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THE ELECTRON CYCLOTRON INSTABILITY

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A B S T R A C T

A model similar to that of Burt and Harris is used to describe the instability of electron cyclotron oscillations. This arises through the interaction of the orbital electrons with the electrons of the background plasma. Stability criteria and growth rates are given for certain limiting situations. The physical mechanism and energetics of the instability which arises when $k_{\perp} \ll k$, are also discussed.

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CONTENTS

1. INTRODUCTION

Investigation of the ion cyclotron resonance instability[1] in mirror machines such as PHOENIX[2] is made difficult by the profusion of physical processes which arise. Burt and Harris[3] have studied unstable cyclotron oscillations for a model in which the plasma forms a cylindrical shell with its axis parallel to a uniform magnetic field. In the equilibrium of this model the ions move in concentric Larmor orbits centred on the axis of the cylinder, and the electrons are taken to be cold. It may be possible however, to isolate the cyclotron phenomenon from the other processes and hence make a detailed experimental study of this instability by means of a small - scale experiment based on a model similar to that of Burt and Harris[3]. Because of the expertise which exists in building and operating electron guns, it is planned to study[4] the cyclotron instability which is expected to arise through electrons orbiting in a cold plasma. Thus electrons are to be injected from a gun so as to spiral about an axis along a field in a mercury plasma. It is expected that under suitable conditions a cyclotron instability will arise through the interaction between the spiralling electrons and the cold background electrons. In the present work we set up and examine a model which we hope will describe some of the features of the envisaged experiment.

In section 2 we consider an annulus of plasma in which there are two groups of electrons. The electrons of the first group orbit about the axis of the annulus with the same angular frequency, while the electrons of the second group have only axial motion in equilibrium. Charge neutrality in equilibrium is provided by ions which also have only axial motions. Using Maxwell's equations and the fluid equations, the method of Harris[5], leads to a dispersion equation in ω which involves general distribution functions for the spread of axial velocities.

The special case of cold ions and electrons is considered in section 3. The dispersion equation is still complicated and it is necessary to consider a number of limiting cases. For $k_\perp \ll k$, we obtain the instability criterion,

$$\ell^2 \omega_{ce}^2 < \omega_{pe}^2 [\beta^{\frac{1}{3}} + (1 - \beta + \mu)^{\frac{1}{3}}]^3 , \qquad \dots (1.1)$$

where β is the fraction of the total number of electrons in the orbital group, ω_{pe} is the plasma frequency, ℓ is the azimuthal wave number, and μ is the ratio of electron to ion mass. Simple expressions for the growth rate of the instability are given for the limiting situations $\beta \ll 1$ and $\beta \sim 1$. These instabilities are analogous to those described by Harris for the ion-cyclotron problem.

For $k_z \ll k$ we neglect the ions, it being shown that these particles are not important in the instabilities considered here. It is shown that for $\ell = 1$ and 2 the situation is stable, whereas for $\ell \geqslant 3$ there are certainly values of ω_{pe} for which instability will set in. We derive definite results only for the limiting case $\beta \ll 1$ (which is probably the one of practical interest), and show that there is a narrow band of ω_{pe} for which instability will occur. For the particular plasma frequency

$$\omega_{pe} = [\ell\ (\ell - 2)]^{\frac{1}{2}}\ \omega_{ce}, \qquad \ldots (1.2)$$

a wave propagated with frequency

$$\omega_R = (\ell - 1)\ \omega_{ce} + O\ (\beta)\ , \qquad \ldots (1.3)$$

and growth rate

$$\omega_I = \pm\ \frac{\ell(\ell - 2)\omega_{ce}}{2(\ell - 1)^{\frac{1}{2}}}\ \beta^{\frac{1}{2}} + O\ (\beta)\ . \qquad \ldots (1.4)$$

In section 4 we examine the physics of the instability which arises when $k_\perp \ll k$. The ions are ignored and β is taken to be small compared with 1. The resulting dispersion equation is identical in form with that obtained for the conventional two-stream electron-ion plasma. The quantities $\frac{\omega_{ce}}{k}$ and β are equivalent to the velocity difference v and μ respectively in the conventional problem. The electron-electron two-stream problem is discussed and the physical picture obtained is used to describe the electron-cyclotron instability.

The section ends with a discussion of the energetics of the instability. It is shown that the unperturbed azimuthal motion of the orbital electrons feed energy into their longitudinal motion and the longitudinal motion of the main body of the electrons in equal amounts.

2. DERIVATION OF THE DISPERSION RELATION

Our model consists of a cylindrical shell of plasma whose axis is parallel to a uniform magnetic field. In the unperturbed state the ions (density N_i) are assumed to have no perpendicular velocity component and hence move independently of the field. The electrons are divided into two groups with densities N_1 and N_2 respectively. It is assumed that in equilibrium the electrons of the former group move in concentric Larmor orbits with their centres on the axis of the cylinder. As with the ions, the second group of electrons does not interact with the field in the equilibrium state. We now use the fluid equations and Maxwell's equations, and extend the method of Harris[(5)] to describe our three-fluid model.

