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THE EFFECT OF SHAPED PLASMA CROSS SECTIONS ON THE IDEAL INTERNAL KINK MODE IN A TOKAMAK

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ABSTRACT

The effect of shaped plasma cross-sections on the stability of the ideal internal kink (m = 1) mode in a Tokamak is studied. Whereas elliptic shaping has a weak destabilising effect it is found that triangular and higher harmonics can significantly increase the maximum poloidal β for stability, above the value obtained for a circular cross-section torus.

AUTHORS NOTE

The present report reproduces the contents of an internal note issued in 1977 whose primary purpose was to understand and combine existing results in the literature. At that time we felt it inappropriate to publish but the note contained material that has proved to be of considerable value to others over the years. A publication in the present format would therefore appear to serve a useful purpose.

I. INTRODUCTION

The effect on the stability of the ideal internal kink mode, i.e. the $m = 1$, $n = 1$ mode, of toroidicity and shaping of the plasma cross-section in a tokamak has been analysed by a number of authors [1-7] with conclusions which are not always in complete agreement.

In ref. [1] Laval examined the effect of elliptic shaping in a particular straight equilibrium, found it to be destabilising and concluded that internal kink modes with growth rates comparable to free boundary kink modes should be possible. Sykes and Wesson [2], comparing circular and elliptic cross-section equilibria (different from that in ref. [1]) found a negligible difference in internal kink growth rates, but found a toroidal stabilising effect [3]. This was confirmed analytically by Bussac et al. [4] who obtained detailed stability results for various current profiles and values of β_p, expressing these results as a limiting β_p (typically ~ 0.1) for stability. Meanwhile Edery et al. [5] found that triangular and quadrupolar shaping of a straight system had a stabilising effect which considerably outweighed the destabilising effect of ellipticity, but repeated the claim of ref. [1] that rapidly growing internal kink modes may occur in elliptic plasmas if q on axis falls below unity. In ref. [6], Young Ping-pao re-examined the question of toroidal stabilisation concluding that, contrary to the results of Bussac et al., toroidal effects destabilise this mode. Finally in ref [7] Berger et al., examined a special toroidal equilibrium numerically and found toroidal stabilisation but also obtained a considerable increase in growth rates for elliptic plasma.

Using the energy principle and following similar lines to papers [4] and [5] we reconsider these points, investigating in particular the effect of plasma shaping on the critical β_p for stability of the internal kink mode in a tokamak torus.

The stabilisation of the internal mode by toroidal effects is only possible when $\beta \leqslant \varepsilon^2$ where $\varepsilon = a/R$ is the inverse aspect ratio, with a and R the plasma minor and major radii. In this case magnetic surfaces are displaced circles where the displacement of their centres $\Delta \sim \varepsilon a$ [8]. If the surfaces are distorted by external windings or a shaped conducting shell, so that they are no longer circular but are modulated with amplitude $S \sim \varepsilon a$, one can expect additional competing contributions to the stability criterion for the internal kink. We consider the equilibrium chracteristics of such shaped cross-sections in Section II, specifying the equilibria by the amplitudes $S^{(n)}$ of the nth harmonic distortion of the plasma surface. In Section III we utilise this information to discuss the stability of the internal kink in terms of the MHD energy principle, investigating, along the lines of refs. [4,5], the coupling of the basic $m = 1$ mode with all the harmonics in the equilibrium.

As far as toroidal effects are concerned we recover the result of ref.[4] rather than those of ref.[6]. The effect of shaping is in agreement with that found in ref. [5], but comparing these effects with toroidal effects, we find that whereas ellipticity is in practice an <u>insignificant</u> destabilising mechanism triangularity can significantly increase the critical value of β_p for stability. The discrepancy between references [1] and [2] is discussed.

II. EQUILIBRIUM PROPERTIES OF SHAPED CROSS-SECTIONS

In this section the equilibrium properties necessary for a study of the stability of the internal kink mode in a large aspect ratio tokamak with $\beta \sim \varepsilon^2$ are obtained, suitably modifying the procedure of ref.[8] to account for shaping. In that paper the equilibrium magnetic surfaces were sought in the form of displaced circles in leading order with elliptic modifications induced by the plasma pressure only appearing in the next order. In the present case however we allow for externally produced elliptic, triangular and indeed all higher harmonic modulations of the plasma surface with amplitude $S^{(n)}(a) \sim \varepsilon a$, where n denotes the harmonic of the modulation, i.e. comparable with the displacement Δ of the basic circular surfaces.

Following ref.[8] we transform from the cylindrical co-ordinate system (R,ϕ,Z) based on the axis of toroidal symmetry to a system (ρ,ω,ϕ) where magnetic surfaces have constant ρ and ω is a poloidal angle. The transformation

$$R = R(\rho,\omega) \qquad Z = Z(\rho,\omega) \tag{1}$$

which conveys all the information about the equilibrium that we require, leads to a metric tensor g_{ij} defined by the element of length

$$(d\ell)^2 = g_{\rho\rho}(d\rho)^2 + 2g_{\rho\omega}\, d\rho d\omega + g_{\omega\omega}(d\omega)^2 + R^2(d\phi)^2. \tag{2}$$

Thus

$$g_{\rho\rho} = \left(\frac{\partial R}{\partial \rho}\right)^2 + \left(\frac{\partial Z}{\partial \rho}\right)^2 \qquad g_{\omega\omega} = \left(\frac{\partial R}{\partial \omega}\right)^2 + \left(\frac{\partial Z}{\partial \omega}\right)^2$$

$$g_{\rho\omega} = \frac{\partial R}{\partial \rho}\frac{\partial R}{\partial \omega} + \frac{\partial Z}{\partial \rho}\frac{\partial Z}{\partial \omega} \tag{3}$$

and the Jacobian $J = \sqrt{\det g_{ij}}$ is given by

$$J = \frac{\partial R}{\partial \omega}\frac{\partial Z}{\partial \rho} - \frac{\partial R}{\partial \rho}\frac{\partial Z}{\partial \omega} \tag{4}$$

In discussing stability we shall require $|\nabla\rho|$, $|\nabla\omega|$ and $\nabla\rho.\nabla\omega$ which are obtained from the inverse of the metric tensor, g^{ij}, through

$$|\nabla\rho|^2 = g^{\rho\rho} \quad , \quad |\nabla\omega|^2 = g^{\omega\omega} \qquad \nabla\rho\cdot\nabla\omega = g^{\rho\omega} \tag{5}$$

The magnetic field $\underset{\sim}{B}$ is represented in the form

$$\underset{\sim}{B} = R_o B_o \,(f(\rho)\ \nabla\phi \times \nabla\rho + g(\rho)\ \nabla\phi) \tag{6}$$

where R_o and B_o are constants, essentially the major axis of the torus and the toroidal magnetic field, introduced to make f and g dimensionless. The safety factor q is then given by

$$q(\rho) = \frac{g(\rho)}{2\pi f(\rho)} \int_o^{2\pi} \frac{d\omega\, J}{R} \tag{7}$$

and the equilibrium equation reads

$$\frac{f}{JR}\left[\frac{\partial}{\partial\rho}\left(\frac{fg_{\omega\omega}}{JR}\right) - \frac{\partial}{\partial\omega}\left(\frac{fg_{\rho\omega}}{JR}\right)\right] + \frac{p'}{R_o^2 B_o^2} + \frac{gg'}{R^2} = 0 \tag{8}$$

where the pressure p, f and g are all functions of ρ only, primes denoting derivatives with respect to ρ.

Employing the standard tokamak ordering $p/B_o^2 \sim \varepsilon^2$, $f \sim \varepsilon$ and $g \sim 1$, equation (8) implies that $g = 1 + \varepsilon^2 g_2$ where the subscript denotes the order in ε. We seek a solution for the equilibrium surfaces in the form

$$R = R_o - \varepsilon\rho\cos\omega - \varepsilon^2\Delta(\rho) + \varepsilon^2 \sum_n S^{(n)}(\rho)\cos(n-1)\omega + \varepsilon^3 P\cos\omega + ..$$

$$Z = \varepsilon\rho\sin\omega + \varepsilon^2 \sum_n S^{(n)}(\rho)\sin(n-1)\omega - \varepsilon^3 P\sin\omega + ..$$

