.)

The difficulty in studying minimum-B fields is that they are geometrically rather complicated and yet, since the field shape is an essential feature it must be treated realistically. It is important to observe, therefore, that confinement and stability can be discussed in minimum-B systems without having to specify their shape (Taylor 1963). This can be done in part using only the basic definition of a magnetic well so that the results apply to any magnetic well system whatever its geometry.

3. LOW FREQUENCY BEHAVIOUR OF ADIABATIC TRAPS

(a) Fluid Model

The simplest description of the properties of magnetic wells is in terms of the fluid model. In this the equilibrium situation is defined by

$$\underset{\sim}{j} \times \underset{\sim}{B} = \text{div} \cdot \underline{\underline{P}} \qquad \ldots (3.1)$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0 \quad , \quad \underset{\sim}{\nabla} \times \underset{\sim}{B} = 4\pi \underset{\sim}{j}$$

where an anisotropic pressure tensor must be used since a mirror system cannot confine an isotropic distribution. These equilibrium equations permit a low pressure $(P_{ij} \ll B^2/8\pi)$ solution if, but only if, the pressure tensor satisfies the conditions

$$\frac{\partial p_{\parallel}}{\partial s} + \frac{(p_{\perp} - p_{\parallel})}{B} \frac{\partial B}{\partial s} = 0 \qquad \ldots (3.2)$$

$$\int \underset{\sim}{\nabla} (p_{\perp} + p_{\parallel}) \cdot \frac{(\underset{\sim}{B} \times \underset{\sim}{\nabla} B)}{B^4} \, ds = 0 \qquad \ldots (3.3)$$

where $p_{\perp}$ and $p_{\parallel}$ are the components of the pressure tensor perpendicular and parallel to the local magnetic field and the integral in (3.3) is taken along a magnetic field line.

A particular class of pressure distribution which satisfy these equilibrium constraints is that in which $p_{\perp}$ and $p_{\parallel}$ are functions only of the field strength B and are connected by

$$Bp'_{\parallel}(B) = p_{\parallel}(B) - p_{\perp}(B) \, . \qquad \ldots (3.4)$$

One of the two functions $p_{\perp}(B)$ $p_{\parallel}(B)$ may be chosen quite arbitrarily, the other is then determined by (3.4). The significance of having closed B = constant isobars is now apparent, for while these equilibria exist in any magnetic field, it is only in fields which possess closed magnetic isobars that they will correspond to closed (confined) plasma configurations.

The stability of these equilibria can be determined by calculating (Bernstein, Frieman, Kruskal and Kulsrud 1958) the change in energy δW resulting from an arbitrary displacement $\delta\underset{\sim}{\xi}(\underset{\sim}{r})$ of the plasma. For the special class of equilibria discussed here it can be shown (Taylor 1963; Taylor and Hastie 1965; Andreoletti 1964) that δW is positive for all displacements $\delta\underset{\sim}{\xi}$ provided only that the pressure is less than some finite threshold depending on the depth of the magnetic well. In fact the maximum stable pressure is given approximately by

$$p_{\perp}^{max} = \frac{1}{8\pi} \left(B^2_{max} - B^2_{min} \right) \qquad \ldots (3.5)$$

where B_{max} is the field strength on the outermost closed magnetic isobar and B_{min} the minimum field strength within the magnetic well.

It may be concluded, therefore, that according to the fluid model a class of stable, confined equilibria exists in magnetic wells and is given by

$$p_\perp = p_\perp(B),\ p_\parallel = p_\parallel(B),\ Bp_\parallel'(B) = p_\parallel(B) - p_\perp(B)\ . \qquad \ldots (3.6)$$

A more comprehensive and detailed description of confinement in magnetic wells is provided by the guiding centre model to which we now turn.

(b) Guiding Centre Model

Confinement in magnetic wells can be discussed in more detail from the particle picture. In this case one introduces a distribution function $f(\mu,\varepsilon,L)$ which is the number of particles having magnetic moment μ and energy ε with guiding centres lying in the flux tube labelled by L. The pressure is related to the distribution by

$$p_\perp = \int f(\mu,\varepsilon,L)\ \frac{\mu B^2}{2(\varepsilon-\mu B)^{\frac{1}{2}}}\ d\mu\ d\varepsilon \qquad \ldots (3.7)$$

$$p_\parallel = \int f(\mu,\varepsilon,L)\ B\ (\varepsilon-\mu B)^{\frac{1}{2}}\ d\mu\ d\varepsilon \qquad \ldots (3.8)$$

Clearly a distribution function $f_0(\mu,\varepsilon)$, depending only on μ and ε, corresponds to pressures which are functions only of B, i.e., to the special class of fluid equilibria described above. A detailed analysis based on the guiding centre model confirms that distributions of the form $f_0(\mu,\varepsilon)$ are stable if $\partial f_0/\partial\varepsilon < 0$ and are confined in magnetic wells.

In fact it can be shown directly (Taylor 1963) that any distribution of the form $f_0(\mu,\varepsilon)$ is stable against all perturbations which do not violate the invariance of μ, provided $(\partial f_0/\partial\varepsilon)_\mu < 0$. This is because such a distribution is a state of lowest energy consistent with the constancy of the generalised entropy

$$S = \int d^3v d^3x\ \ G\ (f,\mu) \qquad \ldots (3.9)$$

where G is any function. This entropy S is clearly invariant so long as μ is invariant, for then both f and μ are constant along each trajectory. In a similar way it can be shown that if a number of invariants $\mu_1,\mu_2\ \ldots$ exist then any distribution which can be written $f(\varepsilon,\mu_1,\mu_2\ \ldots\ldots)$ will be stable if $\partial f/\partial\varepsilon < 0$.

The guiding centre description can conveniently be used to describe the equilibrium and stability of all equilibria in magnetic traps, (Taylor 1964) not just of the special equilibria discussed above. To do this one introduces a representation of the magnetic field in which $\underset{\sim}{B} = \underset{\sim}{\nabla}\alpha \times \underset{\sim}{\nabla}\beta$ so that α and β are constant along each line of force and can be used as coordinates of the line. The guiding centre drifts can then be described in these α,β coordinates. The instantaneous drift is less significant than the average drift over an oscillation between mirror reflections and the equations for this average drift take on a particularly simple form. This is because, in addition to the adiabatic invariant μ associated with the cyclic motion perpendicular to $\underset{\sim}{B}$, there is another invariant associated with the cyclic motion parallel to $\underset{\sim}{B}$ i.e., to the oscillation between mirrors. This is

$$J = \oint p_\parallel\ dq_\parallel = \oint (K-\mu B)^{\frac{1}{2}}\ ds \qquad \ldots (3.10)$$

where K is the total kinetic energy. If one specifies μ,J,α,β then the energy K is implicity determined by (3.10) so that $K = K(\mu,J,\alpha,\beta)$; furthermore, this function K plays the role of a Hamiltonian (Northrop and Teller 1960) for the average drift motion with α,β being canonically conjugate coordinates. The equations for the average drift motion can thus be expressed as

$$\langle\dot{\alpha}\rangle = \frac{\partial K}{\partial \beta}\,, \quad \langle\dot{\beta}\rangle = -\frac{\partial K}{\partial \alpha}\,, \qquad \ldots (3.11)$$

and any equilibrium distribution function can be written

$$F_{eq}(\mu,J,\alpha,\beta) = F(\mu,J,K) \qquad \ldots (3.12)$$

which depends on α,β only through the function $K(\alpha,\beta,\mu,J)$. Calculating the change in energy due to an arbitrary perturbation (Taylor 1964) shows that $F(\mu,J,K)$ is stable if $\partial F/\partial K < 0$, for all K, a result which also follows directly from the generalised entropy argument since μ,J are both invariants.