The equation of motion for group 1 electrons is

$$\frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \nabla) \underline{v} = - \frac{e}{m} (\underline{E} + \frac{\underline{v} \times \underline{B}}{c}) \qquad \dots (2.1)$$

where

$$\underline{B} = (o, o, - B) .$$

We linearise by writing

$$\underline{v} = V_z \underline{e}_z + V_1 \underline{e}_\theta + \underline{v}^1 = \underline{V} + \underline{v}^1 , \qquad \dots (2.2)$$

where

$$V_1 = \omega_{ce} r = - \frac{eB}{mc} r , \qquad \dots (2.3)$$

and the perturbations are proportional to $\exp [i (k_z z - \ell\theta + \omega t)]$. Linearising equation (2.1) we are led to the following forms for the perturbed velocity components:-

$$\left.\begin{aligned} v_r^1 &= \frac{- \frac{ie}{m} E_r^1 (\omega + k_z V_z - \ell \omega_{ce}) - \frac{e}{m} E_\theta^1 \omega_{ce}}{\omega_{ce}^2 - (\omega + k_z V_z - \ell \omega_{ce})^2} \\ v_\theta^1 &= \frac{\frac{e}{m} \omega_{ce} E_r^1 - \frac{ie}{m} E_\theta^1 (\omega + k_z V_z - \ell \omega_{ce})}{\omega_{ce}^2 - (\omega + k_z V_z - \ell \omega_{ce})^2} \\ v_z^1 &= \frac{- \frac{e}{m} E_z^1}{i (\omega + k_z V_z - \ell \omega_{ce})} \end{aligned}\right\} \qquad \dots (2.4)$$

For the electrons of group 2 we put $V_z = 0$ in the linearised velocity terms of equation (2.1). [This amounts to putting $\ell \omega_{ce} = 0$ and replacing ω_{ce} by $- \omega_{ce}$ in equations (2.4)]. We obtain,

$$v_r^1 = \frac{- \frac{ie}{m} E_r^1 (\omega + k_z V_z) + \frac{e}{m} E_\theta^1 \omega_{ce}}{\omega_{ce}^2 - (\omega + k_z V_z)^2} \qquad \dots (2.5)$$

$$v_\theta^1 = \frac{- \frac{e}{m} \omega_{ce} E_r^1 - \frac{ie}{m} E_\theta^1 (\omega + k_z V_z)}{\omega_{ce}^2 - (\omega + k_z V_z)^2} \qquad \dots (2.6)$$

$$v_z^1 = \frac{- \frac{e}{m} E_z^1}{i(\omega + k_z V_z)} \qquad \dots (2.7)$$

The perturbed velocity components for the ions are the same in form as those given by (2.5) - (2.7) with the quantities $m, - e, - \omega_{ce}$, replaced by M, e, ω_{ci} respectively.

The equation of continuity for each type of particle is

$$\frac{\partial n_j}{\partial t} + (\underline{v} \cdot \nabla)\, n = - n \nabla \cdot \underline{v} \;, \qquad \ldots (2.8)$$

and is linearised by taking

$$n = N_j + n_j^1 \qquad j = i,1,2 \;. \qquad \ldots (2.9)$$

The densities N_j are taken to be uniform throughout the cylindrical plasma shell. We introduce the electrostatic potential φ. ($\underline{E}^1 = - \nabla \varphi$). The above equation gives the following expressions for the perturbed densities:-

$$n_1^1 = - \frac{e}{m} N_1 \left[\frac{1}{\omega_{ce}^2 - (\omega + k_z V_z - \ell\, \omega_{ce})^2} \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \varphi}{\partial r}\right) - \frac{\ell^2}{r^2} \varphi \right\} + \frac{k_z^2\, \varphi}{(\omega + k_z V_z - \ell\, \omega_{ce})^2} \right] \qquad \ldots (2.10)$$

$$n_2^1 = - \frac{e}{m} N_2 \left[\frac{1}{\omega_{ce}^2 - (\omega + k_z V_z)^2} \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \varphi}{\partial r}\right) - \frac{\ell^2}{r^2} \varphi \right\} + \frac{k_z^2\, \varphi}{(\omega + k_z V_z)^2} \right] \qquad \ldots (2.11)$$

For n_i^1 we replace N_2, $-e$, m, ω_{ce}^2 by N_i, e, M, ω_{ci}^2 respectively in equation (2.11).

Substitution of these perturbed quantities in Poisson's equation leads to the dispersion equation,

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \varphi}{\partial r}\right) - \frac{\ell^2}{r^2} \varphi - k_z^2 \cdot \frac{1 - G}{1 + W} \varphi = 0 \;, \qquad \ldots (2.12)$$

where

$$G = \sum_{\text{ions}} \frac{4\pi e^2}{M} N_i \cdot \frac{1}{(\omega + k_z V_z)^2} - \sum_{\text{electrons 1}} \frac{4\pi e^2 N_1}{m} \cdot \frac{1}{(\omega + k_z V_z - \ell\, \omega_{ce})^2}$$

$$- \sum_{\text{electrons 2}} \frac{4\pi e^2 N_2}{m} \cdot \frac{1}{(\omega + k_z V_z)^2} \qquad \ldots (2.13)$$

and

$$W = - \sum_{\text{ions}} \frac{4\pi e^2 N_i}{M} \cdot \frac{1}{(\omega + k_z V_z)^2 - \omega_{ci}^2} - \sum_{\text{electrons 1}} \frac{4\pi e^2 N_1}{m} \cdot$$

$$\frac{1}{(\omega + k_z V_z - \ell\, \omega_{ce})^2 - \omega_{ce}^2} - \sum_{\text{electrons 2}} \frac{4\pi e^2 N_2}{m} \cdot \frac{1}{(\omega + k_z V_z)^2 - \omega_{ce}^2} \qquad \ldots (2.14)$$