(9)

where we shall find that we only require information on the $\cos\omega$ harmonic in ε^3 order in discussing stability, although other harmonics are necessary to achieve pressure balance in that order. Substitution of these expressions into the equilibrium equation (8) yields equations for Δ the shift of the centres of the almost circular magnetic surfaces, $S^{(n)}$ the imposed shaping of these surfaces and P which corresponds to a relabelling of the surfaces. Thus with the aid of expressions (3) and (4) we obtain from the coefficients of different harmonics of ω in each order:-

$$\frac{p_2'}{B_o^2} + g_2' + \frac{f_1}{\rho}(\rho f_1)' = 0 \tag{10}$$

$$\Delta'' + \left(\frac{2(\rho f_1)'}{\rho f_1} - \frac{1}{\rho}\right)\Delta' - \frac{2(\rho f_1)'}{R_o\, f_1} - \frac{1}{R_o} - \frac{2\rho}{R_o}\frac{g_2'}{f_1{}^2} = 0 \tag{11}$$

$$S^{(n)''} + \left(\frac{2(\rho f_1)'}{\rho f_1} - \frac{1}{\rho}\right) S^{(n)'} - (n^2 - 1)\frac{S^{(n)}}{\rho^2} = 0 \tag{12}$$

$$P'' + \left(\frac{2(\rho f_1)'}{\rho f_1} - \frac{1}{\rho}\right)P' + \frac{1}{\rho^2} P + \frac{1}{f_1{}^2}\Big[\Big(\frac{p_4}{B_0^2} + g_4\Big)' + g_2 g_2' + \frac{1}{\rho^2}(\rho^2 f_1 f_3)' +$$

$$+ \frac{3\rho^2 g_2'}{2R_o^2} + \frac{2\Delta g_2'}{R_o}\Big] + \frac{3\rho}{2R_o^2} + \frac{\Delta'}{2R_o} - \frac{r\Delta''}{R_o} - \frac{\Delta'^2}{2\rho} + \frac{3\Delta'\Delta''}{2} + \frac{n}{\rho}\Big[S^{(n)'} + (n-1)\frac{S^{(n)}}{\rho}\Big]^2$$

$$+ \frac{(\rho f_1)'}{2\rho f_1}\Big[\frac{3\rho^2}{R_o^2} - \frac{4\rho\Delta'}{R_o} + 3\Delta'^2 + \frac{4\Delta}{R_o} - 3(S^{(n)'})^2 + (n-1)^2\frac{S^{(n)2}}{\rho^2}\Big] = 0 \tag{13}$$

Note that equation (12), unlike the others which have appeared previously is not driven by the plasma pressure and requires an external origin. Thus it describes how shaping at the plasma surface propagates into the plasma. Clearly if q is constant, i.e. a flat current, a case considered previously [9], $f_1 \sim \rho$ and we obtain

$$S^{(n)}(\rho) = \left(\frac{\rho}{a}\right)^{n-1} S^{(n)}(a) \tag{14}$$

However, if the current is not flat, $q' > 0$ and $S^{(n)}$ decreases more rapidly than in equation (14). In Fig.1 we show some examples for $n = 2$, the ellipse, $n = 3$, the triangle, and $n = 4$, the quadrupole, for q profiles

$$q(\rho) = \frac{\rho^2\left[1 - \left(1 - \frac{\rho_o^2}{a^2}\right)^{\nu + 1}\right]}{\rho_o^2\left[1 - \left(1 - \frac{\rho^2}{a^2}\right)^{\nu + 1}\right]} \tag{15}$$

with $\nu = 1$, 2 and 4, together with the flat current case for comparison. These profiles result from $j_\phi \sim j_o\left(1 - \frac{\rho^2}{a^2}\right)^\nu$ and correspond to $q(\rho_o) = 1$; of course equation (12) is independent of the normalisation of q. It is clear from Fig. 1 that the result (14) is qualitatively correct, only being modified by factors of order unity.

In discussing stability it is convenient to use the coordinates (r,θ) rather than (ρ, ω) where [4]

$$r^2 = 2R_o\int_o^\rho d\rho\int\frac{d\omega J}{R} \qquad \Theta = 2\pi\,\frac{\int^\omega \frac{d\omega J}{R}}{\int\frac{d\omega J}{R}} \tag{16}$$

It follows on using expression (4) and (9) that we can identify r with ρ by choosing

$$P = \frac{1}{8}\frac{r^3}{R_o^2} + \frac{r\Delta}{2R_o} - \frac{(n-1)}{2}\frac{S^{(n)2}}{r} \qquad (17)$$

and that

$$\Theta = \omega + (\Delta' + \frac{r}{R_o}) \sin\omega - \frac{1}{n}(S^{(n)'} - (n-1)\frac{S^{(n)}}{r}) \sin n\omega + \ldots \qquad (18)$$

From equations (5) and (9) we obtain, with the aid of results (17) and (18),

$$\left|\nabla r\right|^2 = 1 - 2\Delta'\cos\theta + 2S^{(n)'}\cos n\theta + \frac{\Delta'^2}{2} + \frac{3}{4}\frac{r^2}{R_o^2} + \frac{\Delta}{R_o} + \frac{1}{2}(S^{(n)'})^2$$

$$+ \frac{S^{(n)2}}{2r^2}(n^2-1) + \ldots$$

$$\left|\nabla\theta\right|^2 = \frac{1}{r^2}\{1 + 2\left(\Delta' + \frac{r}{R_o}\right)\cos\theta - 2S^{(n)'}\cos n\theta + \frac{3}{2}(\Delta' + \frac{r}{R})^2 + \frac{1}{4}\frac{r^2}{R^2} + \frac{\Delta}{R} + \frac{\Delta'^2}{2}$$

$$+ \frac{1}{2}(r\Delta'' + 2\Delta' + \frac{r}{R})(r\Delta'' + \frac{r}{R}) + \frac{1}{2}(rS^{(n)''})^2 + \frac{1}{2}(rS^{(n)''})(S^{(n)'} + (n^2-1)\frac{S^{(n)}}{r})$$

$$+ \frac{(3n^2+1)}{2n^2}S^{(n)'2} + \frac{(n^2-1)}{n^2}\frac{S^{(n)}S^{(n)'}}{r} - \frac{(n^2-1)}{2n^2}\frac{S^{(n)2}}{r^2} + \ldots\}$$

$$\nabla r.\nabla\Theta = \frac{1}{r}\{(r\Delta'' + \Delta' + \frac{r}{R_o}) \sin\Theta - \frac{1}{n}(rS^{(n)''} + S^{(n)'} + (n^2-1)\frac{S^{(n)}}{r}) \sin n\Theta + \ldots\}$$

$$R^2 = R_o^2\{1 - \frac{2r}{R_o}\cos\Theta - \frac{2\Delta}{R_o} - \frac{r\Delta'}{R_o} - \frac{1}{2}\frac{r^2}{R_o^2} + \ldots\} \qquad (19)$$

where in second order we do not quote harmonic contributions since these, we shall see, do not contribute in the stability theory of the internal kink. Also we note that equation (7) relates f_1 to q which we prefer to use:

$$\frac{1}{q} = \frac{R_o f_1}{r}\left[1 - g_2 + \frac{f_3}{f_1} + \ldots.\right] \tag{20}$$

III STABILITY OF THE INTERNAL KINK

The mhd energy integral is $\delta W_{mhd} = \delta W + \delta \bar{W}$ where

$$\delta W = \frac{1}{2} \int d\tau \left\{ \left| \delta \underset{\sim}{B} \right|^2 - \underset{\sim}{J} . \delta \underset{\sim}{B}^* \times \underset{\sim}{\xi} + \underset{\sim}{\xi}^* . \nabla p \nabla . \underset{\sim}{\xi} \right\}$$

$$\delta \bar{W} = \frac{1}{2} \int d\tau \gamma p \left| \nabla . \underset{\sim}{\xi} \right|^2 \tag{21}$$

Here $\underset{\sim}{J}$ is the equilibrium current, γ the ratio of specific heats and $d\tau$ the volume element, while the perturbed magnetic field $\delta \underset{\sim}{B}$ is related to the displacement $\underset{\sim}{\xi}$ by $\delta \underset{\sim}{B} = \nabla \times (\underset{\sim}{\xi} \times \underset{\sim}{B})$; using the variables r and Θ, ξ has the form $\underset{\sim}{\xi} = \underset{\sim}{\xi}(r, \Theta) e^{-i\phi}$ for the $n = 1$ mode. Since δW is independent of $\underset{\sim}{\xi} . \underset{\sim}{B}$ it is convenient to let $\underset{\sim}{\xi} = \underset{\sim}{\xi}_p + \eta \underset{\sim}{B}$ with $\eta = \frac{\xi_\phi}{B}$ so that $\xi_{p\phi} = 0$; η can then be chosen to make $\delta \bar{W} = 0$ and it remains to minimise δW with respect to $\underset{\sim}{\xi}_p$. Introducing $\xi_r = g\, \underset{\sim}{\xi}_p . \nabla r$ and $\xi_\theta = g\, r\, \underset{\sim}{\xi}_p . \nabla \theta$, δW can be written [4].

$$\delta W = \frac{1}{2} \int r dr d\theta d\phi \; R_o B_o^2 \left\{ \left| \frac{1}{r} \frac{\partial}{\partial r}(r\xi_r) + \frac{1}{r} \frac{\partial \xi_\theta}{\partial \theta} \right|^2 \right.$$

$$+ \frac{1}{R_o^2} \left| r\nabla\theta \left(\frac{\partial \xi_r}{\partial \phi} + \frac{1}{q} \frac{\partial \xi_r}{\partial \theta}\right) + \nabla r \left(\frac{\partial}{\partial r}\left(\frac{r\xi_r}{q}\right) - \frac{\partial \xi_\theta}{\partial \phi}\right) \right|^2$$

$$+ \frac{1}{fR_o} g' \left[\left|\xi_r\right|^2 r \left(\frac{1}{q}\right)' + \xi_\theta \left(\frac{\partial \xi_r^*}{\partial \phi} + \frac{1}{q} \frac{\partial \xi_r^*}{\partial \Theta}\right) + \xi_\Theta^* \left(\frac{\partial \xi_r}{\partial \phi} + \frac{1}{q} \frac{\partial \xi_r}{\partial \Theta}\right) \right]$$

$$\left. + \frac{1}{fR_o^3 B_o^2} p' \left[\frac{1}{r} \frac{\partial}{\partial r} \left(\frac{R^2 \left|\xi_r\right|^2 r^2}{qg} \right) + \frac{R^2}{g} \left(\xi_\Theta \frac{\partial \xi_r^*}{\partial \phi} + \xi_\Theta^* \frac{\partial \xi_r}{\partial \phi} \right) \right] \right\} \qquad (22)$$