If $\partial F/\partial K < 0$, so that F decreased as K increases, the plasma will be confined only if the contours $K = \text{constant}$ from closed curves in α,β space and surround a minimum in K. Indeed the $K = \text{constant}$ contours play a similar role for the general equilibria to that played by $B = \text{constant}$ contours for the special equilibria. Furthermore, in any minimum-B magnetic field the function K always possess a minimum in α,β space for a very wide range of parameters μ and J. One concludes therefore that magnetic wells provide stable confinement not only for the special equilibria defined by (3.6) but also for a much wider class of equilibria which can be obtained and studied by means of the function K. Some typical $K = \text{constant}$ contours for the minimum B field produced by the configuration in Fig.3 are shown in Fig.5. These exhibit the desired closed contours surrounding a minimum of K at the centre. There are also closed contours surrounding the points X but these are maxima of K and do not represent stable confinement regions.

The relation of the special equilibria $f(\mu,\varepsilon)$ to the more general form $f(\mu,J,K)$ is illuminated when one considers the relaxation of an arbitrary, non-equilibrium, plasma. The final equilibrium is the state of stationary, presumably minimum, energy subject to the condition that the appropriate generalised entropy be conservel. If in the definition of this entropy one regards both μ and J as identifiable parameters of each particle then the corresponding equilibrium is of the form (3.12). If, however, only μ is treated as an identifiable parameter, (corresponding in some sense to retaining only μ as an invariant during the relaxation process) then the only equilibria are those of the form $f(\mu,\varepsilon)$. The "special" equilibria therefore involve a relaxation of the constant entropy constraint and so will generally be states of lower energy.

As a result of the developments described above a satisfactory theoretical understanding of the low frequency behaviour of plasma in mirrors and in magnetic wells has been achieved. In particular, by exploiting the magnetic well concept, the low frequency, long-wavelength instabilities can be overcome. This opens the way, both theoretically and experimentally, to the study of high frequency phenomena, that is of oscillations at frequencies comparable to the cyclotron frequency. Oscillations and instabilities in this regime are expected to be less important than low frequency instabilities in transporting plasma across the confining field. Nevertheless they are still very damaging in mirror systems as they may scatter particles into the loss-cone.

4. HIGH FREQUENCY PHENOMENA (VLASOV MODEL)

For the study of high frequency plasma behaviour the Vlasov equation must be used. Unfortunately, unlike the fluid and guiding centre models, this is too complicated to be applied in realistic geometrical situations. Indeed solutions have only been obtained for such theoretical abstractions as the uniform infinite medium and the plane slab (plasma properties varying only in one direction, perpendicular to the magnetic field). Some features of these idealised models are discussed below.

(a) Infinite Medium

In an infinite medium with uniform magnetic field the equilibrium distribution f_o is an arbitrary function of $v_\perp^2$ and $v_\parallel$. The linearsed equation for small electrostatic perturbations is then

$$\frac{\partial f_1}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial f_1}{\partial \underset{\sim}{x}} + \frac{e}{m} (\underset{\sim}{v} \times \underset{\sim}{B}) \cdot \frac{\partial f_1}{\partial \underset{\sim}{v}} = \frac{-e}{m} \underset{\sim}{E} \cdot \frac{\partial f_o}{\partial \underset{\sim}{v}} (v_\perp^2, v_\parallel) \qquad \ldots (4.1)$$

where f_1 is the perturbation in f and $\underset{\sim}{E} = \underset{\sim}{\nabla} \varphi$ is the perturbed electric field. For a perturbation in the form of a single Fourier mode

$$\begin{Bmatrix} \varphi \\ f_1 \end{Bmatrix} = \begin{Bmatrix} \tilde{\varphi} \\ \tilde{f}_1 \end{Bmatrix} \exp i\,(\omega t + k_x x + k_y y + k_z z)$$

equation (4.1) can be solved (Harris 1961) to give

$$\tilde{f}_1 = \frac{2e}{m} \tilde{\varphi} \sum_{n,m} \frac{J_n(k_\perp a)\; J_m(k_\perp a)}{\omega + k_z v_z + n\Omega} \cdot e^{i(n-m)\,\psi} \left[k_z v_z \cdot \frac{\partial f_o}{\partial v_z^2} + n\Omega \frac{\partial f_o}{\partial v_\perp^2} \right] \qquad \ldots (4.2)$$

where a is the Larmor radius $v_\perp/\Omega$, Ω is the cyclotron frequency and ψ is the azimuthal angle in velocity about the field direction Oz. The perturbed charge density obtained from (4.2) is now inserted into Poisson's equation and yields a dispersion relation (Harris 1961) connecting the frequency ω to the wave number $\underset{\sim}{k}$, namely

$$\frac{k^2}{8\pi^2} = \sum_{i,e} \frac{e^2}{m} \int dv_\perp^2\, dv_\parallel \sum_n \frac{J_n^2(k_\perp a)}{\omega + k_z v_z + n\Omega} \left[k_z v_z \frac{\partial f_o}{\partial v_z^2} + n\Omega \frac{\partial f_o}{\partial v_\perp^2} \right] \qquad \ldots (4.3)$$

In a formal sense this equation provides a complete solution to the infinite medium problem, for the question of stability is now reduced to determining whether, for any real $\underset{\sim}{k}$, equation (4.3) has a complex solution $\omega(\underset{\sim}{k})$ with negative imaginary part (corresponding to growing oscillations). In practice not only is it difficult to solve (4.3) for any given form of f_o but there is, in any case, a wide range of equilibrium distribution functions f_o which are of interest, and for each of which equation (4.3) must be investigated.

If the equilibrium distribution f_o is Maxwellian then, of course, there are no roots of (4.3) corresponding to instability. However, almost any departure of the equilibrium distribution function from the Maxwellian form is a potential, and frequently an actual, source of instability. It is useful, therefore, to note that <u>some</u> departure from the Maxwellian form <u>is</u> possible, without producing any instability. For example, the generalised entropy argument shows that any distribution which is a monotonically decreasing, but not necessarily Maxwellian, function of the energy alone must definitely be stable and this is confirmed by examination of the dispersion equation (4.3).

The dispersion equation also shows that a bi-Maxwellian distribution

$$f_o \sim \frac{1}{\pi^{3/2} a_\perp^{2} a_z} \exp\left[-v_\perp^2/a_\perp^2 - v_\parallel^2/a_z^2\right] \qquad \ldots (4.4)$$

is stable (Shima and Hall 1965) provided temperature ratio $a_\perp^2/a_z^2 < 2$. Whether instabilities arise with larger anisotropy depends on the plasma density and on the ratio of ion to electron temperatures; a large anisotropy will certainly produce instability in dense plasma. For example, the ion distribution function

$$\frac{1}{\pi a_\perp^2} \exp\left(-v_\perp^2/a_\perp^2\right) \cdot \delta(v_z) \qquad \ldots (4.5)$$

has been investigated (Harris 1961) and found to be unstable if the density is such that $\omega_{pe} > \Omega_{ci}$ (where the plasma frequency $\omega_{pe}^2 = 4\pi ne^2/m$ is a measure of density and the cyclotron frequency $\Omega_{ci} = eB/M_i c$ is a measure of the field).

Another source of instability is the loss-cone which exists in any mirror confinement system; because of this the equilibrium distribution function f_o is zero when $v_\perp < \lambda v_\parallel$. This means that the distribution of perpendicular velocities, either at a specified $v_\parallel$ or averaged over all $v_\parallel$, is non-monotonic, being zero both for $v_\perp = 0$ and $v_\perp = \infty$ (see Fig. 6). This property of $f(v_\perp)$ gives rise to a situation similar to that in a maser; higher energy states are preferentially populated and the system relaxes, through an instability, to a state of lower energy. A simple distribution illustrating this loss-cone feature is

$$f_o \sim \delta(v_\perp - v_o)\ \delta(v_z). \qquad \ldots (4.6)$$

For this distribution the dispersion equation (4.3) not only exhibits instabilities driven by the anisotropy, which were already evident using (4.5), but also a new class of instabilities (Harris 1961) with $k_z = 0$ which are of the loss-cone or maser type described above.

