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A CLASS OF METHODS FOR TABULAR INTERPOLATION

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A CLASS OF METHODS FOR TABULAR INTERPOLATION

by

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ABSTRACT

A generalisation of the Neville-Aitken method is described which allows the construction of interpolating functions, other than polynomials, by means of simple recurrence relations. In particular, simple constructions are given for rational functions and trigonometric series which interpolate prescribed function values at non-equispaced positions of the independent variable.

Restrictions imposed by requiring the interpolating functions to be invariant under linear transformations of the coordinates are discussed, and application of the technique to the problem of inverse interpolation is also considered.

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CONTENTS

1. INTRODUCTION

The Neville-Aitken method (Neville, 1934) of successive linear interpolations is an elegant technique for polynomial interpolation in a table of values of a function of one independent variable. The purpose of this paper is to show how an essential feature of the Neville-Aitken process may be generalised so as to permit the construction of interpolating functions other than polynomials, while retaining a basic computational simplicity in the form of, for example, triangle or rhombus recurrence rules.

In the Neville-Aitken method a table of auxiliary interpolating polynomials is constructed, and extended, by forming linear, weighted means of previously completed functions. This may be generalised by introducing non-linear means, thereby allowing the construction of a great variety of forms of interpolating functions.

It will be seen that the tabular nature of the techniques described makes them very convenient for effecting inverse interpolation.

2. TABLES OF INTERPOLATING FUNCTIONS

Consider a set of values $\{f_j;\ j=1,2,\ \ldots\}$ each one corresponding to a value x_j of an independent variable x. The correspondence may be thought of as defined in terms of an originating function $f(x)$ by means of the relation

$$f_j = f(x_j) \quad ; \quad j=1,2,\ \ldots \qquad \ldots (1)$$

The values of $\{x_j;\ j=1,2,\ \ldots\}$ and $\{f_j;\ j=1,2,\ \ldots\}$ will be understood to be finite, although they may be real or complex. We presume that

$$x_j \neq x_k \quad \text{unless} \quad j=k. \qquad \ldots (2)$$

These numbers are arranged in a list, which is to be extended sideways in a triangular fashion, as illustrated in Table 1.

TABLE 1

A TABLE OF INTERPOLATING FUNCTIONS

x_1	f_1				
		f_{11}			
x_2	f_2		f_{12}		
		f_{21}		f_{13}	
x_3	f_3		f_{22}		f_{14} . . . .
		f_{31}		f_{23}	.
x_4	f_4		f_{32}	.	.
		f_{41}	.	.	
x_5	f_5	.	.	.	
.	.	.	.		
.	.	.			
.	.				
.	.				

The quantities $\{ f_{jk}; \ j \geqslant 1, \ k \geqslant 1 \}$ to the right of the column $\{f_j; \ j \geqslant 1 \}$ are to be interpreted either as functions having the property

$$f_{jk}(x_r) = f_r \quad ; \quad j \leqslant r \leqslant j+k \ , \qquad \ldots (3)$$

except, possibly, in special circumstances, or as numerical values of these functions associated with a prescribed value of x.

Since each f_{jk} will be constructed from the values $\{ (x_r, f_r); \ j \leqslant r \leqslant j+k \}$ we shall refer to the set of points $\{ x_r; \ j \leqslant r \leqslant j+k \}$ as the domain, D_{jk}, of f_{jk}. Also, the set of points $\{x_r\}$ satisfying condition (3) will be referred to as the domain of interpolation, D^I_{jk}, of f_{jk}. If equation (3) is satisfied for all points x_r within the domain of f_{jk}, i.e. if

$$D^I_{jk} = D_{jk} \qquad \ldots (4)$$

we shall say that f_{jk} possesses Property I.

By extension, it is reasonable to define

$$f_{jo} = f_j \qquad \ldots (5)$$

and to say that the single point x_j constitutes the domain of interpolation of f_{jo}.

Fig.1 shows how a function $f_{jk}(x)$ stands, in a table of the form of Table 1, in relation to its domain.