The procedure is to insert the large aspect ratio expansion of the equilibrium given in the previous section together with an expansion

$$\underset{\sim}{\xi} = \underset{\sim}{\xi}_o + \varepsilon \underset{\sim}{\xi}_1 + \varepsilon^2 \underset{\sim}{\xi}_2 + \ldots\ldots$$

where

$$\underset{\sim}{\xi}_o = \underset{\sim}{\xi}_o(r) e^{-i\Theta}$$

into this form of the energy integral and then to minimise it order by order. Clearly the leading order term δW_o is minimised to zero by choosing

$$\frac{\partial}{\partial r} (r \, \xi_{r_o}) - \frac{i}{r} \xi_{\Theta o} = 0 \qquad (23)$$

and similarly the contribution of $\underset{\sim}{\xi}_1$ to δW_2 is minimised when

$$\frac{\partial}{\partial r}(r\ \xi_{r_1}) + \frac{1}{r}\frac{\partial}{\partial\Theta}\ \xi_{\Theta_1} = 0 \qquad (24)$$

As is well known the remainder of δW_2 is minimised to zero by choosing ξ_o = constant for $r < r_o$ where $q(r_o) = 1$ and $\xi_o = 0$ for $r > r$; here we write $\xi_{ro} = \xi_o$. The stability is then determined in δW_4 when the toroidal and shaping effects enter.

The contributions to δW_4 fall into three main categories:-

1. The expansions of p, f and g and of R^2, $|\nabla r|^2$, $|\nabla\Theta|^2$ and $\nabla r.\nabla\Theta$ given in equation (19) couple in second significant order with $|\xi_o|^2$ terms. It is now clear why we ignored harmonic contributions to the equilibrium quantities in second order and further that the contributions from different $S^{(n)}$ are decoupled.

2. Contributions from $\underset{\sim}{\xi}_2$: a quadratic term from the first term in δW and linear terms arising from the zero order equilibrium. These terms are thus entirely cylindrical in character with no toroidal or shaping contributions.

3. Quadratic terms in $\underset{\sim}{\xi}_1$ (which satisfies equation (24)) arising from the zero order equilibrium, and linear terms in ξ_1 arising from the coupling with the first order harmonics in the equilibrium. Writing $\xi_{r_1} = \sum_m \xi_1^{(m)} e^{-im\theta}$ we find that the harmonic $S^{(n)}$ is coupled to $m = \pm n + 1$, so that the effects of different $S^{(n)}$ are

again decoupled.

Whereas the contributions from group 1 merely need collecting, those in group 2 require an algebraic minimisation which is achieved by setting [4]

$$\xi_2 = i \frac{r^2}{R_o^2} \left(1 + \frac{1}{q} + \frac{R_o^2 g_2'}{r}\right) \xi_o \underset{\sim}{e}_\Theta \tag{25}$$

This leads to further term in δW_4 of the form of those in group 1.

The terms of group 3 for $n \neq 1$ lead to a contribution

$$\tilde{\delta W}_4 = 2\pi^2 R_o B_o^2 \sum_n \sum_{m=1\pm n} \int_o^a r dr \left\{ \alpha^{(m)2} \left(\left| \frac{r d\xi_1^{(m)}}{dr} \right|^2 + (m^2 - 1) \left| \xi_1^{(m)} \right|^2 \right) \right.$$

$$\left. + \left(D^{(m)} r \xi_1^{(m)} \frac{d\xi_o^*}{dr} + A^{(m)} \xi_1^{(m)} \xi_0^* + \text{c.c.} \right) \right\} \tag{26}$$

where

$$\alpha^{(m)} = \left(\frac{1}{q} - \frac{1}{m}\right)$$

$$D^{(m)} = \alpha^{(m)} \left[S^{(n)'} \left(\frac{m}{m-1} \alpha^{(m)} q \left(\frac{r}{q}\right)' - 1 \right) + m \frac{(m-2)}{m-1} \left(1 - \frac{1}{q}\right) \frac{S^{(n)}}{r} \right]$$

$$A^{(m)} = - S^{(n)'} \frac{q}{m} \left(\frac{r}{q}\right)' + (m-2) \frac{S^{(n)}}{r^2} \left[\frac{m}{q} \left(\frac{r}{q}\right)' - \frac{m+1}{q} + 1 \right] \tag{27}$$

The case $n = 1$, treated in ref.[4], is special in that we get a contribution of the form (26) for $m = 2$ with

$$D^{(2)} = \alpha^{(2)}\left[\Delta'' \, r\alpha^{(2)} + \Delta'\left(\frac{1}{2} + \frac{1}{q}\right) + \frac{r}{2R_o}\right]$$

$$A^{(2)} = r\,\Delta''\left(\frac{1}{4} - \frac{1}{q^2}\right) - \Delta'\left(\frac{2}{q}\left(\frac{r}{q}\right)' + \frac{3}{q}\left(\frac{1}{q} - 1\right) + \frac{1}{4}\right)$$

$$+ \frac{r}{R_o}\left[\frac{1}{q}\left(\frac{r}{q}\right)' + \frac{1}{q^2} - \frac{3}{2q} + \frac{3}{4}\right] \qquad (28)$$

but for $m = 0$ equation (24) implies $\xi_1^{(o)} = 0$ and we obtain a quadratic form in $\xi_{\theta_1}^{(o)}$ which is minimised by

$$\xi_{\theta_1}^{(o)} = -\frac{i}{2}\left(r\Delta'' + 3\Delta' + \frac{r}{R}(1 - 2q)\right)\xi_o \qquad (29)$$

thus leading to more terms like those in groups 1 and 2.

Minimisation of expression (26) results in the driven Euler equations

$$\frac{d}{dr}\left(r^3\alpha^{(m)2}\frac{d\xi_1^{(m)}}{dr} + D^{(m)}r^2\xi_o\right) - (m^2 - 1)\alpha^{(m)2} r\xi_1^{(m)} - A^{(m)} r\xi_o = 0 \qquad (30)$$

which have solutions $\xi_1^{(m)} = \xi^{(m)} + \eta^{(m)}$ where $\xi^{(m)}$ satisfies the homogeneous form of equation (30) and

$$\eta^{(m)} = \xi_o \frac{m}{2n} (S^{(n)'} + (2-m) S^{(n)}) \qquad m = 1 \pm n, \quad n \neq 1$$

$$\eta^{(2)} = - \xi_o (\Delta' + \frac{r}{2 R_o}) \qquad m = 2 \qquad (31)$$

The discontinuity in $\eta^{(m)}$ at $q = 1$ requires a compensating discontinuity in $\xi^{(m)}$ since δB_r must be continuous.

Integrating expression (24) for $\delta\widetilde{W}_4$ by parts in each of the two regions $r > r_o$ and $r < r_o$ separately, with the aid of the Euler equation (30), and then introducing $\xi_1^{(m)} = \xi^{(m)} + \eta^{(m)}$ we obtain

$$\begin{aligned}\delta\widetilde{W}_4 = 2\pi^2 R_o B_o^2 \sum_n \sum_m \Big\{ & - \int_o^{r_o} r dr\, \alpha^{(m)2} \left(\left| r \frac{d\eta^{(m)}}{dr} \right|^2 + (m^2 - 1) \left| \eta^{(m)} \right|^2 \right) \\ & + \eta_o^{(m)} r_o^2 \left(2D_o^{(m)} + 2r_o \alpha_o^{(m)2} \frac{d\eta^{(m)}}{dr} \bigg|_{r_{o-}} - c^{(m)} \alpha_o^{(m)2} \eta_o^{(m)} \right) \\ & + 2 r_o^2 \xi^{(m)}(r_{o-}) \left(D_o^{(m)} - \alpha_o^{(m)2} \eta_o^{(m)} c^{(m)} + r_o \alpha_o^{(m)2} \frac{d\eta^{(m)}}{dr} \bigg|_{r_{o-}} \right) \qquad (32) \\ & + \alpha_o^{(m)2} r_o^2 (b^{(m)} - c^{(m)}) \xi^{(m)2}(r_{o-}) \Big\}\end{aligned}$$

where $b^{(m)} = \frac{r}{\xi^{(m)}} \frac{d\xi^{(m)}}{dr} \Big|_{r_{o-}}$ and $c^{(m)} = \frac{r}{\xi^{(m)}} \frac{d\xi^{(m)}}{dr} \Big|_{r_{o+}}$, the suffix zero refers to values at $r = r_o$ and we have used continuity of $\xi_1^{(m)}$ to relate $\xi^{(m)}(r_{o-})$ to $\xi^{(m)}(r_{o+})$. Finally, we minimise with respect to the amplitudes $\xi^{(m)}(r_{o-})$ as in refs.[4,5]. Summing all contributions to δW_4 from groups (1) to (3) we obtain, with the aid of the equilibrium equations (10)-(13) and some integrations by parts in r, the result