Thus our dispersion equation has the same form as that given by Harris[5], but G and W have now become suitably modified. The summation signs are over the electron and ion velocity groups. These will now be replaced by integrals over distribution functions, and we obtain,

$$1 = \frac{k_z^2}{k^2}\left\{\omega_{pi}^2 \int \frac{f_i\,(V_z)\,dV_z}{(\omega + k_z V_z)^2} + \omega_{pe1}^2 \int \frac{f_{e1}\,(V_z)\,dV_z}{(\omega + k_z V_z - \ell\,\omega_{ce})^2} + \omega_{pe2}^2 \int \frac{f_{e2}\,(V_z)\,dV_z}{(\omega + k_z V_z)^2}\right\}$$

$$+ \frac{k_\perp^2}{k^2}\left\{\omega_{pi}^2 \int \frac{f_i\,(V_z)\,dV_z}{(\omega + k_z V_z)^2 - \omega_{ci}^2} + \omega_{pe1}^2 \int \frac{f_{e1}\,(V_z)\,dV_z}{(\omega + k_z V_z - \omega_{ce})^2 - \omega_{ce}^2}\right.$$

$$\left. + \omega_{pe2}^2 \int \frac{f_{e2}\,(V_z)\,dV_z}{(\omega + k_z V_z)^2 - \omega_{ce}^2}\right\}, \qquad \ldots (2.15)$$

where $k_\perp$ is given by applying the appropriate boundary conditions to the solution

$$\varphi\,(r) = A\,J_\ell\,(k_\perp\,r) + B\,N_\ell\,(k_\perp\,r) \qquad \ldots (2.16)$$

of equation (2.12), and

$$k^2 = k_\perp^2 + k_z^2\,. \qquad \ldots (2.17)$$

3. EXAMINATION OF THE DISPERSION RELATION

For cold ions and electrons $f_i(V_z) = f_{e1}(V_z) = f_{e2}(V_z) = \delta(V_z)$, and equation (2.15) becomes,

$$\frac{1}{\omega_{pe}^2} = \frac{k_z^2}{k^2}\left\{\frac{\mu}{\omega^2} + \frac{1-\beta}{\omega^2} + \frac{\beta}{(\omega - \ell\,\omega_{ce})^2}\right\} + \frac{k_\perp^2}{k^2}\left\{\frac{\mu}{\omega^2 - \omega_{ci}^2} + \frac{\beta}{(\omega - \ell\,\omega_{ce})^2 - \omega_{ce}^2} + \frac{1-\beta}{\omega^2 - \omega_{ce}^2}\right\}, \qquad \ldots (3.1)$$

where

$$\left.\begin{aligned} \omega_{pi}^2 &= \frac{4\pi e^2 N_i}{M} = \frac{m}{M}\cdot\omega_{pe}^2 = \mu\,\omega_{pe}^2 \\ \omega_{pe1}^2 &= \frac{4\pi e^2}{m}\,\beta\,N_i = \beta\,\omega_{pe}^2 \\ \omega_{pe2}^1 &= \frac{4\pi e^2}{m}\,(1-\beta)\,N_i = (1-\beta)\,\omega_{pe}^2 \end{aligned}\right\} \qquad \ldots (3.2)$$

and β is that fraction of the total number of electrons which are in group 1.

We shall now investigate the dispersion equation (3.1) for real k and complex ω.

It will be necessary to consider a number of limiting cases. Thus consider

(a) $k_\perp \ll k$

In this situation we can neglect the second bracket and put $k_z = k$. Equation (3.1) becomes

$$\frac{1}{\omega_{pe}^2} = \frac{\beta}{(\omega - \ell\,\omega_{ce})^2} + \frac{1-\beta+\mu}{\omega^2} . \qquad \dots (3.3)$$

It is simple to show that the condition for instability is

$$\ell^2\omega_{ce}^2 < \omega_{pe}^2 \; [\, \beta^{1/3} + (1-\beta+\mu)^{1/3}\,]^3 . \qquad \dots (3.4)$$

Now consider the limit $\beta \ll 1$. Since $\mu \ll 1$, it is straight forward to derive

$$\omega \approx \ell\,\omega_{ce} \pm \frac{\beta^{1/2}\,\ell\,\omega_{ce}}{\sqrt{\left[\dfrac{\ell^2\omega_{ce}^2}{\omega_{pe}^2} - 1\right]}} , \qquad \dots (3.5)$$

from equation (3.3). We observe that the condition for instability is $\ell^2\omega_{ce}^2 < \omega_{pe}^2$, and that the growth rate is proportional to $\beta^{1/2}$.