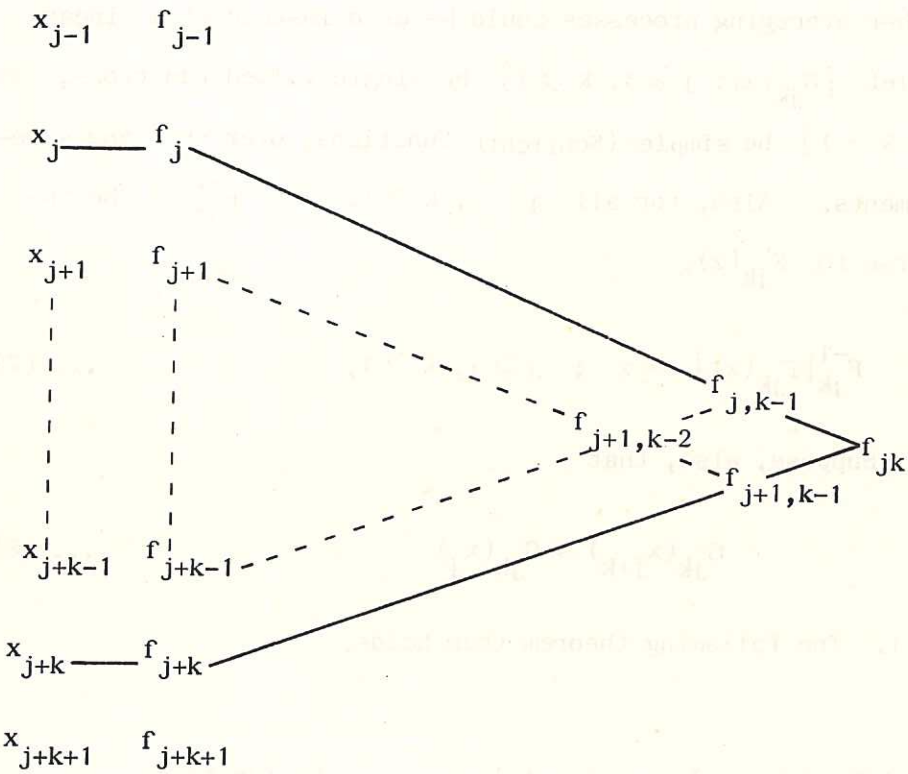

Fig.1

The domain of f_{jk}

3. RECURRENCE FORMULAE

In the Neville-Aitken process a table of the form of Table 1 is constructed by generating each f_{jk} as a linear combination of its earlier neighbours, $f_{j,k-1}$ and $f_{j+1,k-1}$; thus

$$f_{jk}(x) = \frac{(x-x_j)f_{j+1,k-1} + (x_{j+k}-x)f_{j,k-1}}{x_{j+k}-x_j} \qquad \dots (6)$$

It is easily shown that this recurrence relation does indeed lead to a table of functions $\{f_{jk};\ j \geqslant 1,\ k \geqslant 1\}$ which all possess Property I.

Notice that formula (6) gives f_{jk} as a weighted, linear means of $f_{j,k-1}$ and $f_{j+1,k-1}$. This feature of constructing higher order functions as averages of lower order interpolating functions admits of immediate generalisation when we realise that many other averaging processes could be used instead of a linear mean. For example, let $\{G_{jk}(z);\ j \geqslant 1,\ k \geqslant 1\}$ be single valued functions, and let $\{F_{jk}(z);\ j \geqslant 1,\ k \geqslant 1\}$ be simple (Schlicht) functions, over the ranges required by their arguments. Also, for all $j \geqslant 1,\ k \geqslant 1$, let $F^{-1}_{jk}(z)$ be the unique function inverse to $F_{jk}(z)$,

i.e.

$$F^{-1}_{jk}\{F_{jk}(z)\} \equiv z \quad ; \quad j \geqslant 1,\ k \geqslant 1, \qquad \ldots (7)$$

for all required z. Suppose, also, that

$$G_{jk}(x_{j+k}) \neq G_{jk}(x_j) \qquad \ldots (8)$$

for any $j \geqslant 1,\ k \geqslant 1$. The following theorem then holds.

Theorem 1:

For all tables of function values $\{(x_j, f_j);\ j \geqslant 1\}$ the functions $\{f_{jk}(x);\ j \geqslant 1,\ k \geqslant 1\}$ constructed by means of the recurrence formula

$$f_{jk} = F^{-1}_{jk}\left\{\frac{[G_{jk}(x)-G_{jk}(x_j)]\cdot F_{jk}(f_{j+1,k-1}) + [G_{jk}(x_{j+k})-G_{jk}(x)]\cdot F_{jk}(f_{j,k-1})}{G_{jk}(x_{j+k}) - G_{jk}(x_j)}\right\}, \qquad \ldots (9)$$

under the conditions outlined on the previous paragraph, all possess Property I.

The proof is by induction. Suppose the functions $\{f_{j,k-1}, j \geqslant 1\}$ all possess Property I, and consider separately the three cases

$$\begin{aligned} &(i) && x = x_j \\ &(ii) && x = x_r \in D_{j,k-1} \cap D_{j+k,k-1} \\ &(iii) && x = x_{j+k} \end{aligned} \qquad \ldots (10)$$

Case (i):

From equation (9),

$$f_{jk}(x_j) = F_{jk}^{-1}\left\{F_{jk}\left[f_{j,k-1}(x_j)\right]\right\} = f_{j,k-1}(x_j) \; ;$$

but

$$x_j \in D^I_{j,k-1} \, ,$$

therefore $$f_{jk}(x_j) = f_j . \qquad \ldots (11)$$

Case (ii):

Since

$$x_r \in D_{j,k-1} \cap D_{j+1,k-1} = D^I_{j,k-1} \cap D^I_{j+1,k-1}$$

$$f_{j,k-1}(x_r) = f_r = f_{j+1,k-1}(x_r) . \qquad \ldots (12)$$

Hence, from equation (9),

$$f_{jk}(x_r) = F_{jk}^{-1}\left\{\frac{[G_{jk}(x_r)-G_{jk}(x_j)]F_{jk}(f_r) + [G_{jk}(x_{j+k})-G_{jk}(x_r)]F_{jk}(f_r)}{G_{jk}(x_{j+k}) - G_{jk}(x_j)}\right\}$$

i.e.

$$f_{jk}(x_r) = F_{jk}^{-1}\left\{F_{jk}(f_r)\right\} = f_r \qquad \ldots (13)$$

Case (iii):

It follows, by an argument similar to that of Case (i), that

$$f_{jk}(x_{j+k}) = f_{j+1,k-1}(x_{j+k}) = f_{j+k} \qquad \ldots (14)$$

Equations (11), (13) and (14) imply that if all the functions $\{ f_{j,k-1}(x); \; j \geqslant 1 \}$ possess Property I so do all the functions $\{ f_{jk}(x) \; ; \; j \geqslant 1 \}$. However, the functions $\{f_{jo}; \; j \geqslant 1\}$ trivially possess Property I, so the induction is complete.