The distribution (4.6) is an extreme form of loss cone distribution but it has been shown (Rosenbluth and Post 1965) that <u>any</u> distribution of ions possessing a loss cone, (i.e. any ion distribution for which $f_o = 0$ when $v_\perp = 0$), leads to instabilities if $\omega_{pe} > \Omega_{ci}$ and the electrons are cold. If the electron thermal motion is taken into account the threshold for instability is raised (Rosenbluth 1965) to

$$\omega_{pe} > \Omega_{ci} \frac{\langle v_\parallel e\rangle}{\langle v_\perp i\rangle} \qquad \ldots (4.7)$$

where $\langle v_\parallel e\rangle$ and $\langle v_\perp i\rangle$ are the average electron-parallel and ion-perpendicular velocities, and $\langle v_\parallel e\rangle$ exceeds $\langle v_\perp i\rangle$. On the basis of these results one must expect any mirror confined plasma to exhibit a high frequency instability at high plasma density, $\omega_{pe} \gg \Omega_{ci}$.

Fortunately the true situation is not quite as bad as the simple infinite medium analysis implies. To see this one must distinguish between absolute and convective instabilities by studying a localised initial disturbance (a wave packet) instead of a single Fourier mode extending through space. If this wave packet grows at its point of origin the medium is absolutely unstable; if it grows, but at the same time travels through the medium so rapidly that its amplitude at the point of origin decreases, then the medium is only convectively unstable. In an infinite medium both types of instability lead to indefinite growth of an initial disturbance but whether a convective instability will grow indefinitely in a finite medium depends on the boundary (reflection) conditions and the size of the system. It may be that by the time an initial disturbance has been convected through the length of the system, it will not have been amplified sufficiently to outweigh a weak reflection coefficient at the end.

The loss-cone instability referred to above is of the convective type (Rosenbluth and Post 1965) and fortunately it appears that the reflection coefficient R may be very small (Aamodt and Book 1966) so that a relatively large amplification A may be permitted before the produce $RA>1$: however one must also ensure that the final fluctuation level is not too great, otherwise although the system is in principle stable, these fluctuations may themselves produce unacceptable scattering losses. This imposes a more severe restriction on the maximum amplification of a wave packet and therefore on the maximum permitted length of the plasma.

The e-folding length for the convective loss-cone instability can be found from the dispersion equation (4.3) and is (Post and Rosenbluth 1966)

$$2a_i \ (1 + B^2/4\pi nmc^2)^{\frac{1}{2}} \cdot G \qquad \ldots (4.8)$$

where a_i is the average ion Larmor radius, m the electron mass, and G is a factor which depends on the form of the distribution function f_o. If one accepts that about 10 e-folding lengths might be permitted before the amplitude is increased from spontaneous fluctuation level to a significant value, then the maximum permitted length of the plasma is

$$L_{\parallel} = 20 \ a_i \ (1 + B^2/4\pi nmc^2)^{\frac{1}{2}} \ G \ . \qquad \ldots (4.9)$$

At high plasma densities $B^2 \ll 4\pi nmc^2$ and the maximum length is $\sim 20 \ a_i G$.

The function G is extremely sensitive to the exact form of the distribution function. For a sharply peaked distribution it may be very small indeed, perhaps only .05, so that the permitted length is then absurdly small. For smoother loss-cone distributions, such as would be expected to arise in the presence of a small collisional diffusion, G may be larger and values of about 5 have been claculated (Post and Rosenbluth 1966) for realistic loss-cone distribution functions. In this case the critical plasma length would be about 100 ion Larmor radii.

It seems, therefore, that the convective loss-cone instability might be tolerated in a rather short plasma. However, it should be emphasised that the estimate of 100 Larmor radii for the maximum length is sensitive to the exact form of the distribution function.

(b) Plane Slab Model

Slightly more realistic than the infinite medium is the plane slab, which introduces a density gradient perpendicular to the magnetic field and so corresponds a little more closely to true confinement. This density gradient has associated with it a current transverse to the confining magnetic field such that $\underset{\sim}{j} \times \underset{\sim}{B} = \underset{\sim}{\nabla} p$. Unfortunately this current represents an additional departure from the Maxwellian distribution and introduces further instabilities.

To represent a plasma immersed in a uniform magnetic field in the z-direction and with a density gradient in the x-direction one constructs an equilibrium distribution from the three constants of motion $v_\perp^2$, $v_\parallel^2$, $x + v_y/\Omega$. The simplest distribution function constructed from these constants which includes a density gradient is

$$f_o = f_o(v_\perp^2, v_\parallel^2)\left[1 + \varepsilon\ (x + v_y/\Omega)\right] \qquad \ldots (4.10)$$

in which ε is a measure of the density gradient, $\varepsilon = \frac{1}{n}\frac{\partial n}{\partial x}$.

The method of investigation is similar to that of the simpler infinite medium problem. In this case the normal mode solutions of the linearised Vlasov equation are of the form

$$\tilde{f}(x)\ \exp\ i\ (\omega t + k_y y + k_z z)\ , \qquad \ldots (4.11)$$

but the essential features of the problem are illustrated by assuming a perturbation of the form $\sim \exp\ i\ (\omega t + ky)$ and ignoring the x-variations. This is because it is the presence of a transverse current rather than the variation in density itself which is the essential feature. In this way one obtains a dispersion equation

$$\frac{k^2}{8\pi^2} = \sum \frac{J_n^2(ka)}{(\omega + n\Omega)}\left[n\Omega\frac{\partial f_o}{\partial v_\perp^2} + \frac{\varepsilon k}{2\Omega} f_o\right] \qquad \ldots (4.12)$$

The effect of the transverse current associated with the density gradient is embodied in the last term of (4.12) and analysis of this new dispersion equation shows an <u>absolute</u> (non-convective) instability (Post and Rosenbluth 1966) whenever f_o is a loss cone type of distribution and ε exceeds a critical value ε_c defined by

$$\varepsilon_c\, a_i \simeq (0.38)\left(\frac{m}{M}\right)^{2/3}\left(1 + B^2/4\pi nmc^2\right)^{2/3} \qquad \ldots (4.13)$$

This absolute instability will be little affected by finite length and so can be avoided only if the density gradient perpendicular to the magnetic field is everywhere less than that corresponding to ε_c. This imposes a restriction on the size of the plasma in the direction perpendicular to the field which must <u>exceed</u> a value given by

$$L_i \simeq 2.5\left(\frac{M}{m}\right)^{2/3}\left(1 + B^2/4\pi nmc^2\right)^{-2/3} \qquad \ldots (4.14)$$

At high density this is again typically equal to about 100 Larmor radii.

(c) <u>Summary</u>

The overall theoretical position concerning adiabatic mirror containment systems is therefore that they are subject to both low and high frequency instabilities. The low frequency instabilities can be described in realistic detail by the guiding centre model and can be overcome by shaping the magnetic field - magnetic wells. The high frequency instability can be described by the Vlasov model only in idealised situations. Some high frequency instabilities can be controlled by smoothing out the distribution function but the <u>essential</u> loss-cone nature of the distribution function leads to some persistent instabilities. These can be controlled only by imposing restrictions on the size of the plasma - its <u>length</u> (along the field direction) must be <u>less</u> than about 100 Larmor radii and its width (transverse to the field direction) must <u>exceed</u> about 100 Larmor radii.

5. CLOSED SYSTEMS

In closed magnetic confinement systems no attempt is made to restrict the loss of particles along the lines of force as by the mirror effect. Instead one ensures that the lines of force remain within a closed, toroidal, region. In principle, closed systems should provide longer term containment than mirror systems since collisional scattering produces only a slow diffusion across the field instead of a rapid escape through the loss-cone.

