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ADIABATIC INVARIANTS AND THE EQUILIBRIUM OF MAGNETICALLY TRAPPED PARTICLES

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A B S T R A C T

This paper deals with two topics, firstly with the conditions for plasma equilibrium in an arbitrary magnetic field and their relation to the lowest order particle adiabatic invariants, secondly with the form of the higher order contributions to these adiabatic invariants. In part I the equilibrium conditions are investigated in a systematic way: as the time scale of equilibrium is increased the constraints on the distribution function become more severe until they culminate in the requirement that it be a function of the lowest order adiabatic invariants. In part II it is shown that this discussion of equilibrium leads to a convenient and novel way of generating the adiabatic invariants, not just to lowest order but including higher order contributions, for which a recurrence formula is derived. When the first correction to the longitudinal invariant $J = \oint v_{\parallel}\, ds$ is computed some interesting differences are found between the case of particles oscillating between mirrors and that of particles circulating round closed field lines. Part III discusses the effect of electric fields and the extension of the calculations to time dependent magnetic fields, leading to the third adiabatic invariant (the flux invariant). Part IV deals with the case of toroidal magnetic fields possessing magnetic surfaces and the form of longitudinal invariant appropriate in such a field. In the case of small rotational transform a modified line integral for J leads to a convenient description of particle motions in toroidal systems, including the effects of both rotational transform and guiding center drifts.

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CONTENTS

PART I

1. INTRODUCTION

In discussions of low pressure plasma confined by magnetic mirrors two models have been widely used; each model leads to criteria which must be satisfied in equilibrium. In the fluid model the necessary and sufficient conditions[1] for equilibrium are

$$\frac{\partial p_{\parallel}}{\partial s} + \frac{(p_{\perp} - p_{\parallel})}{B}\frac{\partial B}{\partial s} = 0$$

and ... (1.1)

$$\int \nabla (p_{\perp} + p_{\parallel}) \cdot \frac{\underset{\sim}{B} \times \nabla B}{B^4}\, ds = 0$$

where the integral is along the line of force. On the other hand, in the guiding center model the necessary and sufficient condition for equilibrium[2,3] is that the guiding center distribution function F should depend only on μ, J, and the energy ε i.e.

$$F_{eq} \equiv F(\mu, \varepsilon, J) \quad . \qquad \ldots(1.2)$$

where $\mu \equiv v_{\perp}^2/2B$ and

$$J(\mu, \varepsilon, \alpha, \beta) \equiv \int_{\alpha,\beta} [2(\varepsilon - \mu B)]^{\frac{1}{2}}\, ds \qquad \ldots (1.3)$$

are the lowest order adiabatic invariants. [The magnetic field is $\underset{\sim}{B} = \nabla\alpha \times \nabla\beta$ and α, β label a line of force.]

The first part of the present work is an investigation of the relationship between these two equilibrium criteria and of the general rôle of adiabatic invariants in equilibria. The problem is approached through the Vlasov equation when the essential distinction between different approximate equilibria is the time for which the distribution function f can be regarded as stationary. The time scales of interest are investigated systematically by expanding f in powers of m/e, which is equivalent to an expansion in powers of the Larmor radius[4]. As expected, we find that as the time scale is lengthened, increasingly restrictive conditions are imposed on the distribution function, culminating in restrictions equivalent to those of the guiding center model. The fluid conditions do <u>not</u> emerge directly but nevertheless are shown to be appropriate to an intermediate time scale in most circumstances.

In the second part of this paper we show that this discussion of equilibrium leads naturally to a novel and powerful way of obtaining expressions for the adiabatic invariants

$\hat{\mu}$ and $\hat{J}$, not merely to lowest order in m/e but including higher order contributions(7), for which a formal recurrence relation is derived. This is possible because the invariants, while not true constants, are constant to all orders[5]. Hence, within the scope of any expansion $F(\hat{\mu}, \hat{J}, \varepsilon)$ can be regarded as an <u>exact</u> equilibrium and by comparing this form with the results of the direct m/e expansion one can recognise the exact particle invariants $\hat{\mu}$ and $\hat{J}$, even though the concept of such invariants was not originally introduced into the calculations. This method of obtaining invariants by first finding an equilibrium distribution circumvents the necessity for any calculation of orbits, either of particles or of guiding centres, and may have application in other problems.

Using this method we have explicitly calculated the first order corrections μ_1 and J_1 and the second order correction μ_2. An unexpected feature of our results is that the correction J_1 to the longitudinal adiabatic invariant has one form for particles which are trapped between magnetic mirrors and another form for particles which circulate unidirectionally around a closed field line.