Similarly for the limit $1 - \beta \ll 1$, we obtain

$$\omega \approx \pm \frac{(1-\beta+\mu)^{1/2}\,\ell\,\omega_{ce}}{\sqrt{\left[\dfrac{\ell^2\omega_{ce}^2}{\omega_{pe}^2} - 1\right]}} . \qquad \dots (3.6)$$

Thus the stability condition is the same but the growth rate is proportional to $(1-\beta+\mu)^{1/2}$.

(b) $k_z \ll k$

In this situation we can neglect the first bracket in equation (3.1) and put $k_\perp = k$. Thus we have

$$\frac{1}{\omega_{pe}^2} = \frac{\mu}{\omega^2 - \omega_{ci}^2} + \frac{\beta}{(\omega - \ell\,\omega_{ce})^2 - \omega_{ce}^2} + \frac{1-\beta}{\omega^2 - \omega_{ce}^2} . \qquad \dots (3.7)$$

Since μ is very small it is simple to show that there are two roots $\omega \approx \pm\,\omega_{ci}$, and that the corresponding modes are stable. Thus if we assume that the other roots of (3.7) are never close to ω_{ci} then we can ignore the first term on the right hand side of (3.7). To analyse (3.7) it is necessary to consider $\ell = 1$, $\ell = 2$ and $\ell \geqslant 3$ as separate cases:-

(i) $\ell = 1$

For this case equation (3.7) can be written as

$$F(\omega) = -\frac{\beta}{\omega(\omega - 2\omega_{ce})} - \frac{1-\beta}{\omega^2 - \omega_{ce}^2} = -\frac{1}{\omega_{pe}^2}$$

A plot of this function is shown in Fig.1. For any ω_{pe}, the line $-1/\omega_{pe}^2$ always intersects $F(\omega)$ at four real values of ω. Hence for $\ell = 1$ the physical situation is always stable.

(ii) $\ell = 2$

We have

$$F(\omega) = -\frac{\beta}{(\omega - 3\omega_{ce})(\omega - \omega_{ce})} - \frac{1-\beta}{(\omega - \omega_{ce})(\omega + \omega_{ce})} = -\frac{1}{\omega_{pe}^2}$$

Again we plot $F(\omega)$, but the precise form of the curves depends on whether or not $\beta >$ or $< \frac{1}{2}$. (See Fig.2) We observe that $-1/\omega_{pe}^2$ always intersects $F(\omega)$ at three real points, and since the equation

$$F(\omega) = -1/\omega_{pe}^2$$

is a cubic the situation is stable.

(iii) $\ell \geqslant 3$

We have

$$F(\omega) = -\frac{\beta}{[\omega - (\ell+1)\,\omega_{ce}][\omega - (\ell-1)\omega_{ce}]} - \frac{1-\beta}{\omega^2 - \omega_{ce}^2} = -\frac{1}{\omega_{pe}^2}$$

We give a plot of this function in Fig.3. It is straightforward to show that $F(\omega) = 0$ when

$$\omega = \ell\,\omega_{ce}\,(1-\beta) \pm \omega_{ce}\sqrt{[1 - \ell^2\beta(1-\beta)]}\,. \qquad \ldots (3.8)$$

If

$$\ell^2\beta(1-\beta) > 1\,, \qquad \ldots (3.9)$$

then $F(\omega)$ does not cross the ω - axis. It follows that this situation is unstable for large values of ω_{pe}^2 since there are 2 real and 2 complex roots. Note that for any given β (apart from $\beta = 0$, $\beta = 1$), we can always find an ℓ for which (3.9) is satisfied, so that the system is always unstable for some values of ℓ at large ω_{pe}.

If $\ell^2\beta(1-\beta) < 1$, then one of the curves 1 and 2 drops below the ω axis, but not both. Thus there is some finite range of ω_{pe} for which the situation is unstable.

In order to make further progress we examine the case where $\beta \ll 1$. It is simple to show that the stationary values of $F(\omega)$ occur near $\omega = 0$ and $\omega = \omega_{ce}[1 \pm 1]$. $F(0)$ is a minimum and in fact is positive so that curve 1 does not cross the ω axis for small β.

We can show that $F(\omega)$ has a maximum at the point

$$\omega = \omega_{ce}\,[\ell - \sqrt{(1 + \Delta)}]\,,$$

and a minimum at the point

$$\omega = \omega_{ce}\,[\ell - \sqrt{(1 - \Delta)}]\,, \qquad \ldots (3.10)$$

where

$$\Delta = \beta^{\frac{1}{2}}\,\frac{(\ell - 1)^2 - 1}{(\ell - 1)^{\frac{1}{2}}}$$

Substitution of these values gives

$$\left.\begin{aligned} F_{min} &= -\frac{1}{\omega_{ce}^2\,[(\ell - 1)^2 - 1]} + \frac{2\,\beta^{\frac{1}{2}}\,(\ell - 1)^{\frac{1}{2}}}{\omega_{ce}^2\,[\ell - 1)^2 - 1]} \\ F_{max} &= -\frac{1}{\omega_{ce}^2\,[(\ell - 1)^2 - 1]} - \frac{2\,\beta^{\frac{1}{2}}\,(\ell - 1)^{\frac{1}{2}}}{\omega_{ce}^2\,[(\ell - 1)^2 - 1]} \end{aligned}\right\} \qquad \ldots (3.11)$$

From Fig.4 we see that as $\beta \to 0$ these two turning points approach and the region of instability appears.