The obvious closed system would be a simple toroidal field produced by a solenoidal winding on the surface of a torus. However such a toroidal field is necessarily stronger on its inner edge and weaker on its outer edge and so produces gradient-B drift of ions in one direction and of electrons in the other. The ensuing charge imbalance sets up electric fields whose $\underset{\sim}{E} \times \underset{\sim}{B}$ drift carries particles to the walls of the system. A simple toroidal field, in fact, cannot provide equilibrium, still less stable equilibrium.

A toroidal equilibrium can, however, be obtained if an additional component of magnetic field is introduced so that a line of force no longer links up with itself after one circuit around the torus as it does in the simple toroidal field. Instead a line of force passing through a minor cross section at a point P_o will return after one circuit to a point P_1 which is rotated slightly. Successive transits through the cross section will be at P_2, P_3 ...... P_n ...... and if followed indefinitely a single line may thus generate a 'magnetic surface' (Fig.7). In this event the average angle by which the line is rotated about the magnetic axis during one circuit defines the rotational transform (Spitzer 1958) of that surface which, strictly, must be an irrational multiple of 2π. The question of when such magnetic surfaces exist is a difficult one. All one can say in general is that magnetic surfaces exist in an asymptotic approximation when the rotational transform is small, and are exact when the system is invariant under some continuous transformation, as in the case of azimuthal symmetry.

When magnetic surfaces are formed, plasma containment is possible despite the gradient-B drifts, for these will now be averaged over the whole magnetic surface and a drift away from the surface on one part is cancelled by drift towards the surface in another part. Furthermore, since the surface is generated by a single line of force along which electrons flow freely, each surface must be an equipotential and any $\underset{\sim}{E} \times \underset{\sim}{B}$ drifts are tangential to the magnetic surface and can produce no loss.

A feature of closed systems is that they can operate with scalar pressure, indeed they must do so if their full potential for long term (i.e., many collision times) containment is to be realised. In this case each magnetic surface is also a constant pressure surface, for the equilibrium condition $\underset{\sim}{j} \times \underset{\sim}{B} = \underset{\sim}{\nabla} p$ implies immediately that $\underset{\sim}{B} \cdot \underset{\sim}{\nabla} p = 0$.

In practice the rotational transform can be provided in two ways. The obvious method would be to induce a toroidal current in the plasma itself - as in the well known toroidal pinch experiments such as ZETA (Butt, Carruthers, Mitchell, Pease, Thonemann, Bird, Belars and Hartill 1958; Burton, Butt, Cole, Gibson, Mason, Pease, Whiteman and Wilson 1962). However, it is difficult to control the magnetic fields produced in this way and there is a large magnetic energy associated with the plasma current. This energy might be released during plasma motion and so represents an additional and potent source of instability. At present, therefore, toroidal containment can be better studied using fields created mainly by external conductors.

(a) Stellarator

The rotational transform needed for magnetic surfaces can be provided externally by a number of helical conductors wound on the surface of the torus, adjacent conductors carrying current in opposite directions (Fig.8). This arrangement is usually known as a Stellarator (Spitzer 1958). If such a system of helical conductors were wound about a straight axis, instead of being bent into a torus, then the system would be helically invariant and would definitely possess helical magnetic surfaces such as those whose cross section is shown in Fig.9 (which are calculated for a six wire, $\ell = 3$, helix). To what extent these surfaces still exist when the system is made toroidal is a question which at present can be answered only by detailed calculations of each specific case.

Magnetohydrodynamic stability of a Stellarator is ensured, at least at sufficiently low pressure, provided only that the magnetic field also possesses shear, that is that the rotational transform increases with distance from the magnetic axis (Johnson, Oberman, Kulsrud and Frieman 1958). Lines of force on adjacent surfaces then have a different average pitch and are inclined at an angle to one another. This means that it is impossible for a distortion of the plasma to grow without "bending" the lines of force. This would require magnetic energy and at low pressures the plasma is unable to provide this energy. Detailed calculations (Johnson et al 1958), using the fluid model, and in particular the energy principle, confirm that a sheared Stellarator field can indeed provide hydromagnetic stability up to a finite plasma pressure, typically a few percent of the magnetic pressure $B^2/8\pi$. The exact stability conditions must be calculated in each case.

(b) Multipole

Stellarators, that is toroidal systems embodying magnetic surfaces and shear, exist in many forms, but there is also another approach to toroidal confinement. This is exemplified by the 'Multipole' confinement system (Ohkawa and Kerst 1961). In this two or more toroidal hoops, carrying current in the same direction are suspended inside the toroidal containment vessel (Fig.10). The field of these hoop conductors lies entirely in a meridional plane and has the general form shown in Fig.11 which illustrates an octupule (four hoops) device. The lines of force do not generate surfaces, in fact they close upon themselves.

Generally, when lines of force form closed loops instead of surfaces, the condition for hydromagnetic equilibrium is that the pressure (which is constant along each loop) should be a function only of $U \equiv \oint \frac{d\ell}{B}$ where the integral is around the closed loop (Kadmotsev 1958). However, in the axisymmetric Multipole this condition is automatically satisfied by the symmetry; the particle drifts are also in the azimuthal direction and so can produce neither direct particle losses nor electric fields.

In Multipole type containment systems there is, of course, no stabilisation by shear. Instead a form of stabilisation closely related to the Minimum-B principle is exploited. It is, unfortunately, impossible to create a closed toroidal system which is also a true minimum-B system, (essentially because lines of force in a closed system must, at least in part, be convex outwards and a convex field is decreasing). The best that can be done is to ensure that B increases outwards 'on the average'. The appropriate average is proportional to $(\oint \frac{d\ell}{B})^{-1}$, so that the stability criterion can be written (Kadomtsev 1958; Rosenbluth and Longmire 1957) $\nabla p \cdot \nabla U > 0$. This result is closely related to one obtained by averaging the function K (section 3b) for an isotropic distribution of plasma.

Accordingly one will have hydromagnetic stability if the quantity U decreases towards the plasma boundaries; in the Multipole this means towards both the outer boundary and the suspended hoops. In other words one requires a maximum of U on some field line through the central region. The existence of such a maximum in U is automatically ensured in the Multipole by the presence of the null point in the field. On the line of force through this point $U \to \infty$ and the general behaviour of U as a function of the flux is shown in Fig.12. The useful, stable, confinement region is that within ψ_c, which is also indicated in Fig.11. Another useful interpretation of the stability criterion that U possess a maximum relies on the fact that U also represents the volume of a flux tube per unit flux; when this decreases towards the periphery of the plasma then an outward displacement of plasma will involve its compression and so require an increase in energy.

(c) General

A third category of toroidal confinement systems combines some of the features of both Stellarator and Multipole. Like the Stellarator they have magnetic surfaces, but instead of relying on shear for hydromagnetic stability they involve another form of 'minimum average-B'. In this case the average is defined not by the line integral U but by an average over the whole magnetic surface (Lenard 1964). This is again related to the volume within a flux surface and is accordingly often referred to as the 'negative V''' criterion, where $V(\psi)$ is the volume within a flux surface and the primes denote differentiation with respect to the flux ψ within the surface.

In several types of toroidal system then, one can achieve equilibrium and hydromagnetic stability - by shear, by decreasing $\oint \frac{d\ell}{B}$, or by negative V''. As they have no mirror type of loss cone and can operate with almost isotropic distribution functions, they should not be particularly susceptible to high frequency instabilities. One might hope, therefore, that this hydromagnetic stability alone would be sufficient to ensure good confinement in toroidal systems. Unhappily experiments clearly indicate that while there is an improvement in containment when the conditions for hydromagnetic stability are met, there nevertheless remain anomalously large plasma losses in many cases. Although other explanations are not entirely ruled out the loss is believed to be due to some residual low frequency instability.