In Part III we extend the calculation to include the effect of electric fields and time varying magnetic fields. This allows us to derive the third (flux) invariant Φ by a natural extension of the methods of Parts I and II. If the time scale for the field variation is sufficiently slow this third invariant replaces the particle energy as a "constant of the motion" and a solution of Vlasov's equation is of the form $f = f(\mu, J, \Phi)$.

Part IV deals with toroidal magnetic fields possessing magnetic surfaces and the longitudinal invariant (replacing J) appropriate in such a field. One possibility is that J should be replaced by an integral over a magnetic surface, but in the case of small rotational transform a modified line integral is more appropriate. This line integral J^{**} provides a convenient description of particle behaviour in toroidal systems.

2. PLASMA EQUILIBRIUM

In the fluid description of low pressure plasma the necessary and sufficient conditions for equilibrium are those[1] given by equations (1.1), whereas in the guiding center (g.c.) description of the plasma, an equilibrium is described[2,3] by any distribution function of the form (1.2). It may easily be shown, by direct substitution, that a g.c. equilibrium

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(7) It is important to distinguish between the quantities μ and J which are defined e.g. by equation (1.3) and the true adiabatic invariants denoted here by $\hat{\mu}$ and $\hat{J}$. The true invariants are equal to μ, J only to lowest order in the m/e expansion. It might be more appropriate to use μ_0, J_0 instead of μ, J but we wish to avoid excessive subscripts. Thus $\hat{\mu} \equiv \mu + \frac{m}{e}\mu_1 + (\frac{m}{e})^2 \mu_2 \ldots$, $\hat{J} \equiv J + \frac{m}{e} J_1 + (\frac{m}{e})^2 J_2 \ldots$.

distribution automatically satisfies the fluid constraints. On the other hand, there are many g.c. distributions which are not of the form (1.2) but which nevertheless lead to pressures satisfying the conditions for fluid equilibrium. Indeed BenDaniel[6] has pointed out that if the g.c. distribution leads to isotropic pressure it cannot be expressed in the form (1.2) even though the corresponding pressure distribution may satisfy (1.1). One can also construct g.c. distribution functions which lead to anisotropic pressure distributions satisfying (1.1) but which cannot be written in the form (1.2). An example, discussed in Appendix A, is the distribution function

$$F_i = H_i(\mu, \varepsilon)\, Q(\langle J \rangle) \qquad \ldots (2.1)$$

where

$$\langle J \rangle = \sum m_i \int H_i(\mu, \varepsilon)\, J(\mu, \varepsilon, \alpha, \beta)\, d\mu\, d\varepsilon \qquad \ldots (2.2)$$

which cannot, in general, be expressed in the form of a g.c. equilibrium; but nevertheless leads to a pressure tensor satisfying the fluid constraints (1.1). It is clear then, that the fluid and guiding center descriptions are not equivalent and in the following section we will show that this is because they refer to different time scales.

3. LARMOR RADIUS EXPANSION

The physically interesting time scales are those set by the Larmor frequency ω_c, the frequency of motion along the lines of force, $v_{\parallel}/L$ and the frequency with which the guiding center drifts around the system, V_d/L. These three frequencies may also be expressed as

$$\omega_c \quad , \quad \frac{a}{L}\,\omega_c \quad , \quad \frac{a^2}{L^2}\,\omega_c$$

where a is the typical Larmor radius so that the criteria for equilibrium on these various time scales can be systematically examined by means of an expansion in powers of a/L. An equivalent but more convenient procedure is to expand in powers of m/e, so following and extending the procedure of Chandrasekhar, Kaufman and Watson[4]. By imposing order by order the condition that the distribution function be stationary we obtain a sequence of constraints appropriate to equilibrium on the various time scales.

The first step is to introduce an appropriate coordinate system (Appendix B). Each line of force is labelled by α, β where $\underset{\sim}{B} = \nabla\alpha \times \nabla\beta$, and the distance s is measured along each field line from some fixed plane, then (α, β, s) are used as position coordinates. Velocity is expressed in terms of $(\varepsilon, \mu, \varphi)$ where $\varepsilon \equiv \frac{1}{2}(v_\perp^2 + v_\parallel^2)$, $\mu \equiv v_\perp^2/2B$ and φ is the azimuthal angle about the field direction $\underset{\sim}{e}_1$. A feature of these velocity coordinates

is that the transformation from $(\varepsilon, \mu, \varphi)$ to $\underset{\sim}{v}$ is two-valued, since $v_{\parallel} = \pm\,[2(\varepsilon - \mu B)]^{\frac{1}{2}}$. To deal with this we explicitly introduce an extra "coordinate" σ which takes only the values ± 1 and indicates which branch of the square root is to be taken. All quantities are thus functions of $(\alpha, \beta, s, \varepsilon, \mu, \varphi, \sigma)$. Although it has been introduced here purely as a formal device to remove an ambiguity in sign, σ will later play a much more fundamental rôle; in some respects it behaves as a constant of the motion along with ε and μ.