It follows that for $\beta \ll 1$, $\ell^2 \beta < 1$, the system is unstable if

$$F_{max} < -1/\omega_{pe}^2 < F_{min}\,. \qquad \ldots (3.12)$$

We now wish to estimate the growth rate of an unstable wave when the plasma frequency is given by

$$\omega_{pe}^2 = \omega_{ce}^2\,[(\ell - 1)^2 - 1]\,. \qquad \ldots (3.13)$$

The dispersion equation is then

$$\frac{\beta}{(\omega - (\ell + 1)\omega_{ce})\,(\omega - (\ell - 1)\omega_{ce})} + \frac{1 - \beta}{\omega^2 - \omega_{ce}^2} = \frac{1}{\omega_{ce}^2((\ell - 1)^2 - 1)} \qquad \ldots (3.14)$$

We assume that the growth rate is of order $\beta^{\frac{1}{2}}$ and demonstrate that there is indeed an ω_I of this order.

Expanding ω_R and ω_I as series in $\beta^{\frac{1}{2}}$ it is straightforward to show that

$$\omega_R = (\ell - 1)\,\omega_{ce} + O\,(\beta) \qquad \ldots (3.15)$$

and that

$$\omega_I = \pm \frac{\ell(\ell - 2)\omega_{ce}\,\beta^{\frac{1}{2}}}{2(\ell - 1)^{\frac{1}{2}}} + O\,(\beta)\,. \qquad \ldots (3.16)$$

(c) $k_\perp^2 \sim \beta^{\frac{1}{2}} k^2$

For $\ell = 1$ we have shown from the dispersion equation (3.1) that for $k_\perp \ll k$ (retaining only the first bracket) there can be instability, whilst for $k_z \ll k$ (retaining only the second bracket) the situation is stable. In this section we ask whether under condition of instability one of the terms in the 'stable bracket' can be of the same order as the terms in the 'unstable bracket' and so modify the instability. For small β and assuming $\omega \sim \omega_{ce}$ the ordering $(\omega - \omega_{ce}) \sim (\frac{k_\perp}{k})^2 \sim \beta^{\frac{1}{2}}$ with $\beta \ll 1$ is such that the terms in

$$\frac{1}{\omega_{pe}^2} = \frac{\beta}{(\omega - \omega_{ce})^2} + \frac{1}{\omega_{ce}^2} + \frac{k_\perp}{k^2} \cdot \frac{1}{2\omega_{ce}\,(\omega - \omega_{ce})}\,, \qquad \ldots (3.17)$$

are of the same order.

It follows that the condition for instability is

$$\omega_{ce}^2 < \left(1 - \frac{1}{16\beta}\left(\frac{k_\perp}{k}\right)^4\right)\omega_{pe}^2\,. \qquad \ldots (3.18)$$

Thus we observe that the condition for instability previously obtained for $\ell = 1$ has been made slightly more difficult to satisfy.

4. PHYSICS OF THE INSTABILITY

In this section we attempt to set up a 'physical picture' of the electron-electron Harris instability as described by the dispersion equation

$$\frac{1}{\omega_{pe}^2} = \frac{1}{\omega^2} + \frac{\beta}{(\omega - \ell\,\omega_{ce})^2}\,, \qquad \ldots (4.1)$$

where we have taken $k_\perp \ll k_z$, $\beta \ll 1$ and have ignored the motion of the ions. This is identical in form with the dispersion equation

$$\frac{1}{\omega_{pe}^2} = \frac{1}{\omega^2} + \frac{\mu}{(\omega - kv)^2} \qquad \ldots (4.2)$$

for the conventional two-stream electron-ion problem.

The stationary (in the unperturbed system) electrons behave like the electrons in the conventional two-stream problem. The behaviour of the orbiting electrons is analogous to that of the ions in the electron-ion problem. This is because the orbital electrons in the first problem and the ions in the second problem have low characteristic response frequencies compared with the stationary electrons. In the case of the orbiting electrons it is low because their small number density is comparatively ineffective in producing electric fields, whereas in the case of the ions it is low because of their large mass.

We consider the two-stream problem for two parallel streams of electrons. The dispersion equation is

$$\frac{1}{\omega_{pe}^2} = \frac{1-\beta}{(\omega - kv_2)^2} + \frac{\beta}{(\omega - kv_1)^2} \; . \qquad \ldots (4.3)$$

It follows that when $\beta \ll 1$, $\frac{\omega}{k} \approx v_1$, and for instability

$$\frac{1}{\omega_{pe}^2} < \frac{1}{k^2 (v_2 - v_1)^2} \; . \qquad \ldots (4.4)$$

For the very unstable situation

$$\omega = kv_1 \pm i \, \beta^{\frac{1}{2}} \, k \, (v_2 - v_1) \; . \qquad \ldots (4.5)$$

We now move to the rest frame of group 1 electrons (small group) i.e. $v_1 = 0$. The main physical role of particles 1 is to select the wave-frame.