Thus, working in a direction of increasing k, a table of functions possessing Property I may be generated by repeated applications of recurrence formulae of the form of equation (9). Moreover, we can construct several different tables, using different choices for the functions $\{ F_{jk}(z), \; G_{jk}(z) \; ; \; j \geqslant 1, \; k \geqslant 1 \}$ and then combine these, by averaging over entries in corresponding positions in the different tables, to form yet another table composed of functions all possessing Property I. Once again, the averaging methods need not be linear, and may be different at different locations (j,k) in the tables.

For some purposes it may be necessary to use functions $F_{jk}(z)$ in formula (9) which are not simple everywhere in the regions accessible to their arguments. In that case there will exist tables of function values $\{(x_j,f_j);\ j \geqslant 1\}$ for which $D^I_{jk} \neq D_{jk}$ for some (j,k), but nevertheless recurrence (9) may often produce a useful table of interpolating functions. If, for example, we use an $F_{jk}(z)$ which is simple everywhere except at a finite number of isolated singularities it could happen that, arising out of the domain $\{x_j;\ 1 \leqslant j \leqslant J,\}$ any necessary evaluation of F_{jk} and F^{-1}_{jk} is well defined, and then the functions $\{f_{jk}; 1 \leqslant j \leqslant J,\ 1 \leqslant k \leqslant J-1\}$ will still all possess Property I. An illustration of this possibility is provided by example (v) of the next section, which will always generate tables of functions possessing Property I unless one of the given values f_r is equal to zero. It seems desirable to investigate such cases individually in order to determine the precise conditions under which recurrence (9) generates a function $f_{jk}(x)$ which fails to interpolate one of the given function values at a point within its domain.

4. SOME SPECIAL RECURRENCE FORMULAE

It is clear that, by varying the choices of $F_{jk}(z)$ and $G_{jk}(z)$ in equation (9), it is possible to construct a wide variety of tables of interpolating functions. Actually, the difficult problem is not one of interpolation, as such, in a given table of function values, but one of choosing the form of the interpolating function to be consistent with known, or assumed, general properties of the originating function $f(x)$. For example, $f(x)$ may be known to be periodic, or non-negative, or it may possess not more than a certain number of singularities of a known type, etc. The following examples illustrate possible choices of $F_{jk}(z)$ and $G_{jk}(k)$ which may be useful in special circumstances.

(i) $F_{jk}(z) \equiv z \equiv G_{jk}(z)$; $j \geqslant 1, k \geqslant 1.$

This gives rise to the Neville-Aitken process for polynomial interpolation, which is, of course, the motivation for the construction of recurrence formula (9).

(ii) $F_{jk}(z) \equiv z, \quad G_{jk}(z) \equiv \frac{1}{z}$; $j \geqslant 1, k \geqslant 1.$

Notice that a choice of $G_{jk}(z)$ to be independent of j and k is equivalent to performing a transformation on the independent variable.

$$x' = G(x) \qquad \ldots (15)$$

In this case we find that interpolating functions of the form

$$f_{jk}(x) = \sum_{r=0}^{k} a_{jr} \cdot x^{-r} \qquad \ldots (16)$$

are generated; i.e. polynomials in inverse powers of x. Clearly the more general case of equation (15) would lead to interpolating functions in the form of polynomials in $G(x)$.

(iii) $F_{jk}(z) \equiv z$; $j \geqslant 1, k \geqslant 1$

$$\left.\begin{aligned} G_{jk}(z) &\equiv z \quad ; \ k \text{ odd} \\ &\equiv \frac{1}{z} \quad ; \ k \text{ even} \end{aligned}\right\} j \geqslant 1.$$

This leads to a table of interpolating functions which in general contain both positive and negative powers of x; thus;

$$\begin{aligned} f_{jk} &= \sum_{r=-\frac{k}{2}}^{\frac{k}{2}} a_{jr} \cdot x^{r} \ ; \ k \text{ even} \\ &= \sum_{r=\frac{1-k}{2}}^{\frac{k+1}{2}} a_{jr} \cdot x^{r} \ ; \ k \text{ odd} \end{aligned} \qquad \ldots (17)$$

(iv) $F_{jk}(z) \equiv z$; $j \geqslant 1, k \geqslant 1$

$$\left.\begin{aligned} G_{jk}(z) &\equiv e^{i\lambda z} \quad ; \ k \text{ odd} \\ &\equiv e^{-i\lambda z} \quad ; \ k \text{ even} \end{aligned}\right\} j \geqslant 1,$$

where λ is a real constant, independent of j and k. By comparison with case (iii) we see that this choice of $F_{jk}(z)$ and $G_{jk}(z)$ must give rise to a table of interpolating functions $f_{jk}(x)$ which in general contain both positive and negative powers of $e^{i\lambda x}$, up to a degree whose modulus is $\frac{k}{2}$ when k is even and $\frac{k+1}{2}$ when k is odd. Now let us construct a second table using the same $F_{jk}(z)$, but taking

$$\left.\begin{aligned} \bar{G}_{jk}(z) &\equiv e^{-i\lambda z} \quad ; \; k \text{ odd} \\ &\equiv e^{i\lambda z} \quad ; \; k \text{ even} \end{aligned}\right\} \; j \geqslant 1.$$

We next form a third table by taking arithmetic means of quantities in corresponding positions in the previous two tables, and we see that if the $\{f_j; j \geqslant 1\}$ are real quantities this third table will consist of interpolating functions of the form

$$\begin{aligned} f_{jk}(x) &= \sum_{r=0}^{\frac{k}{2}} \left\{ \alpha_{jr} \cdot \text{Cos}(r\lambda x) + \beta_{jr} \cdot \text{Sin}(r\lambda x) \right\} ; \quad k \text{ even} \\ &= \sum_{r=0}^{\frac{k+1}{2}} \left\{ \alpha_{jr} \cdot \text{Cos}(r\lambda x) + \beta_{jr} \cdot \text{Sin}(r\lambda x) \right\} ; \quad k \text{ odd} \end{aligned} \qquad j \geqslant 1, \quad \ldots (18)$$

where the $\{\alpha_{jr}\}$ and $\{\beta_{jr}\}$ are real quantities.