When expressed in these coordinates the Vlasov equation

$$\frac{\partial f}{\partial t} + \underset{\sim}{v} \cdot \nabla f + \frac{e}{m} (\underset{\sim}{v} \times \underset{\sim}{B}) \cdot \frac{\partial f}{\partial \underset{\sim}{v}} = 0$$

can be written

$$\frac{\partial f}{\partial \varphi} = \lambda \left(D f + \frac{1}{B} \frac{\partial f}{\partial t}\right) \qquad \ldots (3.1)$$

where $\lambda \equiv m/e$ can be regarded as a formal expansion parameter equivalent to a/L. The operator D is defined by:

$$\begin{aligned} Df = \frac{1}{B} \Big\{ & \sigma q \frac{\partial f}{\partial s} + c_{\perp} \left(\nabla f - \underset{\sim}{V} \frac{\partial f}{\partial \mu}\right) \cdot (\underset{\sim}{e}_2 \cos\varphi + \underset{\sim}{e}_3 \sin\varphi) + \mu\sigma q \frac{\partial f}{\partial \mu} \left[(\rho_3 - \rho_2)\cos 2\varphi + (\tau_2 - \tau_3)\sin 2\varphi\right] \Big\} \\ + \frac{1}{B} \Big\{ & \sigma q (\tau_1 + \tfrac{1}{2}(\tau_2 + \tau_3)) + \left(\frac{q^2}{c_{\perp}} \sigma_1 - c_{\perp}\sigma_2\right)\cos\varphi + \left(\frac{q^2}{c_{\perp}} \rho_1 - c_{\perp}\sigma_3\right)\sin\varphi + \tfrac{1}{2} \sigma q \big[(\tau_2 - \tau_3)\cos 2\varphi \\ & + (\rho_2 - \rho_3) \sin 2\varphi\big] \Big\} \frac{\partial f}{\partial \varphi} \qquad \ldots (3.2) \end{aligned}$$

where

$$q \equiv [2(\varepsilon - \mu B)]^{\frac{1}{2}} \quad , \quad c_{\perp} \equiv (2\,\mu B)^{\frac{1}{2}} \quad , \quad \underset{\sim}{V} \equiv \frac{\mu}{B} \nabla B + \frac{q^2}{B} \underset{\sim}{e}_1 \; \nabla \underset{\sim}{e}_1 \quad . \qquad \ldots (3.3)$$

The unit vectors $\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3$ are orthogonal, with $\underset{\sim}{e}_1 \equiv \underset{\sim}{B}/B$, and the coefficients ρ_i, σ_i, τ_i are related to the curvature and torsion of the lines of force as described in Appendix B. The velocity $\underset{\sim}{V}$ is related to the usual guiding center drift $\underset{\sim}{V}_d$ by

$$\frac{e}{m} \underset{\sim}{V}_d = (\underset{\sim}{e}_1 \times \underset{\sim}{V}) \quad . \qquad \ldots (3.4)$$

We shall look for solutions of (3.1) in the form

$$f = f_0 + \lambda f_1 + \lambda^2 f_2 + \ldots \qquad \ldots (3.5)$$

The time dependence of f_0 may be regarded as being of any order in λ, depending on the time scale one wishes to investigate. We shall regard $\partial f_0/\partial t$ as negligible to successively higher orders in λ and so obtain criteria for equilibrium on successively longer time scales.

(a) Zero Order

In the lowest order we have

$$\frac{\partial f_o}{\partial \varphi} = \frac{m}{eB} \frac{\partial f_o}{\partial t} \qquad \ldots (3.6)$$

indicating merely that if f_o is to be stationary on the time scale of the Larmor period then the appropriate constraint is that it must be independent of the azimuthal angle φ.