For particles 1 we have

$$i \, \omega \, v_1^1 = - \frac{e}{m} E \; , \qquad \ldots (4.6)$$

and for particles 2,

$$i \, k \, v_2 \, v_2^1 = - e \, E/m \; . \qquad \ldots (4.7)$$

We also have the continuity equations

$$\left.\begin{aligned} n_2^1 \, kv_2 &= - n_2 \, kv_2^1 \\ \omega \, n_1^1 &= - n_1 \, k \, v_1^1 \end{aligned}\right\} \qquad \ldots (4.8)$$

and Poisson's equation

$$i \, k \, E = - 4\pi e \, (n_1^1 + n_2^1) \; . \qquad \ldots (4.9)$$

From equations (4.6) - (4.9) we have the phase diagram shown in Fig.5.

It is seen from the diagram that the unstable perturbation is one in which there is a local increase (say) in the density of electrons of type 2 and a smaller decrease in the density of electrons of type 1 resulting in a negative charge concentration. As the type 2 electrons pass through this charge their velocity is reduced by the resulting electric field. This leads to a further increase in their density. The particles of type 1 which are at rest with respect to the wave, are expelled from the region of negative charge and their density is further reduced.

These ideas can be carried over to the cyclotron instability. In this case we have a small group of electrons (group 1) orbiting in the magnetic field while group 2 (the large group) is stationary. A wave now propagates through the system such that its direction of propagation is almost along the field. If we consider the motions of the particles relative to the wave-frame, then a particle of group 1 will trace a helix such that it will remain in phase with the wave. Type 2 electrons flow through the wave with a relative velocity ω_{ce}/k. Thus in the wave-frame the instability is physically similar to the usual two-stream situation. We end this section by making an examination of the energetics involved in this instability. Assuming $\ell = 1$, $\mu \ll 1$, $\beta \ll 1$, and $\omega_{ce}^2 \ll \omega_{pe}^2$ (i.e. very unstable), then it follows from (3.5) that

$$\omega = \omega_{ce} \left\{ 1 - \beta - i\, \beta^{\frac{1}{2}} \right\}. \qquad \ldots (4.10)$$

We consider our plasma to consist of cold ions (these are not expected to play any important role in the physics), cold electrons, and a group of orbiting cold electrons, with densities $N, \beta N$, and $(1 - \beta) N$ respectively. Their densities are taken to be uniform through the annular plasma. Wave propagation is essentially along the magnetic field. The perturbed radial and azimuthal velocity components are neglected, it being expected that only the perturbed longitudinal velocity is significant in determining the perturbed densities.

Assuming the real part of the frequency and certain features of the model, we aim to deduce, using an energy balance, the growth rate which is given by (4.10). This consistency will indicate that our basic ideas concerning the model and the energy exchange are correct.

We take the perturbed z - component of the electric field to be given by

$$E_z^1 = E_{oz} \cos\,(kz - \theta + \omega_R\, t)\, e^{\gamma t} \qquad \ldots (4.11)$$

The other perturbed quantities will have both in - phase and out - of phase components, e.g.

$$E_\theta^1 = E_{\theta 1} \operatorname{Cos}\,(k\, z - \theta + \omega_R\, t)\, e^{\gamma t} + E_{\theta 2} \operatorname{Sin}\,(k\, z - \theta + \omega_R\, t)\, e^{\gamma t} \qquad \ldots (4.12)$$

$$\begin{bmatrix} v^{1}_{zi} \\ v^{1}_{zes} \\ v^{1}_{ze} \\ n^{1}_{i} \\ n^{2}_{es} \\ n^{1}_{e} \end{bmatrix} = \begin{bmatrix} V_{i1} \\ V_{es1} \\ V_{e1} \\ N_{i1} \\ N_{es1} \\ N_{e1} \end{bmatrix} (\cos\,(kz - \theta + \omega_R t)\; e^{\gamma t}) + \begin{bmatrix} V_{i2} \\ V_{es2} \\ V_{e2} \\ N_{i2} \\ N_{es2} \\ N_{e2} \end{bmatrix} (\mathrm{Sin}\,(kz - \theta + \omega_R t)\; e^{\gamma t}), \qquad \ldots (4.13)$$

where s denotes the orbital group.

Linearising the equations of motion and continuity for the three groups of particles, substituting (4.13) and carrying out certain eliminations, we obtain the following expressions for the amplitudes:-

$$\left.\begin{aligned} N_{es1} &= \frac{-\beta N (\omega_R - \omega_{ce})\, V_{es1} + \beta N k \gamma V_{es2}}{(\omega_R - \omega_{ce}) \;+ \gamma^2} \\ V_{i1} &= \frac{\frac{e}{M} \gamma E_{oz}}{\omega_R^2 + \gamma^2} \\ V_{es1} &= \frac{-\frac{e}{m} \gamma E_{oz}}{(\omega_R - \omega_{ce})^2 + \gamma^2} \\ V_{es2} &= -\frac{-\frac{e}{m} (\omega_R - \omega_{ce}) E_{oz}}{(\omega_R - \omega_{ce})^2 + \gamma^2} \\ V_{e1} &= \frac{-\frac{e}{m} E_{oz}}{\omega_R^2 + \gamma^2} \end{aligned}\right\} \qquad \ldots (4.14)$$