Thus, $f_{jk}(x)$ is a trigonometric series with period $\frac{2\pi}{\lambda}$, and of least order necessary in general for interpolating the $k+1$ consecutive points $\{(x_r, f_r); j \leqslant r \leqslant j+k\}$. When k is odd $f_{jk}(x)$ possesses $k+2$ assignable constants - one more than may seem strictly necessary for interpolating $k+1$ general points - although when k is even $f_{jk}(x)$ possesses exactly $k+1$ assignable constants. However, it is worth noting that if, when k is odd, either of the terms in $\text{Sin}\{(k+1)\frac{\lambda x}{2}\}$ or $\text{Cos}\{(k+1)\frac{\lambda x}{2}\}$ is discarded then the resulting form of function will not be sufficiently general to interpolate all given sets of $k+1$ points.

In practice, of course, only the first of the above three tables need be constructed, since the rest of the procedure is only a device for fitting the process of neglecting imaginary parts into the general scheme for constructing tables of interpolating functions.

(v) $F_{jk}(z) \equiv \frac{1}{z}$, $G_{jk}(z) \equiv z$; $j \geqslant 1$, $k \geqslant 1$.

It is easy to verify that this choice of $F_{jk}(z)$ and $G_{jk}(z)$ leads to interpolating functions of the form

$$f_{jk}(x) = \left\{ \sum_{r=0}^{k} a_{jr} \cdot x^r \right\}^{-1} , \qquad \ldots (19)$$

i.e. to meromorphic functions with all their poles in the finite part of the complex plane and all their zeros at infinity. This is in contrast with case (i) which gives rise to meromorphic functions with all their zeros in the finite part of the complex plane and all their poles at infinity. Notice that $F_{jk}(z)$ is simple everywhere except at zero, and that no $f_{jk}(x)$ can be constructed which interpolates a point (x_t, f_t) if f_t equals zero.

(vi) $F_{j1}(z) \equiv z \;\; ; \;\; j \geqslant 1,$

$$F_{jk}(z) \equiv \frac{1}{z - f_{j+1,k-2}} \;\; ; \;\; j \geqslant 1, \; k \geqslant 2$$

$$G_{jk}(z) \equiv z \;\; ; \;\; j \geqslant 1, \; k \geqslant 1.$$

Notice again that when $k \geqslant 2$, $F_{jk}(z)$ is simple everywhere except where $z = f_{j+1,k-2}$.

Under this choice the quantities $\{f_{j1} ; j \geqslant 1\}$ are formed, as in case (i), by

$$f_{j1} = \frac{(x - x_j) \cdot f_{j+1} + (x_{j+1} - x) \cdot f_j}{x_{j+1} - x_j} , \qquad \ldots (20)$$

while for $k \geqslant 2$ we have the rhombus rule

$$f_{jk} = f_{j+1,k-2} + \frac{x_{j+k} - x_j}{\dfrac{x - x_j}{f_{j+1,k-1} - f_{j+1,k-2}} + \dfrac{x_{j+k} - x}{f_{j,k-1} - f_{j+1,k-2}}} \qquad \ldots (21)$$

It is obvious by inspection of formula (21) that the functions $f_{jk}(x)$ are in general rational if $k \geqslant 2$, but the form of the dependence upon k of the degrees of the polynomials constituting numerator and denominator of $f_{jk}(x)$ is by no means clear. However, it may be shown (Stoer 1961) and (Larkin, 1966) that $f_{jk}(x)$ may be written in the form

$$f_{jk}(x) = \frac{\sum_{r=0}^{\frac{k}{2}} p_{jkr} \cdot x^r}{\sum_{r=0}^{\frac{k}{2}} q_{jkr} \cdot x^r} \;\; ; \;\; j \geqslant 1, \; k \text{ even}, \qquad \ldots (22)$$

and

$$f_{jk}(x) = \frac{\sum_{r=0}^{\frac{k+1}{2}} p_{jkr} \cdot x^r}{\sum_{r=0}^{\frac{k-1}{2}} q_{jkr} \cdot x^r} \quad ; \quad j \geqslant 1, \text{ k odd,} \qquad \ldots (23)$$

where the p_{jkr} and q_{jkr} are constants. Hence, the form of $f_{jk}(x)$ is identical to the rationalised form of the Thiele continued fraction (Thiele, 1909)

$$g_{jk}(x) = a_0 + \cfrac{x-x_j}{a_1 + \cfrac{x-x_{j+1}}{a_2 + \cdots \cfrac{x-x_{j+k-1}}{a_k}}} \qquad \ldots (24)$$

which interpolates the $k+1$ points $\{(x_r, f_r); j \leqslant r \leqslant j+k\}$. However, it may be shown (e.g. Hildebrand, 1956) that if this rational interpolating function exists it must be unique, which implies that

$$f_{jk}(x) \equiv g_{jk}(x) \qquad \ldots (25)$$

It is also shown (Stoer 1961) and (Larkin 1966) that rational interpolation functions of the more general form

$$f_{jk}(x) = \frac{\sum_{r=0}^{s} p_{jkr} \cdot x^r}{\sum_{r=0}^{t} q_{jkr} \cdot x^r}, \qquad \ldots (26)$$

where

$$s + t = k, \qquad \ldots (27)$$

may be constructed by judicious use of recurrence relations (20) and (21), and the recurrence associated with case (v), which is

$$f_{jk} = \frac{x_{j+k} - x_j}{\frac{x-x_j}{f_{j+1,k-1}} + \frac{x_{j+k}-x}{f_{j,k-1}}} \qquad \ldots (28)$$