6. RESIDUAL INSTABILITIES IN CLOSED SYSTEMS

The origin of the residual instabilities is still under investigation. One possibility is that they are due to small non-ideal effects such as resistivity which have been neglected in the ideal hydromagnetic model. Although the resistivity is small it can, like viscosity, be dominant in a thin layer. If such a layer, (analogous to a boundary layer but occurring anywhere within the plasma), is formed then new unstable motions can occur which would otherwise be prohibited in the fluid model (Furth, Killeen and Rosenbluth 1963; Johnson, Greene and Coppi 1963; Roberts and Taylor 1965). Essentially this is because resistance relaxes the constraint $\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B} = 0$ which implied, in particular, $E_{\parallel} = 0$. Other physical phenomena such as electron inertia can produce a similar effect. With a resistance η, instabilities are found in which the resistive layer has a thickness proportional to $\eta^{1/3}$ and which have growth rates proportional to $\eta^{1/3}$. Although slow, these instabilities are nevertheless much more rapid than normal resistive diffusion, which is proportional to η itself, when the resistivity is small. However as the resistivity is reduced, e.g., by increasing temperature, the resistive layer becomes so thin that other non-ideal effects, such as viscosity and the finite Larmor radius corrections, become important. When all these effects are included in the theory they tend to cancel one another - in the sense that stability of a reasonably high shear Stellarator is restored at sufficiently high plasma temperatures (Coppi and Rosenbluth 1966; Coppi 1964; Stringer 1967). If resistance were the cause of the anomalous loss, therefore, one would expect it to disappear at high plasma temperature.

Another possible origin for the anomalous loss which we will examine in more detail, is the so called "drift wave instability" (Rudakov and Sagdeev 1960; Mikhailovski and Rudakov 1963; Krall and Rosenbluth 1965; Kadomtsev and Timofeev 1962; Galeev, Oraevski and Sagdeev 1962). This is an example of a class of low frequency instabilities which cannot be described by a simple fluid model, either because they are of short wavelength or because they depend on wave-particle resonance. They are a specifically plasma phenomena and to study them one must again turn to the Vlasov equation and the one dimensional plane slab model.

(a) Drift Wave Instabilities

Because toroidal systems can, potentially, confine plasma for many interparticle collision times the distribution functions of most interest are those of near Maxwellian form. An appropriate equilibrium distribution function constructed from the constants of motion of the plane slab model (section 4b) is thus

$$f_o = n_o \left(\frac{m}{2\pi\kappa T}\right)^{3/2} \left[1 + \varepsilon\ (x + \frac{v_y}{\Omega})\right] \exp - \left[\frac{mv^2}{2\kappa T}\right] \quad \text{... (6.1)}$$

which again represents plasma with a density gradient $\varepsilon = \frac{1}{n}\frac{\partial n}{\partial x}$, and is simply a special case of (4.10) chosen to represent nearly Maxwellian plasma. The normal mode solutions of the linearised Vlasov equation for the perturbed distribution will be of the form

$$\tilde{f}(x) \exp i\ (\omega t + k_y y + k_z z). \quad \text{... (6.2)}$$

However the essential features can again be illustrated by regarding $\tilde{f}(x)$ as constant because it is again the departure of f_0 from the Maxwellian velocity distribution, rather than the fact that the density varies, which is important. The linearised Vlasov equation can now be solved to give the dispersion equation (Krall and Rosenbluth 1965)

$$k_y^2 = \sum_{i,e} \frac{1}{\lambda^2} \left\{ -1 + \left[1 + \frac{\kappa T k_y}{m\omega\Omega} \left(\frac{1}{n}\frac{dn}{dx} \right) \right] \sum_n \frac{\omega e^{-Z} I_n(Z)}{(\omega + n\Omega)} \left[1 + W\left(\frac{\omega + n\Omega}{k_z (2\kappa T/m)^{\frac{1}{2}}} \right) \right] \right\} \quad \ldots (6.3)$$

where $\lambda^2 = \kappa T/4\pi n e^2$, $Z = k_y^2 \, \kappa T/m\Omega^2$ and W is defined by

$$W(x) = \frac{-1}{\pi^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{y e^{-y^2}}{(x + y)} \, dy \, . \quad \ldots (6.4)$$

This form of the dispersion equation describes both high and low frequency modes, but for the low frequency drift modes, $\omega \ll \Omega$, only the $n = 0$ term need be retained.

This dispersion equation (6.3) leads to unstable waves whose real frequency is given by $\omega \sim k_y v_J$, where v_J is the velocity corresponding to the diamagnetic current introduced by the density gradient;

$$v_J = \frac{1}{n}\frac{dn}{dx}\frac{\kappa T}{eB} \quad \ldots (6.5)$$

The growth rate of these waves depends on the perpendicular wave number and is greatest when $k_y a_i \sim 1$ (a_i is the mean ion Larmor radius).

The basic oscillations $\omega \sim k_y v_J$ can be derived from a simple model in which the ions move across the field with their $\underset{\sim}{E} \times \underset{\sim}{B}$ drift while the electrons distribute themselves according to a Boltzmann distribution $\exp^{-e\varphi/\kappa T}$. The perturbed ion density δn_i is then given by

$$i\omega \, \delta n_i = \frac{-ik_y \varphi}{B} \frac{dn_o}{dx} \quad \ldots (6.6)$$

and the perturbed electron density δn_e by

$$\delta n_e = {}^{-e\varphi n_o/\kappa T} \quad \ldots (6.7)$$

so that charge neutrality requires the frequency to be $k_y v_J$, where v_J is given by (6.5). That this wave is unstable is due to its interaction with electrons moving parallel to the field. In a uniform plasma this interaction would result only in damping - the famous Landau damping (Landau 1946) - but in inhomogeneous plasma there are directions of propagation for which wave amplification occurs.

Because the drift wave arises solely from the existence of a density gradient, and in the simple model is always unstable, it seems to rule out stable confinement. Fortunately, however, the true situation is again not as discouraging as this result would indicate. This is because the drift instability is again convective, being convected both across and along the magnetic field. Accordingly it is susceptible to stabilisation (Krall and Rosenbluth 1965; Mikhailovskaya and Mikhailovsky 1963) both by finite length effects and, more importantly, by "shear", that is by a change in the direction of the magnetic field as one moves across the field in the direction of the density gradient.

The effect of shear can be incorporated into the plane slab model by taking the magnetic field to be

$$B = (0, \frac{x}{L_s} B, B)$$

so that lines of force at different values of x are inclined at an angle to one another. The parameter L_s is referred to as the "shear length", being the distance in the direction of the density gradient over which the direction of the field changes by about $\pi/4$.

For a discussion of the influence of "shear" on the drift wave instability the x-dependence of the normal mode is essential, thus

$$\varphi \sim \tilde{\varphi}(x) \exp i (\omega t + k_y y + k_z z) \qquad \dots (6.7)$$

However $\tilde{\varphi}$ and the equilibrium density are assumed to vary slowly over the scale of the Larmor radius. The linearised Vlasov equation can then be solved for the perturbation f_1 and the perturbed charge density introduced into Poissons equation. This now takes the form

$$a_i^2 \frac{d^2\tilde{\varphi}(x)}{dx^2} - Q(\omega,k,x)\, \tilde{\varphi}(x) = 0 \qquad \dots (6.8)$$

where a_i is the ion Larmor radius. Putting $Q(\omega,k,x) = 0$ would lead to the 'local' dispersion equation (6.3) discussed earlier.

In the usual analysis the frequency $\omega(k)$ is determined, e.g., as in the W.K.B. method, by the requirement that $\tilde{\varphi}(x)$ be a localised solution, i.e., decaying as $x \to \pm \infty$. The full analysis indicates that a very small shear should completely stabilise the drift wave instability, (Krall and Rosenbluth 1965; Mikhailovskaya and Mikhailovski 1963) for no localised, W.K.B. solution can be found with unstable $\omega(k)$ if the shear length L_s satisfies

$$L_s \frac{1}{n}\frac{dn}{dx} < 8 \left(\frac{a_i}{n}\frac{dn}{dx} \right)^{-1} \qquad \dots (6.9)$$

a condition which is easily met. It is usual to introduce an equilibrium scale length L defined by $1/L = \frac{1}{n}\frac{dn}{dx}$ when (6.9) becomes

$$\delta \equiv \frac{L}{L_s} > \frac{1}{8}\frac{a_i}{L} \qquad \dots (6.10)$$

showing that the shear parameter δ, the change in direction of the field over one density scale length, only needs to exceed a fraction of the already small quantity a_i/L for stability.