(b) First Order

In the next order

$$\frac{\partial f_1}{\partial \varphi} = D f_o + \frac{1}{B} \left(\frac{\partial f_o}{\partial t} \right)_1 \qquad \ldots (3.7)$$

so that if f is again stationary we have

$$f_1 = g_1 (\alpha, \beta, s, \mu, \varepsilon, \sigma) + \int d\varphi \, D f_o \qquad \ldots (3.8)$$

and, since f_1 must be single valued,

$$\langle D f_o \rangle = 0 \qquad \ldots (3.9)$$

where g_1 is an arbitrary function of the indicated variables only, and the angular brackets denote the average over the angle φ. Since f_o is already independent of φ (3.9) becomes

$$\langle D f_o \rangle \equiv \sigma \frac{q}{B} \frac{\partial f_o}{\partial s} = 0 \quad . \qquad \ldots (3.10)$$

Hence the requirement that $\partial f_o / \partial t = 0$ in this order, (corresponding to equilibrium on the time scale $L/V_{\parallel}$) imposes the constraint that f_o must be independent of both φ and s, and that f_1 be given by

$$f_1 = g_1 + \int D f_o \quad . \qquad \ldots (3.11)$$

(here and henceforth, if the variable of integration is not explicitly indicated it is understood to mean integration with respect to φ; other variables of integration will be explicitly indicated). Equation (3.10) indicates, through its Lagrangian subsidiary equations, that in this order, α, β, μ, ε, σ are all "constants of the motion" in accordance with the physical picture of a g.c. tied to a line of force. [But note that neither here nor elsewhere do we introduce the g.c. concept directly.]

(c) Second Order

In this order we have

$$\frac{\partial f_2}{\partial \varphi} - \frac{1}{\lambda B} \frac{\partial f_o}{\partial t} = D f_1 = D g_1 + D \int D f_o \qquad \ldots (3.12)$$

so that if f is stationary in this order we have

$$f_2 = g_2 (\alpha \beta s, \mu \varepsilon \sigma) + \int D g_1 + \int D \int D f_o \quad . \qquad \ldots (3.13)$$

To ensure that f_2 be single valued we must have

$$\langle D g_1 \rangle + \langle D \int D f_o \rangle = 0 \qquad \ldots (3.14)$$

or

$$\frac{\sigma q}{B} \frac{\partial g_1}{\partial s} + \langle D \int D f_o \rangle = 0 \qquad \ldots (3.15)$$

so that

$$g_1 = h_1 (\alpha, \beta, \mu, \varepsilon, \sigma) - \sigma \int_{s_o}^{s} \frac{B ds}{q} \langle D \int D f_o \rangle \qquad \ldots (3.16)$$

where h_1 is arbitrary.

Equation (3.16) leads to a new constraint on f_o; in its simplest form this would arise, e.g., from the requirement that g_1 be single valued in s and would be

$$\oint \frac{B ds}{q} \langle D \int D f_o \rangle = 0 \qquad \ldots (3.17)$$

where the integral is around a closed field line. If $\varepsilon > \mu B$ everywhere along the field line then the constraint <u>does</u> take the simple form (3.17). This is applicable when particles circulate unidirectionally around a closed field line without undergoing mirror reflection. In this case the particle streams moving in either direction, represented by $\sigma = \pm 1$, are independent and the constraint (3.17) applies to each direction individually.

When particles are reflected by magnetic mirrors the situation is less simple because $\varepsilon < \mu B$ over part of the range of s and q becomes imaginary. Physically the two streams represented by $\sigma = \pm 1$ are no longer independent and it is the coupling between them which now leads to a constraint. Mathematically this constraint arises because the two branches ($\sigma = \pm 1$) of the distribution function coincide whenever $\varepsilon = \mu B$ and f_o is independent of σ at these "turning points". However f_o does not vary with s so it must be independent of σ whenever it refers to particles trapped between mirrors.

Similarly g_1 is independent of σ at a turning point (but not elsewhere) and the change in g_1 between turning points must be the same for both $\sigma = \pm 1$. This leads, from (3.16) to the condition that

$$+\int_A^B \frac{B ds}{q} \left[\langle D \int D f_o \rangle \right]_{\sigma=+1} = -\int_A^B \frac{B ds}{q} \left[\langle D \int D f_o \rangle \right]_{\sigma=-1} \qquad \ldots (3.18)$$

or

$$\sum_{\sigma=\pm 1} \int \frac{Bds}{q} \left\langle D \int D\, f_o \right\rangle = 0 \qquad \ldots (3.19)$$

where the integral is between turning points. For brevity we introduce the operator

$$L\, f \equiv \left\langle D \int D\, f \right\rangle \qquad \ldots (3.20)$$

then we can summarize the second order constraints on f_o as:

(i) For particles which circulate round a closed field line:

$$\left[\oint \frac{Bds}{q} L\, f_o \right]_{\sigma=+1} = \left[\oint \frac{Bds}{q} L\, f_o \right]_{\sigma=-1} = 0 \quad . \qquad \ldots (3.21)$$

(ii) For particles trapped between mirrors:

$$\sum_{\sigma=\pm 1} \left[\int \frac{Bds}{q} L\, f_o \right] = 0 \quad \text{and} \quad \left[f_o \right]_{\sigma=+1} = \left[f_o \right]_{\sigma=-1} \quad . \qquad \ldots (3.22)$$

In fact the operator L is independent of σ but we leave these constraints in the general form so that we may refer to them in connection with other operators which will arise later.