Thus, if we choose

$$\text{(vii)}\quad \left.\begin{aligned} F_{jk}(z) &\equiv z &&; \quad 1 \leqslant k \leqslant i \\ &\equiv \frac{1}{z - f_{j+1,k-2}} &&; \quad k \geqslant i+1 \end{aligned}\right\} \; j \geqslant 1.$$

$$G_{jk}(z) \equiv z \quad ; \quad j \geqslant 1, \quad k \geqslant 1$$

in general we obtain interpolating functions of the form

$$f_{jk} = \frac{P_{jk}(x)}{Q_{jk}(x)} \qquad \ldots (29)$$

where $P_{jk}(x)$ and $Q_{jk}(x)$ are polynomials. Moreover, writing $\deg\{P\}$ for the degree of the polynomial $P(x)$, it turns out that

$$\left.\begin{aligned} \deg\{P_{jk}\} &\leqslant k \\ \deg\{Q_{jk}\} &= 0 \end{aligned}\right\} ; \quad j \geqslant 1, \; 0 \leqslant k \leqslant i \qquad \ldots (30)$$

and

$$\left.\begin{aligned} \deg\{P_{jk}\} &\leqslant k - \left[\frac{k-i+1}{2}\right] \\ \deg\ Q_{jk} &\leqslant \left[\frac{k-i+1}{2}\right] \end{aligned}\right\} ; \quad j \geqslant 1, \; k \geqslant i+1, \qquad \ldots (31)$$

where the symbol $[y]$ indicates "the largest integer $\leqslant y$."

The choice

$$\text{(viii)}\quad \left.\begin{aligned} F_{jk}(z) &\equiv \frac{1}{z} &&; \quad 1 \leqslant k \leqslant i \\ &\equiv \frac{1}{z-f_{j+1,k-2}} &&; \quad k \geqslant i+1 \end{aligned}\right\} \; j \geqslant 1$$

$$G_{jk}(z) \equiv z \quad ; \quad j \geqslant 1, \; k \geqslant 1$$

leads to functions similar in form to those of the previous example, except that

$$\left.\begin{aligned} \deg\{P_{jk}\} &= 0 \\ \deg\{Q_{jk}\} &\leqslant k \end{aligned}\right\} ; \quad j \geqslant 1, \; 0 \leqslant k \leqslant i \qquad \ldots (32)$$

and

$$\left.\begin{aligned} \deg\{P_{jk}\} &\leqslant \left[\frac{k-i+1}{2}\right] \\ \deg\{Q_{jk}\} &\leqslant k - \left[\frac{k-i+1}{2}\right] \end{aligned}\right\} ; \quad j \geqslant 1, \; k \geqslant i+1 . \qquad \ldots (33)$$

Stoer, (1961) also introduces a convenient technique for constructing the values of the coefficients in the polynomials which constitute numerators and denominators of the rational functions generated by the recurrences in cases (vi), (vii) and (viii).

5. APPLICATION OF INVARIANCE CONDITIONS

In view of the wide choice available for the functions $\{F_{jk}, G_{jk};\ j \geqslant 1,\ k \geqslant 1\}$ in recurrence relation (9) it is of interest to consider how the field may be narrowed by placing reasonable restrictions upon properties of the interpolating functions $\{f_{jk};\ j \geqslant 1,\ k \geqslant 1\}$.

One feature which is usually, although not invariably, desirable in an interpolating function $f_{jk}(x)$ is the following:-

(a) The form of the function shall be independent of the origin of the x coordinate; that is, it shall be invariant under an arbitrary, uniform translation α, in the x direction, of the given tabular values $\{f_r; j \leqslant r \leqslant j+k\}$. Writing in the explicit dependence of f_{jk} upon x and $\{x_r; j \leqslant r \leqslant j+k\}$, this may be expressed as

$$f_{jk}(x+\alpha,\ x_r+\alpha) = f_{jk}(x, x_r) \qquad \ldots (34)$$

where the dependence upon x_r denotes dependence on <u>all</u> $x_r \in D_{jk}$.

<u>Theorem 2</u>

<u>A necessary and sufficient condition that recurrence formula (9) generate a table of functions which satisfy condition (a) for all given values</u> $\{f_r; j \leqslant r \leqslant j+k\}$ <u>is that, for all</u> r <u>and</u> s <u>satisfying</u> $j \leqslant r \leqslant j+k-s,\ 1 \leqslant s \leqslant k$,

<u>either</u>
$$G_{rs}(z) \equiv A_{rs} z + B_{rs}$$
<u>or</u>
$$G_{rs}(z) \equiv A_{rs} e^{B_{rs} z} + C_{rs} \qquad \ldots (35)$$

<u>where</u> A_{rs}, B_{rs} <u>and</u> C_{rs} <u>are constants</u>.