However, this result is much <u>too</u> optimistic, for whilst it may be true that there are no indefinitely growing exponential modes when (6.9) is satisfied, this does not prevent the initial growth of a wave packet to a substantial amplitude before its eventual decay (Kadomtsev 1966). A complete analysis of the behaviour of a wave packet indicates that it may grow to an unacceptably large amplitude, before finally decaying, unless the shear length is reduced well below the value given by (6.9). How small the shear length must be depends on how much importance is attached to waves of very short transverse wavelength. If only disturbances with transverse wavelengths much greater than an ion Larmor radius are important in producing plasma loss then the necessary shear (Coppi, Laval, Pellat and Rosenbluth 1966) is of order.

$$\delta \sim \left(\frac{m}{M}\right)^{\frac{1}{2}} \qquad \ldots (6.11)$$

Thus instead of being small in the ratio of Larmor radius to scale length (a/L), the required shear is now small only as the square root of the electron : ion mass ratio.

If disturbances of all transverse wavelengths are included then still more shear is required for stability. One estimate (Frieman and Rutherford 1967) is that the necessary shear is then of order $\delta \sim \left(\frac{m}{M}\right)^{\frac{1}{3}}$; that is small only as the cube root of the mass ratio.

An analysis of the effect of finite length, which in toroidal systems means the effect of imposing periodicity along each line of force to simulate toroidal confinement, indicates that the drift mode will be stabilised (Krall and Rosenbluth 1965) if the length of the lines of force is less than

$$L_c \simeq 2(2\pi)^{3/2} \left(\frac{1}{n}\frac{dn}{dx}\right)^{-1} .$$

Therefore line closure may overcome the drift instability only if the length of the line does not exceed about 30 times the scale length of the density gradient.

Stabilisation of the drift wave becomes still more difficult (Coppi, Rosenbluth and Sagdeev 1966; Kadomtsev 1966) when there is a gradient in ion temperature as well as in density. In fact if

$$\frac{1}{T}\frac{dT}{dx} > 2\,\frac{1}{n}\frac{dn}{dx}$$

then an unstable W.K.B. normal mode solution exists unless the shear exceeds the prohibitively large value corresponding to δ of order unity, that is with the shear length comparable to the density gradient scale length. It is impossible to meet the requirement in a Stellarator but it might be possible in a toroidal pinch.

(b) Summary

The overall theoretical position on stability of toroidal containment systems is that while hydromagnetic stability can be provided, at least at low plasma pressure, in several ways, this alone is not sufficient - possibly because of resistive instabilities or drift wave instabilities.

Resistive instabilities should disappear at sufficiently high temperature and, in the absence of severe temperature gradients, the drift instabilities can be overcome by shear. Much more shear is required for this than for hydromagnetic stability - the shear length must be less than ~ 10 times the density gradient scale length. Alternatively stabilisation may be achieved if the length of the closed lines of force is less than ~ 30 times the density gradient scale length.

7. CONCLUSIONS

Some of the theoretical plasma models used to investigate plasma confinement and stability have been described and illustrated by their application to the problems of containment in magnetic mirrors and in toroidal systems. They lead to the following conclusions.

The low-frequency behaviour of plasma confined by mirrors can be described in detail and appears to agree with experiment. In particular, the low-frequency long-wavelength instabilities, to which mirror systems are prone, can be completely controlled by appropriate design of the magnetic field (magnetic well). Studies of the high-frequency behaviour indicate that highly peaked, or highly anisotropic, distributions are always unstable. These instabilities can be alleviated, and some can be eliminated, by using a smoother, (that is a less peaked or less anisotropic) distribution. They are also alleviated by raising the electron temperature. However, the essential loss-cone character of the distribution function in a mirror system is responsible for two instabilities which persist even for distributions which are as well smoothed as is compatible with the existence of a loss cone. These instabilities impose restrictions on the dimensions of the plasma. To control or limit them to an acceptable level, the plasma length (parallel to the field) must be <u>less</u> than about 100 ion Larmor radii (depending sensitively on the exact distribution) and the plasma width (perpendicular to the field) must be <u>more</u> than about 100 ion Larmor radii.

Although these restrictions may eventually be overcome or may prove acceptable, the inevitable loss cone remains a potential source of instability and in any case results in poor containment even with a fully stable plasma. Consequently, although mirror systems have been and continue to be a valuable means of investigating plasma containment and stability, the closed toroidal systems appear to offer the better long term prospect for plasma containment.

Closed toroidal systems are of two types, those which depend on the existence of magnetic surfaces and those in which the lines of force close upon themselves. The conditions for hydromagnetic equilibrium are different in the two cases; when magnetic surfaces exist the constant pressure surfaces coincide with them; when the lines of force close the constant pressure surfaces are the surfaces upon which $U = \oint \frac{d\ell}{B}$ is constant. Magnetic surfaces certainly exist when symmetry makes one coordinate in the problem redundant - otherwise their existence is proved only in an asymptotic sense or to the extent that numerical calculations are relevant.

Hydromagnetic stability of these toroidal systems can be achieved, in the one case by shear in the magnetic field, and in the other by ensuring that U decreased towards the periphery of the plasma (minimum average B). There are, however, further low-frequency instabilities possibly due to resistance or to drift waves. The former can be overcome by increasing temperature and the latter by increasing the shear or by restricting the length of closed lines of force.

For shear stabilisation, calculations of simple models indicate that the shear length must be less than about 10 times the scale length of the transverse density gradient and in the case of closed lines their length must be less than about 30 times the scale length of the transverse density gradient.

At the present time, therefore, the favoured routes toward long term stable plasma are via the high-shear systems and the minimum-average-B closed-line system exemplified by the Multipole.

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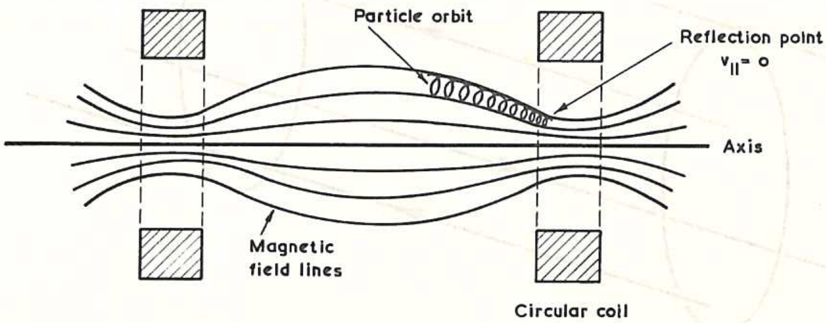


Fig. 1 Mirror Containment System (CLM-L 14)