$$\delta W_4 = 2\pi^2 R_o B_o^2 \left|\xi_o\right|^2 \frac{r_o^4}{R_o^4} \left(\delta W^{(T)} + \sum_{n>1} \delta W^{(n)}\right) \tag{33}$$

where the toroidal contribution δW_T is given by ref.[4]:

$$\delta W^{(T)} = \frac{8s(b^{(2)}-c^{(2)})+\frac{9}{4}(b^{(2)}-1)(1-c^{(2)})-6(b^{(2)}-1)(c^{(2)}+3)(\beta_p+s)-4(c^{(2)}+3)(b^{(2)}+3)(\beta_p+s)^2}{16(b^{(2)}-c^{(2)})} \tag{34}$$

with

$$s = \int_o^{r_o} \frac{dr}{r_o}\left(\frac{r}{r_o}\right)^3\left(\frac{1}{q^2}-1\right)$$

$$\beta_p = \frac{2}{B_p^2(r_o)} \int_o^{r_o} dr\left(-\frac{dp}{dr}\right)\left(\frac{r}{r_o}\right)^2$$

and the higher harmonic contributions by ref.[5]:

$$\delta W^{(n)} = -\frac{1}{4} \sum_{m=1\pm n} \frac{R_o^2}{r_o^2} \frac{\left(s_o^{(n)\prime} - (m-2)\frac{s_o^{(n)}}{r_o}\right)^2 (m+1+b^{(m)})(m+1+c^{(m)})}{(b^{(m)}-c^{(m)})} \tag{35}$$

The quantities $b^{(m)}$ and $c^{(m)}$ must be obtained from a solution of the homogeneous form of equation (30). Thus $b^{(m)}$ is calculated from the solution within $r<r_o$ for which $\xi^{(m)}(o) = 0$, while $c^{(m)}$ is calculated from the solution in $r > r_o$ which has boundary condition $\frac{d\xi^{(m)}}{dr} = 0$ at the singular point $q(r) = m$, if such a point exists in the plasma, or $\xi^{(m)}=0$ at the plasma surface if not. Thus in general numerical computation is required for specified q profiles. We consider the profiles (15) with $\nu = 1$, 2 and 4 as an example and calculate the $b^{(m)}$ and $c^{(m)}$ for the cases $n = 1$ (toroidal effects), $n = 2$ (elliptical shaping), $n = 3$ (triangular shaping) and $n = 4$ (quadrupole shaping). Combining these results with corresponding solutions of the equilibrium equation(12) we calculate $\delta W^{(n)}$. Table 1 shows results for $n= 2$, 3 and 4 as r_o varies, where each $S^{(n)}$ is normalised so that $S^{(n)}(a) = a^2/R$. The toroidal contribution can be expressed as $\delta W^{(T)}= \gamma_o + \gamma_1\beta_p + \gamma_2\beta_p^2$ and we tabulate γ_o, γ_1 and γ_2 in Table 2. In all cases wall stabilisation is evident as $r_o \to a$ when there are no surfaces with $q = n + 1$ within the plasma

For the situation in which $r_o \ll a$ it is possible to obtain some analytic results [4]. We take

$$q = 1 - \Delta q \left(1 - \left(\frac{r}{r_o}\right)^{\lambda}\right) \tag{36}$$

as a representation of q near the axis so that $q(o) = 1 - \Delta q$ with $\Delta q \ll 1$. In this case we can solve for $\xi^{(m)}$ as an expansion in Δq as long as $r \ll a$, the leading order solution behaving as r^{m-1} or r^{-m-1}. In the region $r < r_o$ we must take the solution r^{m-1} and calculate the Δq correction from which $b^{(m)}$ can be calculated to sufficient accuracy. In calculating $\xi^{(m)}$ for $r > r_o$ we have some combination of the two solutions r^{m-1} and r^{-m-1} determined by the boundary conditions at the point $q(r)=m$ or $r = a$, both of which are far from r_o. However the influence of the solution r^{m-1} at $r= r_o$ will be negligible in comparison with that of the solution r^{-m-1} and calculating the Δq correction to this latter solution we obtain $c^{(m)}$. Relating $s^{(n)'}(r_o)$ to $s^{(n)}(r_o)$ through the equilibrium equation (12), we obtain, provided $\lambda > 4$,

$$\delta W_4=\lambda\Delta q\left\{\frac{13}{16(\lambda+4)} - \frac{3\beta_p^2}{4-\lambda} - \frac{12\lambda^2(\Delta q)^2\, s^{(2)^2}(r_o)}{(4+\lambda)^2\,(6-\lambda)}\frac{R_o^2}{r_o^4}\right.$$

$$\left.+\sum_{n>2}\frac{R_o^2}{r_o^4}\,\frac{n(n-1)(n-2)}{(n-1+\frac{\lambda}{2})}\,s^{(n)}(r_o)^2\right\} \qquad (37)$$

where we see that the elliptic contribution $n=2$ is destabilising and very small. This is because the $m=1-n$ contribution, which is in general a strong stabilising term and is responsible for the sum with $n>2$, has $b=0$ exactly in the elliptic $n=2$ case, leaving only the destabilising $m=n+1$ contribution which is of order $(\Delta q)^3$ for all n since

$$c^{(n+1)} = -(n+2) + 0(\Delta q) \quad \text{and} \quad s^{(n)'} - (n-1)\frac{s^{(n)}}{r} \sim 0(\Delta q).$$

Note that in expression (37) we expect $S^{(n)}(r_o) \sim (r_o/a)^{n-1}S^{(n)}(a)$ so that $(R_o/r_o)\ S^{(n)}(r_o) \sim (r_o/a)^{n-2}(R_o/a)\ S^{(n)}(a)$.

IV DISCUSSION

It is clear from the numerical results and the analytic result (25) that, although the elliptic term n=2 is destabilising, all the higher harmonics are stabilising. Moreover, the elliptic term is numerically much smaller than the toroidal terms when $S^{(2)}(a) \sim a^2/R$ and indeed also much smaller than the triangular and quadrupole terms when $S^{(3)}(a) \sim S^{(4)}(a) \sim a^2/R$ even though $S^{(n)}(r_o) \sim (r_o/a)^{n-1}S^{(n)}(a)$.

The weakness of the elliptic terms is in accord with numerical results of Sykes and Wesson [2]. They treated a straight elliptic cylinder so that in comparing with their results, we may take the cylindrical limit of δW_4

$$\delta W_4 = 2\pi\, R_o B_o^2 \left|\xi_o\right|^2 \left(\frac{r_o}{R_o}\right)^4 (\delta W^{(c)} + \delta W^{(2)})$$

where the elliptic contributions $\delta W^{(2)}$, are given by equation (23) with n=2, and the cylindrical contribution $\delta W^{(c)}$ is given by [4].

$$\delta W^{(c)} = - \beta_p(r_o) - \int_0^{r_o} \frac{dr\ r^3}{r_o^4} \left(\frac{1}{q^2} + \frac{2}{q} - 3\right)$$

which is cancelled by toroidal effects in a real torus [4]. The numerical results were for $\beta_p \sim 1$ and it was found that with $b/a = 2$, corresponding to $S^{(2)} = 1/3$, only 20% changes in growth rate were obtained. The weakness of elliptic effects helps to reconcile this numerical result with analytic work of Laval[1] who treated finite ellipticity, i.e $S^{(2)}(a) \sim a$, and found the possibility of instability in δW_2. However the weakness of the numerical coefficients means that the δW_4 contributions can dominate this δW_2 contribution in practice. In this context we should comment on the analytic result concerning a current profile which was constant for $r<r_c$ and zero for $r_c < r < a$ in ref.[1]. The growth rate was evaluated in terms of the elliptic deformation of the <u>resonant surface</u> r_o; for the most unstable case $r^*_o = a/\sqrt{3}$ and was ostensibly comparable with external kink growth rates when $S^{(2)}(r^*_o) \simeq a$. However for such a current profile

$$S^{(2)}(r) \sim (r/r_c)^3 + (r_c/r)$$

with the consequence that $S^{(2)}(r_o^*)/S^{(2)}(a) \sim 1/3\sqrt{3}$. Thus even if $S^{(2)}(a) \simeq a$, the growth rate of this internal kink would be much less than that of external kink modes.

The numerical results of Berger et al [7] indicated a substantial effect of ellipticity on the internal mode. However the equilibrium they studied corresponds, in a large aspect ratio expansion, to a flat current, shear of order ε^2 being generated solely by toroidal effects [8]. Such an equilibrium could be studied analytically for finite ellipticity but the stability result would arise in δW_6 because of the vanishing of

$s^{(2)'} - s^{(2)}/r$ for a flat current. Nevertheless this corresponds to a growth rate comparable to those arising here from δW_4 [4,6].