The proof is as follows:

(i) Sufficiency:

Suppose the functions $\{f_{rs};\ j \leqslant r \leqslant j+k-s,\ 1 \leqslant s \leqslant k-1\}$ all satisfy condition (a). If either of the forms for G_{jk}, in equations (35), is substituted into (9) for the purpose of constructing $\{ f_{jk}(x+\alpha, x_r+\alpha)\}$ it is clear that the explicit dependence on α disappears from the formula. However, since $f_{j,k-1}$ and $f_{j+1,k-1}$ satisfy condition (a) they cannot contribute any implicit dependence on α, so f_{jk} must also satisfy condition (a). The induction is completed by noting that the functions $\{ f_{jo}; j \geqslant 1 \}$ all trivially satisfy condition (a), thus establishing the sufficiency of equations (35).

(ii) Necessity:

Once again suppose that the functions $\{ f_{rs}; j \leqslant r \leqslant j+k-s, 0 \leqslant s \leqslant k-1\}$ all satisfy condition (a), and notice that, since $f_{j,k-1}$ depends on x_j and f_j and $f_{j+1,k-1}$ does not, $F_{jk}(f_{j,k-1})$ and $F_{jk}(f_{j+1,k-1})$ may vary independently. Thus, given that

$$f_{j+1,k-1}(x+\alpha, x_r + \alpha) \equiv f_{j+1,k-1}(x,x_r) \qquad \ldots (36)$$

for all $x_r \in D_{jk}$, it is clear that f_{jk} will not satisfy condition (a) for all given base points $\{ x_j; j \geqslant 1 \}$ unless the expression

$$\frac{G_{jk}(x+\alpha) - G_{jk}(y+\alpha)}{G_{jk}(z+\alpha) - G_{jk}(y+\alpha)} \qquad \ldots (37)$$

is independent of α, for all x,y and z. Let us now discard the suffices and write

$$\frac{G(x+\alpha) - G(y+\alpha)}{G(z+\alpha) - G(y+\alpha)} = H(x,y,z) \qquad \ldots (38)$$

Transposing y and z in equation (38) yields

$$\frac{G(z+\alpha) - G(x+\alpha)}{G(z+\alpha) - G(y+\alpha)} = H(x,z,y) \qquad \ldots (39)$$

We now consider the special circumstances

$$\left.\begin{aligned} y &= x - h \\ z &= x + h \end{aligned}\right\} ; \qquad \ldots (40)$$

and note that subtracting equation (38) from (39), and letting h tend to zero, leads to the result

$$\frac{G''(x+\alpha)}{G'(x+\alpha)} = 2.\lim_{h \to 0} \left\{ \frac{H(x,x+h,x-h)-H(x,x-h,x+h)}{h} \right\} = K(x), \text{ say.} \qquad \ldots (41)$$

However, from equation (41),

$$\frac{\partial K}{\partial x} = \frac{\partial K}{\partial \alpha} = 0 \qquad \ldots (42)$$

so that K must be a constant, say λ.

Thus

$$\frac{G''(z)}{G'(z)} = \lambda \qquad \ldots (43)$$

Now, if $\lambda = 0$ the solution of equation (43) is

$$G(z) = Az + B \qquad \ldots (44)$$

and if $\lambda \neq 0$ the solution is

$$G(z) = Ae^{\lambda z} + B, \qquad \ldots (45)$$

and these two forms for $G(z)$ are precisely those required by equations (35). This establishes the "necessity" part of the theorem and completes the proof.

Another condition which could be imposed is the following:-

(b) A function $f_{jk}(x)$ which is intended to interpolate the points $\{(x_r,f_r);\ j \leqslant r \leqslant j+k\}$ shall be invariant under the application of an arbitrary, constant scale factor α to the quantities x and $\{x_r;\ j \leqslant r \leqslant j+k\}$. This may be expressed as

$$f_{jk}(\alpha x,\alpha x_r) = f_{jk}(x,x_r) \qquad \ldots (46)$$

where, as in equation (34), the x_r dependence indicates dependence on <u>all</u> $x_r \in D_{jk}$. By arguments similar to those of Theorem 2 it may be shown that:-

<u>Theorem 3</u>

<u>A necessary and sufficient condition that recurrence formula (9) generate a table of functions which satisfy condition (b) for all given values</u> $\{f_r; j \leqslant r \leqslant j+k\}$ <u>is that, for all r and s satisfying</u> $j \leqslant r \leqslant j+k-s,\ 1 \leqslant s \leqslant k$,

<u>either</u>

$$G_{rs}(z) = A_{rs}\log(z) + B_{rs}$$

<u>or</u>

$$G_{rs}(z) = A_{rs}z^{B_{rs}} + C_{rs} \qquad \ldots (47)$$

where A_{rs}, B_{rs} and C_{rs} are constants.

Conditions (a) and (b) above express invariance of the interpolation functions $\{f_{jk}(x);\ j \geqslant 1,\ k \geqslant 1\}$ under translation in the x direction and x-scale magnification respectively, and they both result in restrictions upon the form of the functions $\{G_{jk}(z);\ j \geqslant 1,\ k \geqslant 1\}$. Similar conditions may be imposed on the behaviour of the interpolating functions under translation or scale magnification in the f direction, and these will in general result in restrictions upon the form of the functions $\{F_{jk}(z);\ j \geqslant 1,\ k \geqslant 1\}$. Thus, we could impose the conditions:-

(c) A function $f_{jk}(x)$ which is intended to interpolate the points $\{(x_r,f_r);\ j \leqslant r \leqslant j+k\}$ shall be uniformly translated by an amount α in the f direction as a result of adding an arbitrary constant α to all of the ordinate values $\{f_r;\ j \leqslant r \leqslant j+k\}$. This may be expressed as

$$f_{jk}(x,f_r + \alpha) = \alpha + f_{jk}(x,f_r) \qquad \text{... (48)}$$

where, here and in condition (d) below, the f_r dependence is intended to represent dependence on all the ordinates $\{f_r;\ j \leqslant r \leqslant j+k\}$, and