From $\nabla \times \underline{E} = 0$, we have

$$E_{\theta 1} = -\, E_{oz}/kr, \; E_{\theta 2} = 0 \,. \qquad \ldots (4.15)$$

The rate of transfer of energy between the groups of particles and the field is given by

$$\overline{\underline{j}^1 \cdot \underline{E}^1} + \overline{\frac{\partial}{\partial t} (E^1)^2} = 0 \,, \qquad \ldots (4.16)$$

where the bar denotes an average over one wavelength of the wave. Ignoring the radial and azimuthal electric fields as compared with the longitudinal components, we get

$$e \left[- V_{\perp} N_{es1} E_{\theta 1} + N V_{i1} E_{oz} - \beta N V_{es1} E_{oz} - N v_{e1} E_{oz} \right] + \frac{\gamma}{4\pi} E^2_{oz} = 0 \qquad \ldots (4.17)$$

where $V_\perp$ is the unperturbed azimuthal velocity of the electrons. Substituting (4.14) and (4.15) this can be written,

$$\frac{2\,\beta\,\omega_{ce}\,(\omega_R - \omega_{ce})}{[(\omega_R - \omega_{ce})^2 + \gamma^2]^2} + \frac{\beta}{(\omega_R - \omega_{ce})^2 + \gamma^2} + \frac{1}{\omega_R^2 + \gamma^2}\,[1 + \mu] + \frac{1}{\omega_{pe}^2} = 0 \qquad \ldots (4.18)$$

Now we assume $\gamma^2 \ll \omega_R^2 \approx \omega_{ce}^2$, and from (4.10) we have

$$(\omega_R - \omega_{ce})^2 = \beta^2\,\omega_{ce}^2\,.$$

Since $\mu \ll 1$ the motion of the ions can be neglected. Then equation (4.18) becomes

$$\gamma^4\,[1 + \frac{\omega_{ce}^2}{\omega_{pe}^2}] + \beta\,\omega_{ce}^2\,\gamma^2 - 2\,\beta^2\,\omega_{ce}^4 = 0\,. \qquad \ldots (4.19)$$

Now we have assumed our model to be very unstable and hence $\omega_{ce}^2 \ll \omega_{pe}^2$. Thus we may neglect the second term in the bracket and this amounts to assuming that the rate of change of energy stored in the field is negligible. Solving equation (4.19) leads to

$$\gamma = \pm\,\beta^{\frac{1}{2}}\,\omega_{ce}\,, \qquad \ldots (4.20)$$

which is the result obtained in (4.10) by the usual method of solving the dispersion equation. The last term in equation (4.19) represents the rate of energy transfer from the unperturbed orbital motion to the longitudinal oscillations of both the orbital and non-orbital electrons. The rates of energy absorption by these two groups are given by the second and first terms in (4.19) respectively and these are equal in magnitude.

We now summarise the physics of our model. A small group of electrons is orbiting about the magnetic field. The ions in the main-body of the plasma play no part because of their large mass. For a very unstable situation the change of energy stored in the field is negligible. Only the perturbed longitudinal velocity components are important in determining the perturbed densities. The unperturbed azimuthal motion of the orbiting electrons feeds energy into their longitudinal motion and into the longitudinal motion of the main-body of the electrons in equal amounts.

The energetics of the more familiary non-homogeneous electron-ion Harris instability can be discussed in exactly the same way. The growth rate $\omega_I = \pm\,\mu^{\frac{1}{2}}\,\omega_{ci}$ is obtained and it is found that energy is distributed to the longitudinal motion of the ions and electrons in equal amounts.

5. REFERENCES

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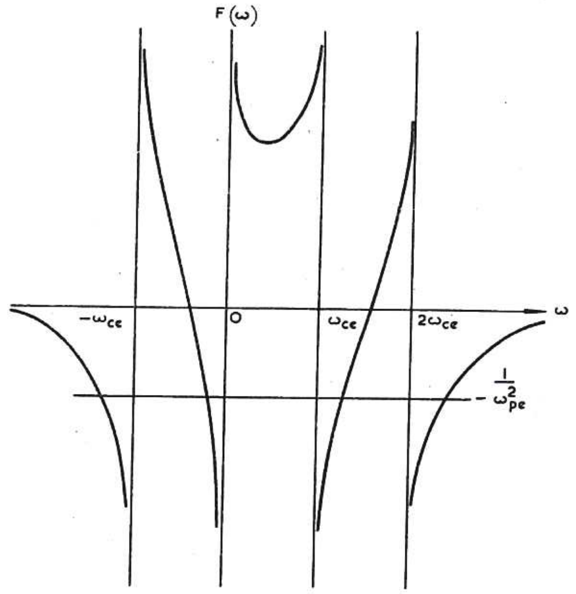

Fig. 1 (CLM-M44)
Plot of the dispersion relation for the physical situation $k_z \ll k$ and $\ell = 1$