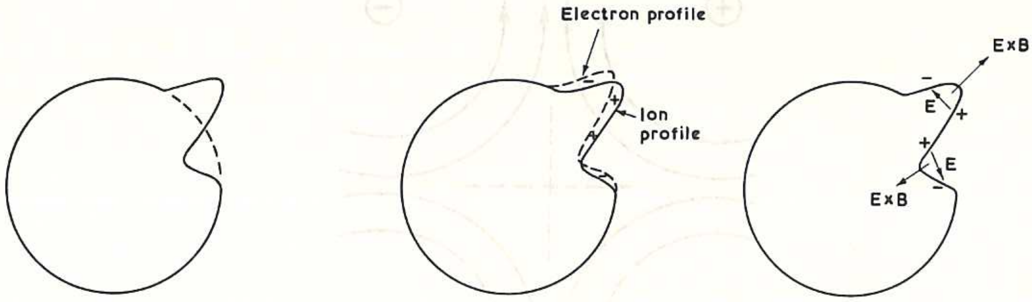
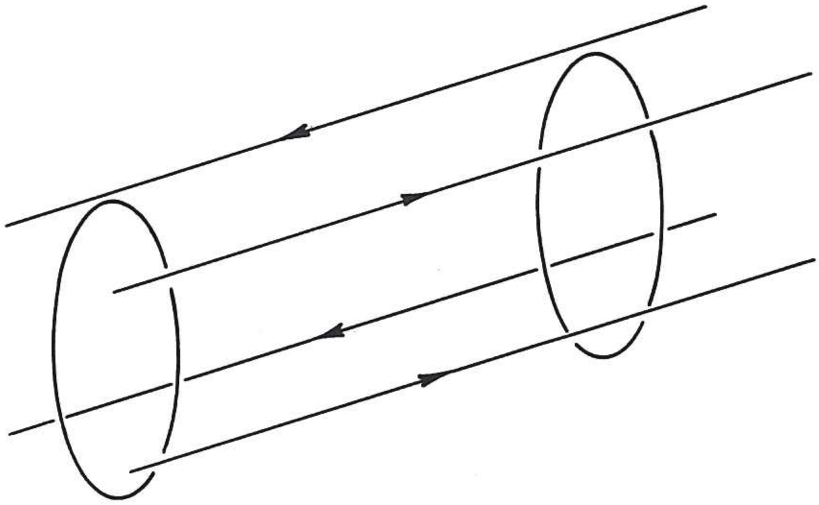


Fig. 2 (CLM-L 14)
Development of Flute Instability; (a) Initial Disturbance)
(b) Effect of Ion and Electron azimuthal drifts
(c) Resulting E × B drifts increase amplitude



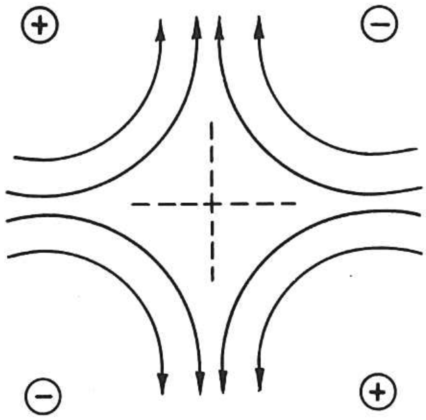
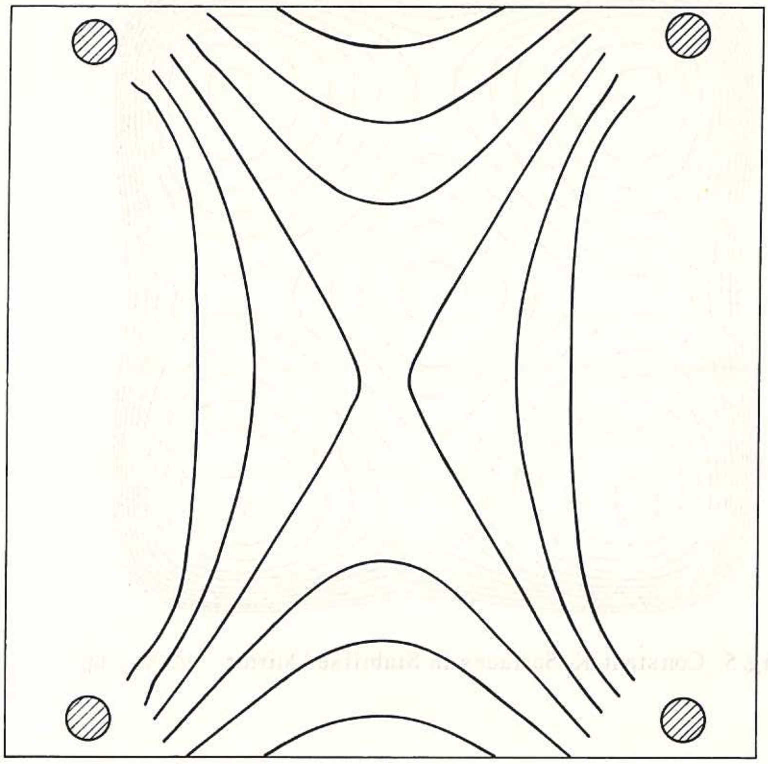


Fig. 3 (CLM-L 14)
(a) Stabilising Bars added to Mirror Coils (b) Field of Stabilising Bars



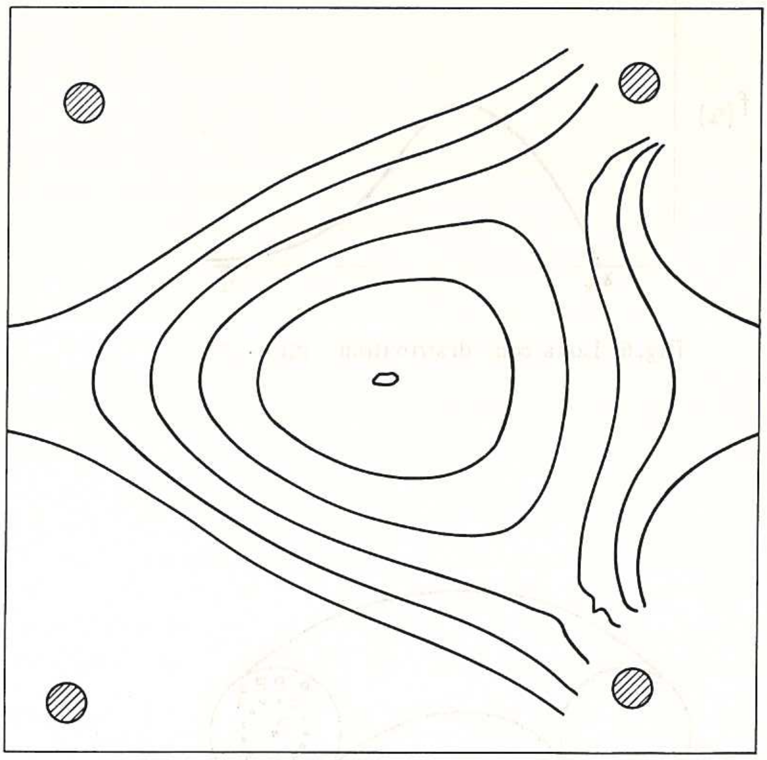


Fig. 4 (CLM-L 14)

Magnetic Isobars in Stabilised Mirror; (a) Small current in stabilising rods (b) Large current in stabilising rods

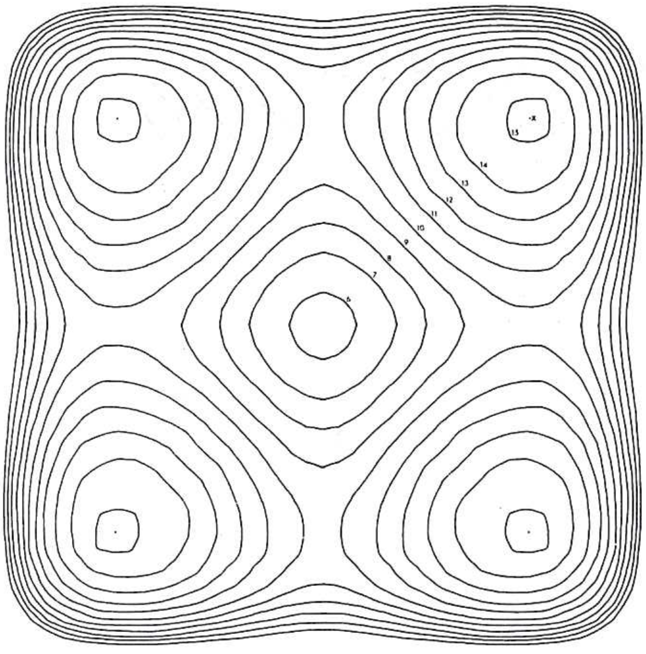


Fig. 5 Constant K Surfaces in Stabilised Mirror (CLM-L14)