(d) A function $f_{jk}(x)$ which is intended to interpolate the points $\{(x_r,f_r);\ j \leqslant r \leqslant j+k\}$ shall be multiplied by an amount α as a result of multiplying all of the ordinate values $\{f_r;\ j \leqslant r \leqslant j+k\}$ by an arbitrary constant α. This may be expressed as

$$f_{jk}(x,\alpha f_r) = \alpha f_{jk}(x,f_r) \qquad \text{... (49)}$$

Transformation conditions (c) and (d) allow rather more variety in the $\{F_{jk}\}$ and $\{G_{jk}\}$ than do invariance conditions (a) and (b). This arises because, even though $G_{jk}(z)$ may depend explicitly on any of the quantities $\{f_{rs};\ j \leqslant r \leqslant j+k-s, 0 \leqslant s \leqslant k\}$ since these quantities are strictly invariant under the transformations mentioned in conditions (a) and (b) the proofs of Theorems 2 and 3 are not invalidated; however the situation is rather more complicated if $F_{jk}(z)$ depends explicitly on one or more of the $\{f_{rs};\ j \leqslant r \leqslant j+k,\ 0 \leqslant s \leqslant k\}$. The simplest case is covered by the following theorems.

Theorem 4:

If none of the functions $\{F_{rs}(z), G_{rs}(z);\ j \leqslant r \leqslant j+k-s,\ 1 \leqslant s \leqslant k\}$ depends explicitly on any of the quantities $\{f_{rs};\ j \leqslant r \leqslant j+k-s,\ 0 \leqslant s \leqslant k\}$ then a necessary and sufficient condition that recurrence (9) generate a table of functions which satisfy condition (c) for any given values $\{f_r; j \leqslant r \leqslant j+1\}$ is that, for all r and s satisfying $j \leqslant r \leqslant j+k-s,\ 1 \leqslant s \leqslant k$,

either

$$\left.\begin{aligned} F_{rs}(z) &= A_{rs}z + B_{rs} \\ &\text{or} \\ F_{rs}(z) &= A_{rs}\cdot e^{B_{rs}z} + C_{rs} \end{aligned}\right\} \quad \ldots (50)$$

where A_{rs}, B_{rs} and C_{rs} are constants.

(i) Sufficiency:

Suppose all the functions $\{f_{rs};\ j \leqslant r \leqslant j+k-s,\ 0 \leqslant s \leqslant k-1\}$ satisfy condition (c). We then require that

$$F_{jk}(f_{jk}+\alpha) = \beta F_{jk}(f_{j+1,k-1}+\alpha) + (1-\beta)F_{jk}(f_{j,k-1}+\alpha), \quad \ldots (51)$$

for any β, independently of α.

It is clear, by trying either of expressions (50), that they both are indeed sufficient for the satisfaction of equation (51). However, the quantities $\{f_r;\ j \leqslant r \leqslant j+k\}$ trivially satisfy condition (c), so the induction is complete.

(ii) Necessity:

Supposing again that all the functions $\{f_{rs};\ j \leqslant r \leqslant j+k-s,\ 0 \leqslant s \leqslant k-1\}$ satisfy condition (c), notice that equation (51) may be rearranged in the form

$$\frac{F_{jk}(f_{jk}+\alpha) - F_{jk}(f_{j,k-1}+\alpha)}{F_{jk}(f_{j+1,k-1}+\alpha) - F_{jk}(f_{j,k-1}+\alpha)} = \beta \quad \ldots (52)$$

which must be independent of α. However, equation (52) is of the same form as equation (38), so the same reasoning may be used to show that $F_{jk}(z)$ must indeed be of the form required by the Theorem.

Similarly, it may be shown that

Theorem 5:

If none of the functions $\{F_{rs}(z); j \leqslant r \leqslant j+k-s,\ 1 \leqslant s \leqslant k\}$ depends explicitly on any of the quantities $\{f_{rs}; j \leqslant r \leqslant j+k-s,\ 0 \leqslant s \leqslant k\}$ than a necessary and sufficient condition that recurrence (9) generate a table of functions which satisfy condition (d) for any given values $\{f_r; j \leqslant r \leqslant j+k\}$ is that, for all r and s satisfying $j \leqslant r \leqslant j+k-s,\ 1 \leqslant s \leqslant k$

$$\left.\begin{array}{ll} \text{either} & F_{rs}(z) = A_{rs}\log(z) + B_{rs} \\ \text{or} & F_{rs}(z) = A_{rs} z^{B_{jk}} + C_{rs} \end{array}\right\} \quad \ldots (53)$$

where A_{rs}, B_{rs} and C_{rs} are constants.

Corollary:

If none of the functions $\{F_{rs}; j \leqslant r \leqslant j+k-s,\ 1 \leqslant s \leqslant k\}$ depends explicitly on any of the quantities $\{f_{rs}; j \leqslant r \leqslant j+k-s,\ 0 \leqslant s \leqslant k\}$ the only function constructed by application of recurrence formula (9) which satisfies all of conditions (a), (b), (c) and (d) is the unique polynomial of degree less than or equal to k which interpolates the points $\{(x_r, f_r); j \leqslant r \leqslant j+k\}$.