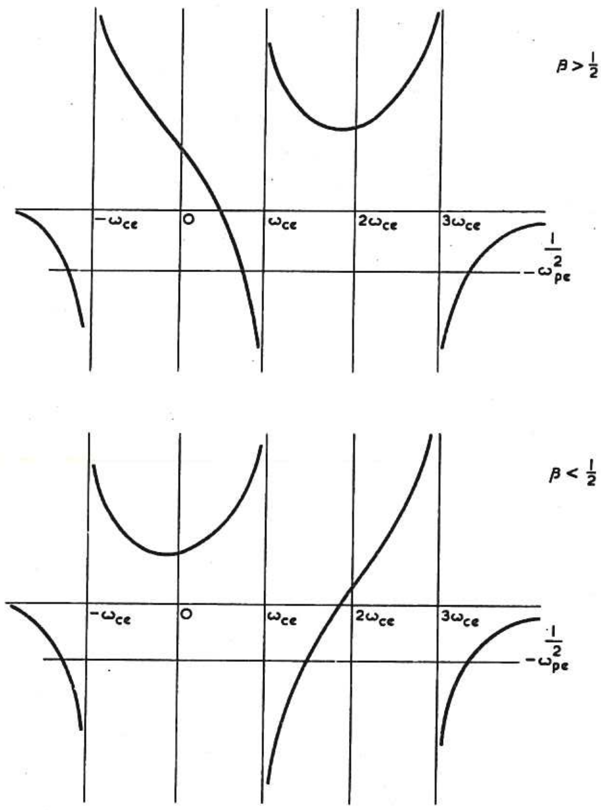

Fig. 2 (CLM-M44)
Plot of the dispersion relation for the physical situation $k_z \ll k$ and $\ell = 2$

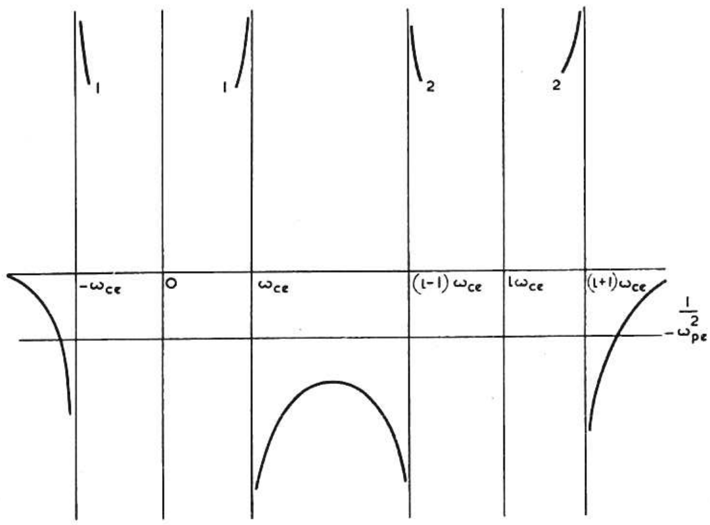

Fig. 3 (CLM-M 44)
Plot of the dispersion relation for the physical situation $k_z \ll k$ and $\ell \geqslant 3$

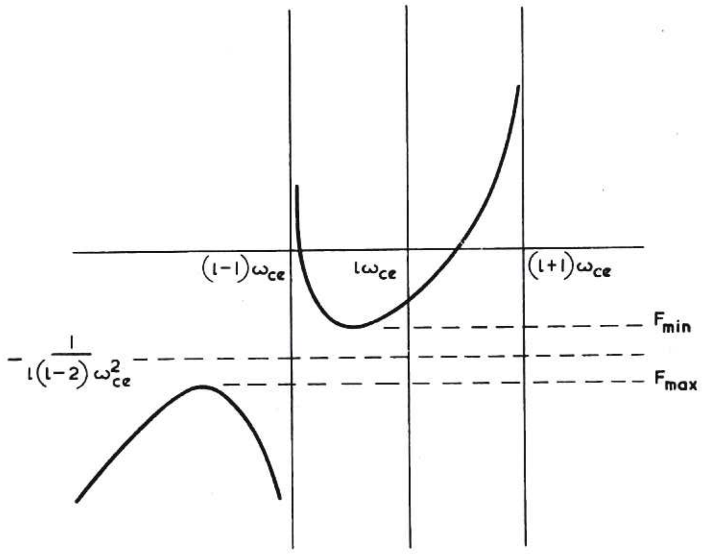

Fig. 4 (CLM-M 44)
Plot of the dispersion relation for the physical situation $k_z \ll k$, $\ell \geqslant 3$ and $\beta \ll 1$, in the neighbourhood of $\omega = \ell\omega_{ce}$

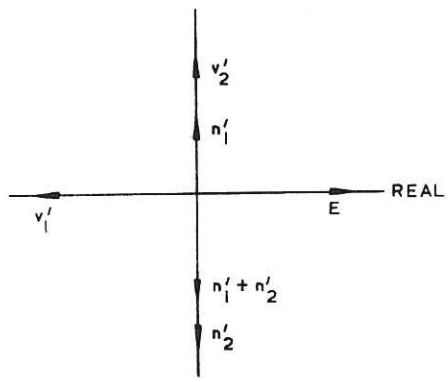

Fig. 5 (CLM-M 44)
Diagram showing the phase relationships between the perturbed quantities for the two-stream (electron-electron) instability