The stabilisation possible from the higher harmonics can be viewed as an increase in the critical β_p. Thus ignoring the small elliptic contribution and retaining only the triangular term from the higher harmonics we find from equation (25), taking $\lambda=2$, that the stability criterion becomes

$$\beta_p^2 < 0.09 + 1.33 \left(\frac{R_o}{r_o^2} s^{(3)}(r_o)\right)^2 .$$

If $s^{(3)}(r_o) \sim (r_o/a)^2 s^{(3)}(a)$, as for a flat current, we have approximately

$$\beta_p^2 < 0.09 + 1.33 \left(\frac{R_o}{a} \frac{s^{(3)}(a)}{a}\right)^2$$

and thus with $s^{(3)}(a) > a^2/R_o$ we have $\beta_{p\ crit} < 1$, as opposed to $\beta_{p\ crit} = 0.3$ in the circular case. This result, based on $r_o \to 0$ is borne out by the more complete numerical results. Following ref.[4] we can use expression (21) to define $\beta_{p\ crit}$ such that $\delta W_4 = 0$ and in Figs. 2-3 we consider the effect of $s^{(3)}(a)\ R_o/a^2$ and $s^{(4)}(a)\ R_o/a^2$ on this value as a function of r_o for various values of ν. These results are summarised in Fig.4 where we plot $\beta^*_{p\ crit}$ the critical value of β_p which ensures stability <u>everywhere</u> to the internal kink. Quadrupolarity is seen to be less important than comparable triangularity because of its much weaker penetration into the plasma. After an initial increase in $S^{(4)}(a)$ the most unstable position for r_o is near the axis where the quadrupolarity is negligible so that $\beta_{p\ crit}$ cannot increase further. Provided the current is not too peaked (i.e. $\nu = 1,2$) we can ensure stability against the internal kink for $\beta_p \sim 1$ with values $1 \lesssim S^{(3)}(a)\ R_o/a^2 \lesssim 2$ while in the peaked current case $\nu = 4$ the extremely low value $\beta_p = 0.02$ for the circular case can be increased to $\beta_p \sim 0.5$. It is clear that triangularity in particular shows a considerable stabilising influence. Note that $q(o)$ falls below 1/2 when $r_o > 0.8$ for $\nu = 2$ and when $r_o > 0.6$ for $\nu = 4$ so that these regions would correspond to an unstable $m = 1, n = 2$ mode.

The ordering scheme we have used with $S^{(n)}(a) \sim a^2/R$ chosen to provide maximal ordering, i.e. showing shaping competing with toroidicity, is indeed in accordance with practical situations; e.g. it is expected that elliptical plasmas will have $b/a \lesssim 2$ which leads to $S^{(2)} \lesssim a/3$ consistent with $a/R_o \leqslant 1/3$.

In practical situations the $m = 1$ resistive mode may be more important than the ideal mode considered here [10], but the characteristics of the resistive mode are likely to be influenced by the properties of δW and so these results will also be relevant to a study of resistive modes.

CONCLUSIONS

Using the ideal mhd energy principle we have investigated the influence of plasma shaping in competition with toroidal effects on the stability of the internal kink (m = 1) in a tokamak. It is found that ellipticity has an extremely weak destabilising effect, but that all higher harmonics are stabilising. In particular, the triangular shaping effectively propagates from the plasma surface to the interior and provides strong stabilisation leading to the prospect of stability for $\beta_p < \beta_{p\ crit}$ with $\beta_{p\ crit} \sim 1$. In such a situation the ideal internal kink modes will remain stable as q_o, the value of the safety factor on aixs, is depressed below unity until $q_o = 1/2$ when an $m = 1$, $n = 2$ mode would become unstable. In the absence of resistive effects one can then envisage a mode of tokamak operation with $1 > q_o > 1/2$ and $q_a \gtrsim 1$, where q_a is only limited by the Kruskal-Shafranov value.

For the circular cross section tokamak our results coincide with those of Bussac et al [4] rather than those of Young-Ping Pao [6].

ACKNOWLEDGEMENTS

We are pleased to acknowledge help with numerical computations from A. Sykes.

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TABLE I

$\delta W^{(2)}$, $\delta W^{(3)}$ and $\delta W^{(4)}$ as r_o varies for ν = 1, 2 and 4

ν	r_o	$\delta W^{(2)}$	$\delta W^{(3)}$	$\delta W^{(4)}$
1	0.1	$-5.25\ 10^{-10}$	$9.82\ 10^{-7}$	$2.59\ 10^{-8}$
	0.2	$-1.46\ 10^{-7}$	$6.53\ 10^{-5}$	$6.88\ 10^{-6}$
	0.3	$-4.36\ 10^{-6}$	$7.90\ 10^{-4}$	$1.88\ 10^{-4}$
	0.4	$-5.09\ 10^{-5}$	$4.84\ 10^{-3}$	$2.06\ 10^{-3}$
	0.5	$-3.45\ 10^{-4}$	$2.07\ 10^{-2}$	$1.39\ 10^{-2}$
	0.6	$-1.52\ 10^{-3}$	$7.18\ 10^{-2}$	$6.99\ 10^{-2}$
	0.7	$-3.96\ 10^{-3}$	$2.21\ 10^{-1}$	$2.94\ 10^{-1}$
	0.8	$3.02\ 10^{-3}$	$6.56\ 10^{-1}$	1.13
	0.9	$1.22\ 10^{-1}$	2.13	4.47
2	0.1	$-2.71\ 10^{-9}$	$1.10\ 10^{-6}$	$2.71\ 10^{-8}$
	0.2	$-7.81\ 10^{-7}$	$7.43\ 10^{-5}$	$7.33\ 10^{-6}$
	0.3	$-2.42\ 10^{-5}$	$9.29\ 10^{-4}$	$2.07\ 10^{-4}$
	0.4	$-2.93\ 10^{-4}$	$5.95\ 10^{-3}$	$2.37\ 10^{-3}$
	0.5	$-2.05\ 10^{-3}$	$2.68\ 10^{-2}$	$1.70\ 10^{-2}$
	0.6	$-9.38\ 10^{-3}$	$0.99\ 10^{-1}$	$9.13\ 10^{-2}$
	0.7	$-2.80\ 10^{-2}$	$3.29\ 10^{-1}$	$4.16\ 10^{-1}$
	0.8	$-2.34\ 10^{-2}$	1.08	1.75
	0.9	$3.56\ 10^{-1}$	4.01	7.48
4	0.1	$-1.07\ 10^{-8}$	$9.31\ 10^{-7}$	$2.17\ 10^{-8}$
	0.2	$-3.21\ 10^{-6}$	$6.55\ 10^{-5}$	$6.11\ 10^{-6}$
	0.3	$-1.01\ 10^{-4}$	$8.68\ 10^{-4}$	$1.84\ 10^{-4}$
	0.4	$-1.22\ 10^{-3}$	$5.98\ 10^{-3}$	$2.29\ 10^{-3}$
	0.5	$-8.50\ 10^{-3}$	$2.93\ 10^{-2}$	$1.81\ 10^{-2}$
	0.6	$-3.73\ 10^{-2}$	$1.18\ 10^{-1}$	$1.08\ 10^{-1}$
	0.7	$-1.05\ 10^{-1}$	$4.41\ 10^{-1}$	$5.38\ 10^{-1}$
	0.8	$-1.04\ 10^{-1}$	1.65	2.42
	0.9	$7.72\ 10^{-1}$	6.34	9.78

TABLE II

The coefficients λ_0, λ_1 and λ_2 where $\delta W^T = \lambda_0 + \lambda_1\beta_p + \lambda_2\beta_p^2$ as r_o varies for ν = 1, 2 and 4

ν	r_o	λ_o	λ_1	λ_2
1	0.1	$1.36\ 10^{-7}$	$-6.57\ 10^{-9}$	$-1.57\ 10^{-6}$
	0.2	$8.85\ 10^{-6}$	$-2.14\ 10^{-6}$	$-1.25\ 10^{-4}$
	0.3	$1.03\ 10^{-4}$	$-6.15\ 10^{-5}$	$-1.56\ 10^{-3}$
	0.4	$5.95\ 10^{-4}$	$-6.22\ 10^{-4}$	$-8.51\ 10^{-3}$
	0.5	$2.34\ 10^{-3}$	$-3.32\ 10^{-3}$	$-2.76\ 10^{-2}$
	0.6	$7.29\ 10^{-3}$	$-1.09\ 10^{-2}$	$-5.83\ 10^{-2}$
	0.7	$2.00\ 10^{-2}$	$-2.01\ 10^{-2}$	$-7.26\ 10^{-2}$
	0.8	$5.35\ 10^{-2}$	$6.20\ 10^{-3}$	$1.54\ 10^{-2}$
	0.9	$1.56\ 10^{-1}$	$2.28\ 10^{-1}$	$3.90\ 10^{-1}$
2	0.1	$2.72\ 10^{-7}$	$-2.85\ 10^{-8}$	$-3.40\ 10^{-6}$
	0.2	$1.77\ 10^{-5}$	$-9.09\ 10^{-6}$	$-2.66\ 10^{-4}$
	0.3	$2.03\ 10^{-4}$	$-2.62\ 10^{-4}$	$-3.31\ 10^{-3}$
	0.4	$1.12\ 10^{-3}$	$-2.67\ 10^{-3}$	$-1.82\ 10^{-2}$
	0.5	$3.97\ 10^{-3}$	$-1.49\ 10^{-2}$	$-6.18\ 10^{-2}$
	0.6	$1.02\ 10^{-2}$	$-5.64\ 10^{-2}$	$-1.53\ 10^{-1}$
	0.7	$2.70\ 10^{-2}$	$-1.06\ 10^{-1}$	$-1.96\ 10^{-1}$
	0.8	$8.51\ 10^{-2}$	$-6.73\ 10^{-2}$	$-8.89\ 10^{-2}$
	0.9	$3.09\ 10^{-1}$	$3.91\ 10^{-1}$	$3.82\ 10^{-1}$
4	0.1	$5.45\ 10^{-7}$	$-1.22\ 10^{-7}$	$-7.26\ 10^{-6}$
	0.2	$3.49\ 10^{-5}$	$-3.69\ 10^{-5}$	$-5.38\ 10^{-4}$
	0.3	$3.80\ 10^{-4}$	$-9.25\ 10^{-4}$	$-5.82\ 10^{-3}$
	0.4	$1.85\ 10^{-3}$	$-7.80\ 10^{-3}$	$-2.65\ 10^{-2}$
	0.5	$5.27\ 10^{-3}$	$-3.57\ 10^{-2}$	$-7.48\ 10^{-2}$
	0.6	$9.07\ 10^{-3}$	$-1.13\ 10^{-1}$	$-1.59\ 10^{-1}$
	0.7	$7.23\ 10^{-3}$	$-2.70\ 10^{-1}$	$-2.73\ 10^{-1}$
	0.8	$1.02\ 10^{-1}$	$-1.87\ 10^{-1}$	$-1.45\ 10^{-1}$
	0.9	$5.57\ 10^{-1}$	$6.21\ 10^{-1}$	$3.94\ 10^{-1}$