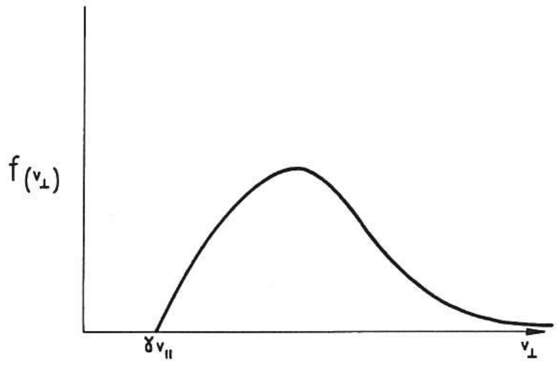


Fig. 6 Loss cone distribution (CLM-L14)

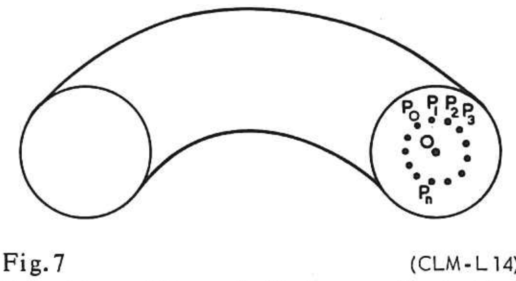


Fig. 7 (CLM-L14)
Formation of Magnetic Surfaces. Single line of force intersects minor cross section at P_0, P_1, . . . P_n . . . on successive circuits around torus.

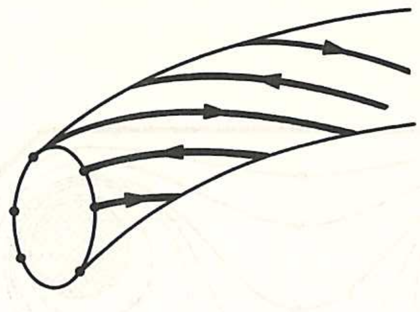


Fig. 8 (CLM-L 14)
Helical Windings to produce Rotational Transform and Magnetic Surfaces (Stellarator).

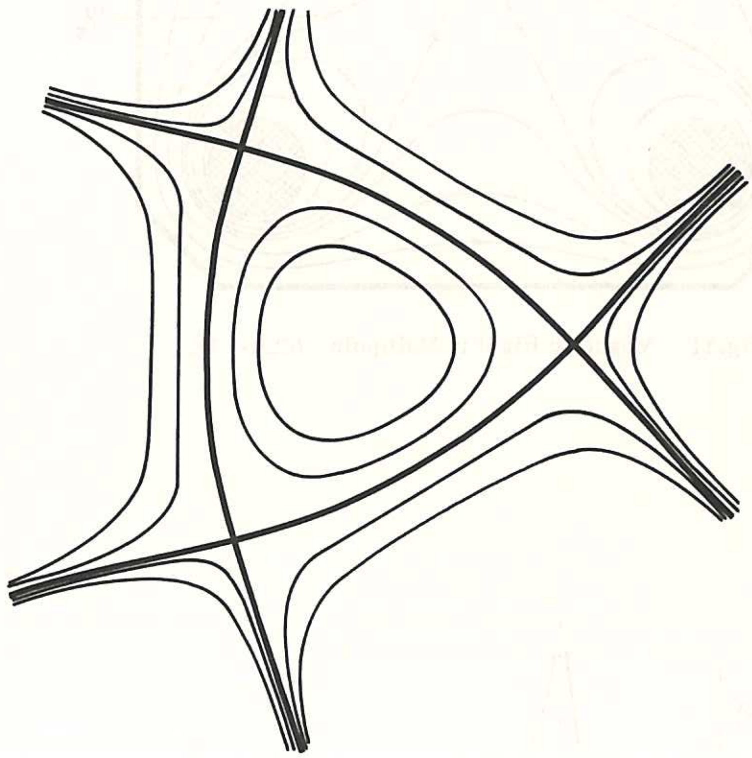


Fig. 9 (CLM-L 14)
Cross section through Magnetic Surfaces produced by Helical Windings

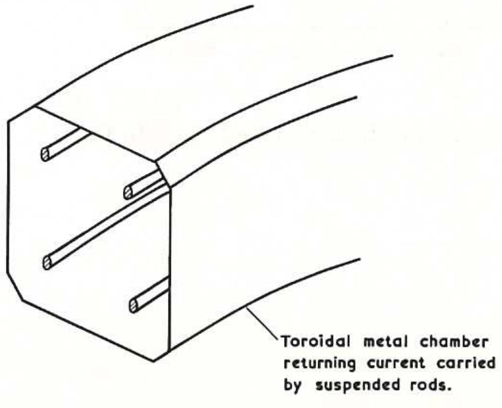


Fig. 10 Toroidal Multipole Configuration (CLM-L 14)

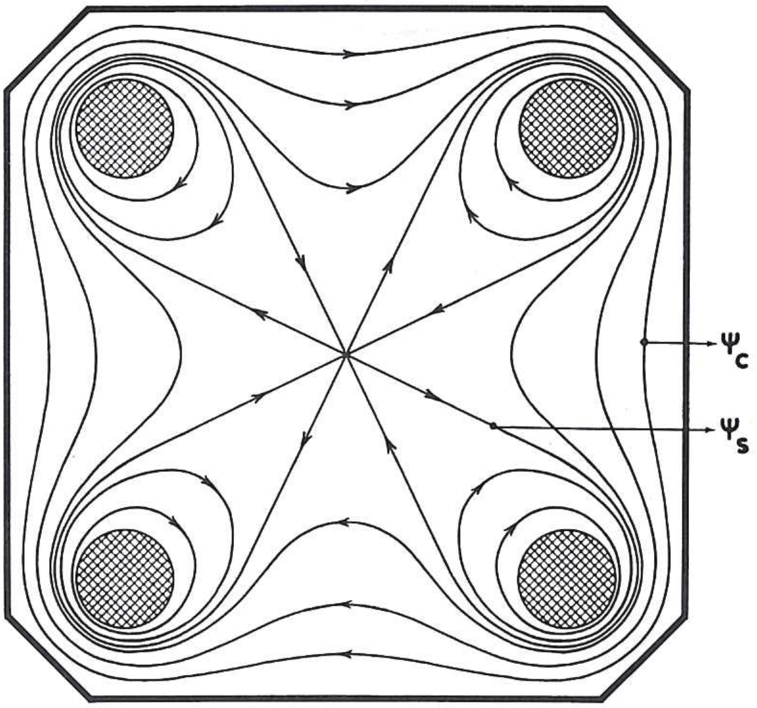


Fig. 11 Magnetic Field in Multipole (CLM-L14)

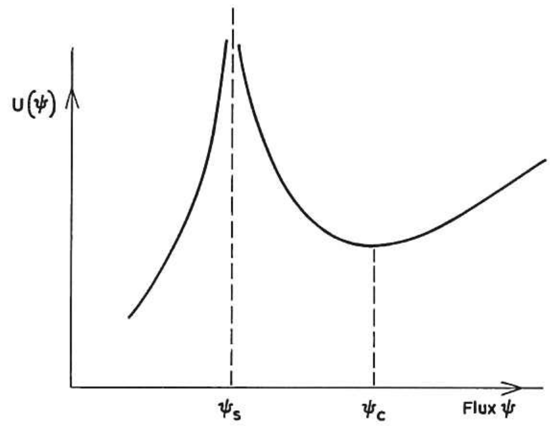


Fig. 12 (CLM-L14)
Value of U as function of flux enclosed by field line