If $F_{jk}(z)$ is allowed to depend explicitly on one or more of the quantities $\{f_{rs}; j \leqslant r \leqslant j+k-s,\ 0 \leqslant s \leqslant k\}$ a much wider range of function forms may be constructed which satisfy conditions (c) and (d). For example, if we consider recurrence (21), where

$$F_{jk}(z, f_{j+1,k-2}) = F^*_{jk}(z - f_{j+1,k-2}) \text{, say,} \quad \ldots (54)$$

it is clear that, provided the functions $\{f_{r1}; j \leqslant r \leqslant j+k-1\}$ satisfy condition (c), all functions in the table constructed by using the form (54) in recurrence (9) will automatically satisfy the condition. It only remains to ensure that the $\{f_{r1}; j \leqslant r \leqslant j+k-1\}$ also satisfy condition (d), and to restrict $F^*_{jk}(\theta)$ to either of the forms in (53), when all the functions $\{f_{rs}; j \leqslant r \leqslant j+k-s, 0 \leqslant s \leqslant k\}$ will satisfy both of conditions (c) and (d). Examples (vi), (vii) and (viii) in section 4 fall into this category. Similarly, if we take

$$F_{jk}(z,f_{j+1,k-2}) = F^*_{jk}\left(\frac{z}{f_{j+1,k-2}}\right) \qquad \ldots (55)$$

and ensure that the functions $\{ f_{r1};\ j \leqslant r \leqslant j+k-1 \}$ all satisfy condition (d), all functions in the table constructed by using the form (55) in recurrence (9) will satisfy condition (d). Furthermore, if the $\{ f_{r1};\ j \leqslant r \leqslant j+k-1\}$ also satisfy condition (c) and $F^*_{jk}(\theta)$ is of either of the forms in equation (50), all the functions $\{ f_{rs};\ j \leqslant r \leqslant j+k-s,\ 0 \leqslant s \leqslant k \}$ will satisfy both conditions (c) and (d).

As indicated in section 4, the forms chosen for the functions $\{ F_{rs}, G_{rs};\ j \leqslant r \leqslant j+k-s,\ 1 \leqslant s \leqslant k\}$ should be governed by those features of the originating function $f(x)$ which it is desired to preserve in the interpolating function $f_{jk}(x)$. Conditions (a), (b), (c) and (d) probably represent the most usual invariance properties which may be required for an interpolation function, but other invariance properties may be desirable in special circumstances.

6. INVERSE INTERPOLATION

Tables of the form of Table 1 lend themselves very conveniently to inverse interpolation. For example, suppose we have two estimates x_1 and x_2 of a zero of $f(x)$ improved estimates may be found by the following iterative process. We construct Table 2, regarding x as a function of the independent variable f, thus:

TABLE 2

INVERSE INTERPOLATION BY TABULAR CONSTRUCTION

f_1	x_1				
		x_{11}			
f_2	x_2		x_{12}		
		x_{21}		x_{13}	
f_3	x_3		x_{22}		x_{14}
		x_{31}		x_{23}	
f_4	x_4		x_{32}		
		x_{41}			
f_5	x_5				
f_6	x_6				

Using a recurrence formula analogous to (9), with the roles of x and f transposed, we determine x_{11} by interpolation between the points (f_1, x_1) and (f_2, x_2) at the position where $f=0$. We then set

$$x_3 = x_{11} , \qquad \dots (56)$$

compute

$$f_3 = f(x_3) , \qquad \dots (57)$$

and thence construct x_{21} and x_{12} from the recurrence formula. Continuing, we set

$$x_4 = x_{12} , \qquad \dots (58)$$

compute

$$f_4 = f(x_4) \qquad \dots (59)$$

and proceed to compute x_{31}, x_{22} and x_{13} from the recurrence formula. The process is terminated when the modulus of

$$f_{j+1} = f(x_{j+1}) = f(x_{1,j-1}) \qquad \dots (60)$$

is considered small enough.

Of course, it may not be desirable to extend Table 2 further to the right than some prescribed column, say the k^{th}. In that case the sequence of estimates of the required zero of $f(x)$ is generated by

$$\left.\begin{aligned} x_{j+1} &= x_{1,j-1} \quad ; \quad 2 \leqslant j \leqslant k+1 \\ &= x_{j-k,k} \quad ; \quad j \geqslant k+2 \end{aligned}\right\} \qquad \dots (61)$$

The Neville-Aitken form of the above inverse interpolation process is described, for example by Hildeband (1956). For a given function $f(x)$ it may be possible to devise a recurrence relation for which the above process converges particularly quickly. However, for general purposes, either the Neville-Aitken process or recurrence (21) may be used. Both methods are attractive, in the case of single zeros, giving nearly quadratic convergence at the expense of only one function evaluation per iteration step (Ostrowski, 1960). Table 3 illustrates the use of recurrence (21) for inverse interpolation

TABLE 3

RATIONAL INVERSE INTERPOLATION; ITERATION TO A ZERO OF $x(x-1)$

j	f_j	x_j	$k = 1$	2	3	4
1	-0.25	0.5				
			0.75			
2	0.75	1.5		1.125		
			0.90		1.0125	
3	-0.1875	0.75		0.98780488		1.00015244
			0.96428571		0.99984761	
4	0.140625	1.125		0.99954296		0.99999998
			1.00137363		0.99999993	
5	0.01265625	1.0125		1.00000021		
			1.00000188			
6	0.00015246	1.00015244				
		0.99999998				

Starting from the estimates $x_1 = 0.5$, $x_2 = 1.5$ five iterations are required in order to reach an accuracy of 2 parts in 10^8. Notice that $x_1 = 0.5$ is a particularly bad choice for an initial estimate of the zero at $x = 1$, since it lies midway between two zeros, at a point where $f'(x) = 0$. Notice also that, for the purposes of automatic computation, only the last computed diagonal line of auxiliary quantities need be retained from one step in order to initiate the next.

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