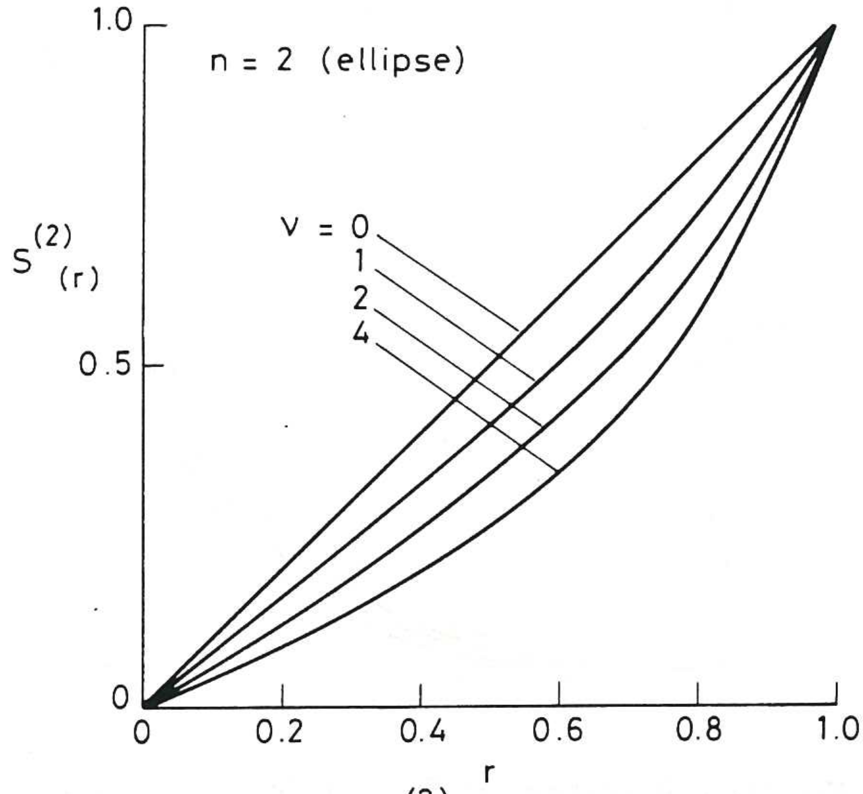

Fig. 1 (a). Profiles of $S^{(2)}(r)$ for $\nu = 0, 1, 2, 4$

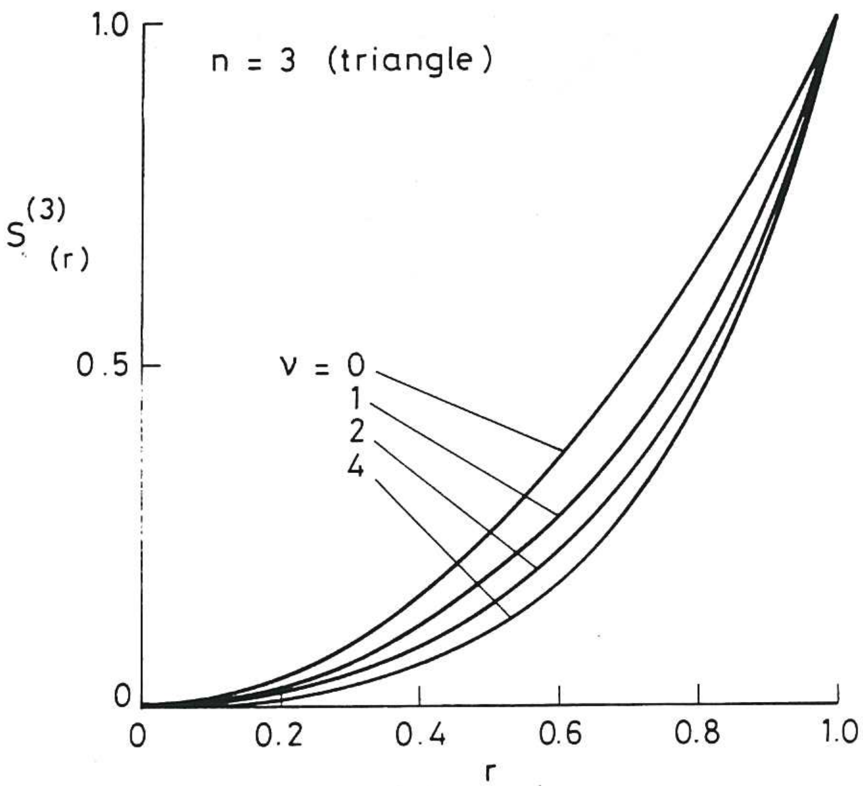

Fig. 1(b). Profiles of $S^{(3)}(r)$ for $\nu = 0, 1, 2, 4$

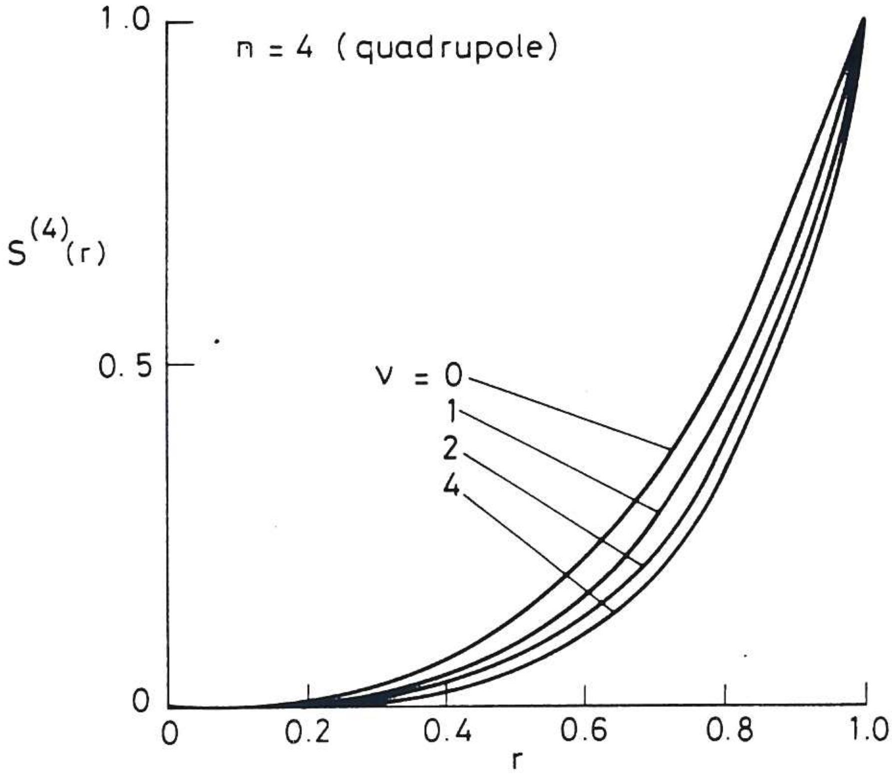

Fig.1(c). Profiles of $S^{(4)}(r)$ for $\nu = 0, 1, 2, 4$

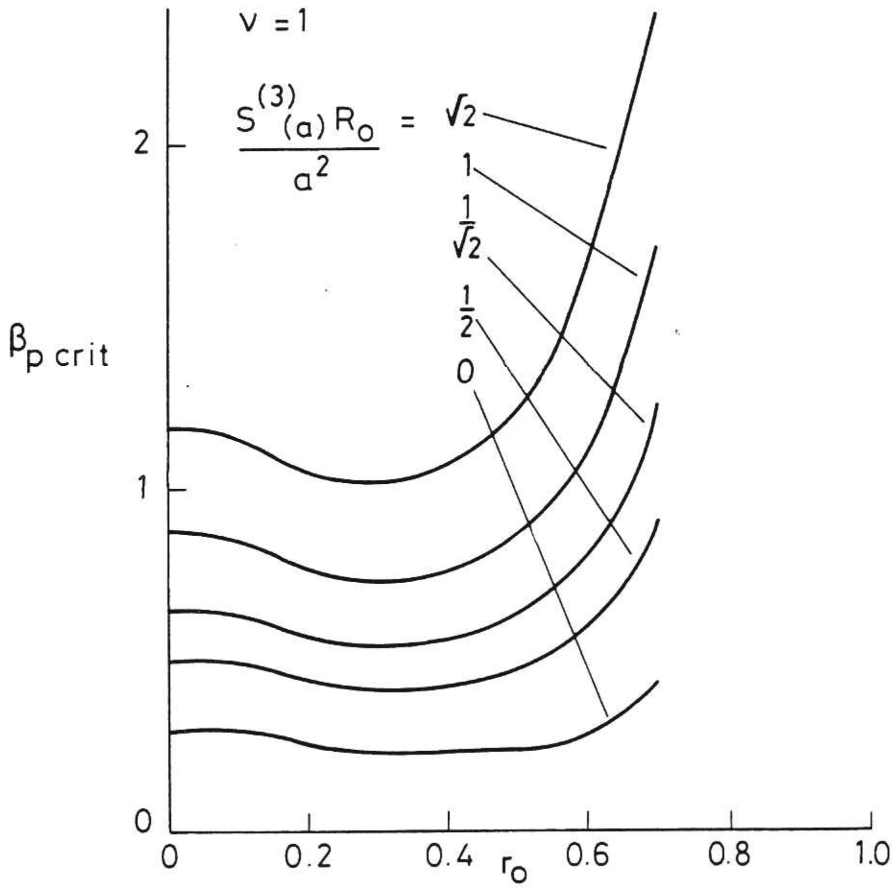

Fig. 2(a). $\beta_{p\,crit}$ as a function of r_o for various values of $\frac{S^{(3)}(a)\,R_o}{a^2}$ for $\nu = 1$

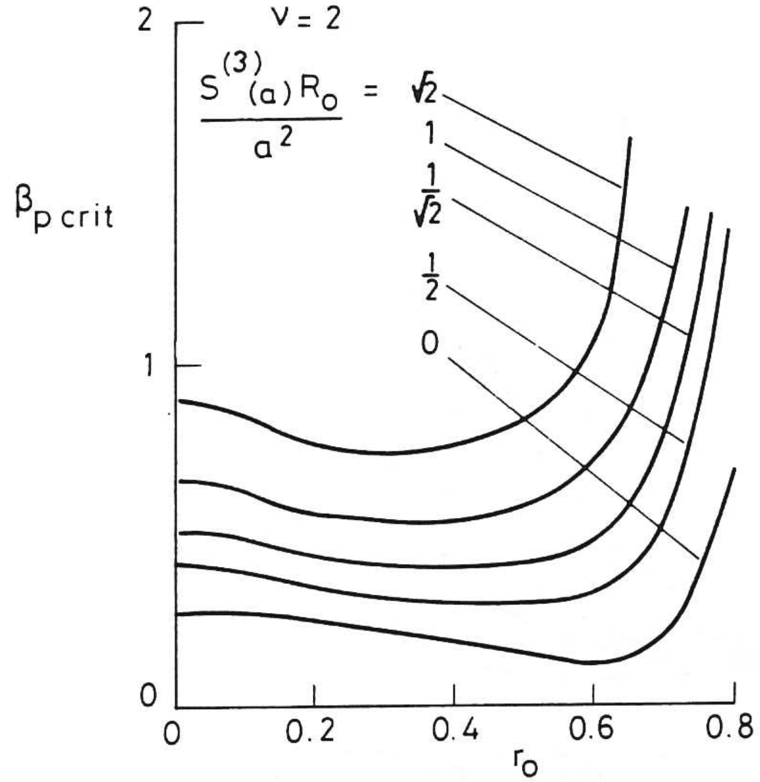

Fig. 2(b) $\beta_{p\,crit}$ as a function of r_o for various values of $\frac{S^{(3)}(a)R_o}{a^2}$ for $V = 2$

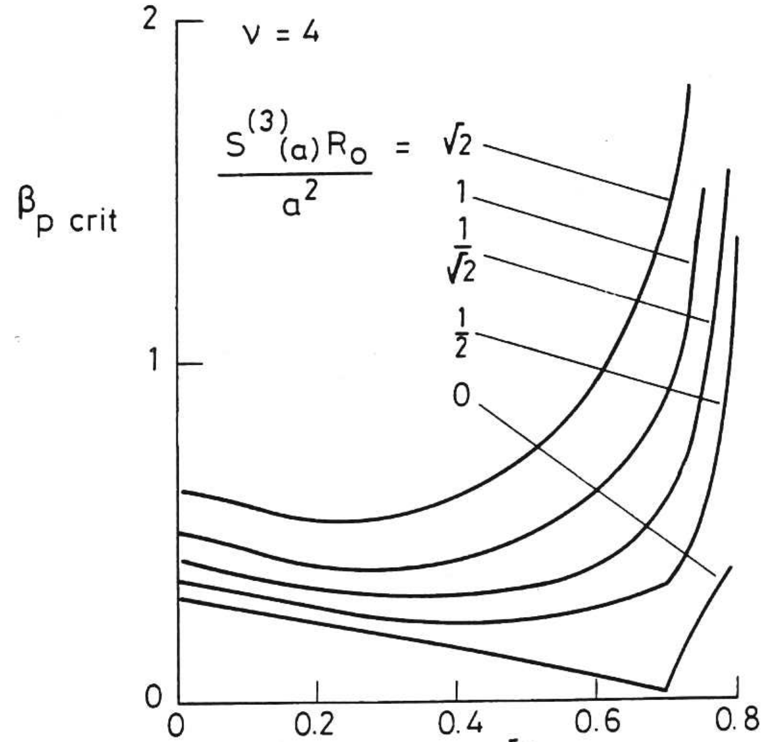

Fig. 2(c) $\beta_{p\,crit}$ as a function of r_o for various values of $\frac{S^{(3)}(a)R_o}{a^2}$ for $V = 4$

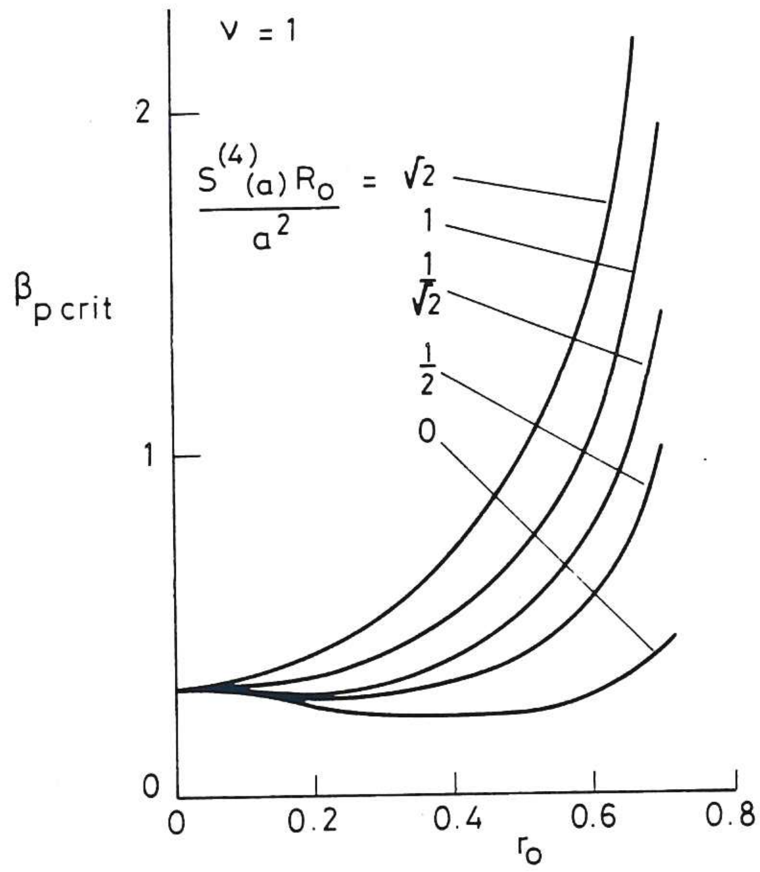

Fig. 3(a). $\beta_{p\,crit}$ as a function of r_o for various values of $\dfrac{S^{(4)}(a)\,R_o}{a^2}$ for V = 1

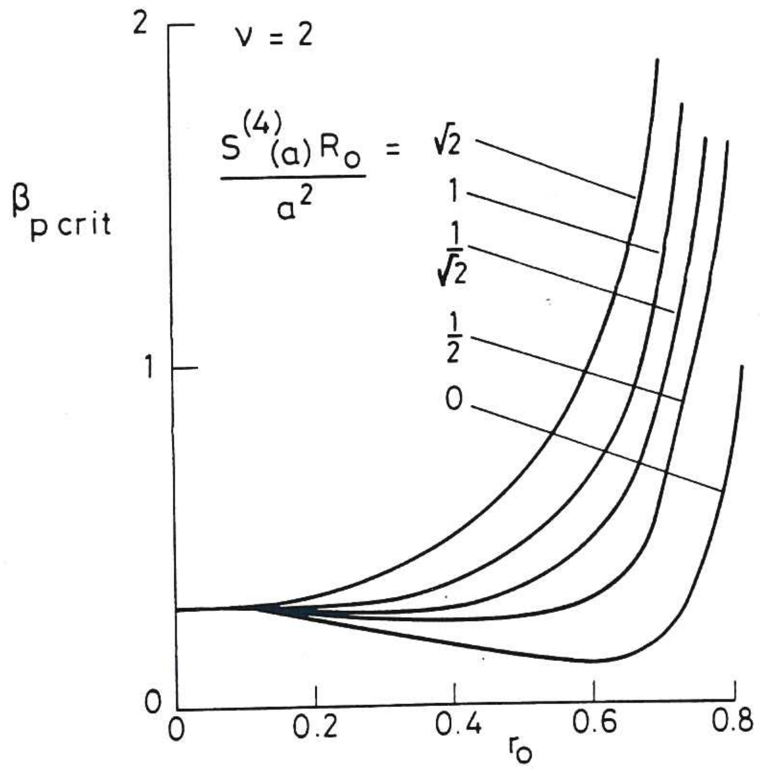

Fig. 3(b). $\beta_{p\,crit}$ as a function of r_o for various values of $\dfrac{S^{(4)}(a)\,R_o}{a^2}$ for V = 2

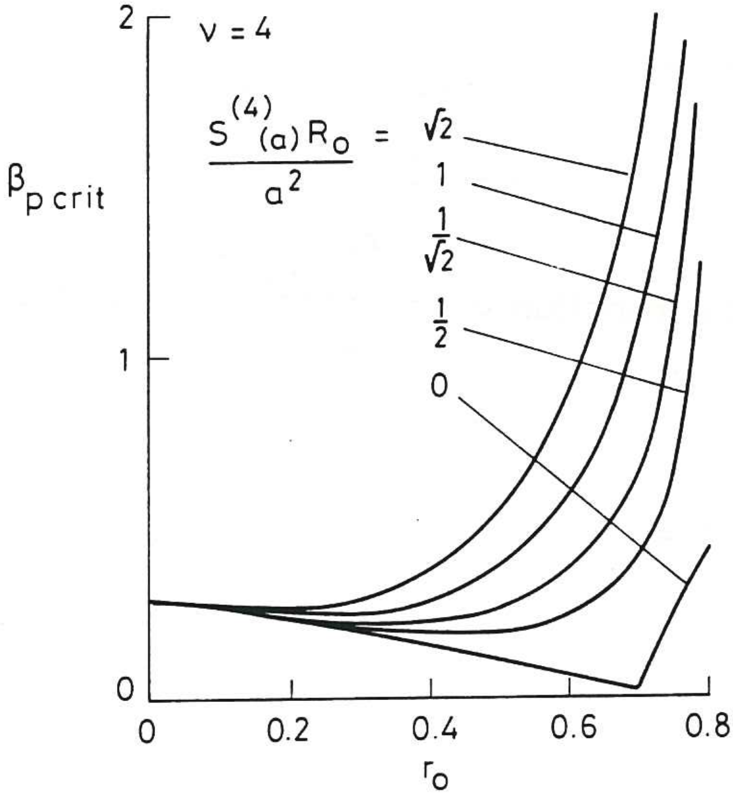

Fig. 3(c). $\beta_{p\,crit}$ as a function of r_o for various values of $\frac{S^{(4)}(a) R_o}{a^2}$ for $\nu = 4$

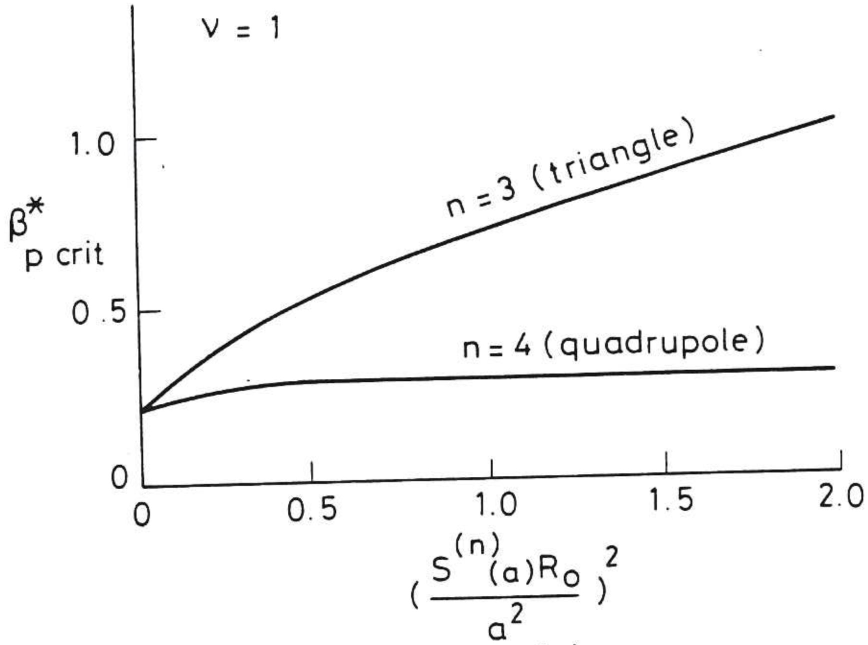

Fig. 4 (a). $\beta^*_{p\,crit}$ as a function of $\left(\frac{S^{(n)}(a)R_o}{a^2}\right)^2$ for n =2 and 3 with V = 1

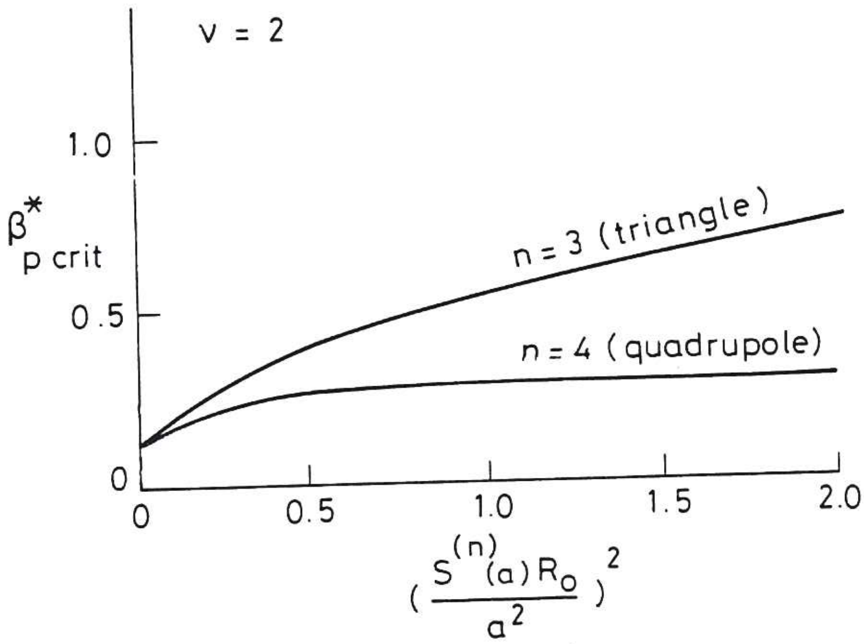

Fig. 4 (b). $\beta^*_{p\,crit}$ as a function of $\left(\frac{S^{(n)}(a)R_o}{a^2}\right)^2$ for n =2 and 3 with V = 2

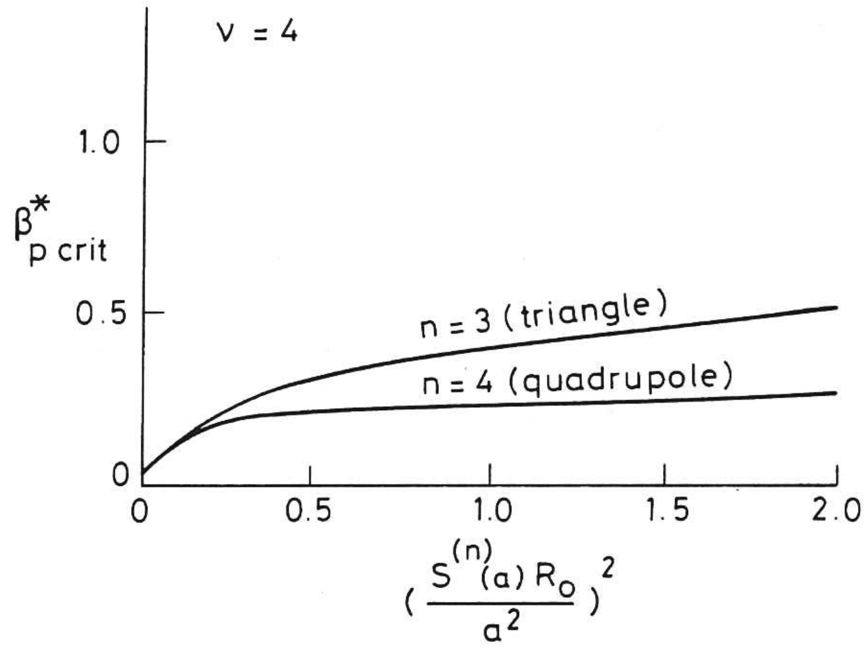

Fig. 4 (c) $\beta^*_{p\,crit}$ as a function of $\left(\frac{S^{(n)}(a)\,R_o}{a^2}\right)^2$ for $n=2$ and 3 with $\nu = 4$