The operator L is discussed in the Appendix; when operating on a function such as $f_o\ (\mu,\ \varepsilon,\ \alpha,\ \beta)$, i.e. one independent of s, φ it can be expressed as

$$L\, f_o = \frac{eV_d \cdot \nabla f_o}{mB} + \mu \left(\frac{\partial f_o}{\partial \mu} \right)_{\alpha\beta} \frac{q}{B} \frac{\partial}{\partial s} \left(\frac{q\, \underset{\sim}{e}_1 \cdot \mathrm{curl}\, \underset{\sim}{e}_1}{B} \right) \qquad \ldots (3.23)$$

where $\frac{eV_d}{m}$ is the drift velocity defined earlier. Inserting this expression for $L\, f_o$ into the constraints (3.21) and (3.22) one finds that either constraint can be concisely expressed in the α, β, coordinate system as

$$\left(\frac{\partial f_o}{\partial \alpha} \frac{\partial J}{\partial \beta} - \frac{\partial f_o}{\partial \beta} \frac{\partial J}{\partial \alpha} \right) = 0 \qquad \ldots (3.24)$$

where J is defined by

$$J\ (\alpha,\ \beta,\ \mu,\ \varepsilon) = \oint q\ ds \quad , \qquad \ldots (3.25)$$

the integration being around a closed field line or between turning points as appropriate.

According to equation (3.24) f_o does not depend on α, β individually but only on $J\ (\alpha,\ \beta,\ \mu,\ \varepsilon)$; the lower order constraints already make f_o independent of φ and s and we therefore conclude that it must be of the g.c. equilibrium form.

So far, then, we have shown that if equilibrium is to persist on the L/V_d time scale:-

$$f_o \equiv f_o\,(J,\ \mu,\ \varepsilon,\ \sigma) \qquad \ldots (3.26)$$

$$f_1 = h_1\,(\alpha,\ \beta,\ \mu,\ \varepsilon,\ \sigma) - \sigma \int_{s_o}^{s} \frac{B ds}{q}\ L\, f_o + \int D\, f_o \qquad \ldots (3.27)$$

$$f_2 = g_2\,(\alpha,\ \beta,\ s,\ \mu,\ \varepsilon,\ \sigma) + \int D\, f_1 \qquad \ldots (3.28)$$

where f_o, h_1 and g_2 are arbitrary. For particles trapped between mirrors f_o is independent of σ and it is convenient to take the lower limit of integration, s_o, at a turning point; then h_1 is also independent of σ. For particles which are not reflected by mirrors (circulating particles) the choice of s_o must be left arbitrary.

If we had retained time dependence of f_o in this order we would have obtained, instead of (3.24), the equation

$$\frac{e}{m}\frac{\partial f_o}{\partial t}\frac{\partial J}{\partial \varepsilon} + \left(\frac{\partial f_o}{\partial \alpha}\frac{\partial J}{\partial \beta} - \frac{\partial f_o}{\partial \beta}\frac{\partial J}{\partial \alpha}\right) = 0 \quad . \qquad \ldots (3.29)$$

and the Lagrangian subsidiary equations then give the time derivatives

$$\bar{\dot{\alpha}} = +\frac{m}{e}\left(\frac{\partial J}{\partial \beta}\right)\left(\frac{\partial J}{\partial \varepsilon}\right)^{-1} \quad , \quad \bar{\dot{\beta}} = -\frac{m}{e}\left(\frac{\partial J}{\partial \alpha}\right)\left(\frac{\partial J}{\partial \varepsilon}\right)^{-1} \qquad \ldots (3.30)$$

and so $\dot{J} = 0$. (The $\bar{\dot{\alpha}}$ and $\bar{\dot{\beta}}$ are of course, just the first order guiding center drift velocities averaged over the oscillation between mirrors or around the line of force.) Consequently μ, J and ε are the appropriate constants of the motion on the drift time scale and f_o is a function of these constants. For circulating particles σ is also a "constant of the motion" and so also appears in f_o.

(d) <u>Third Order</u>

So far in our analysis we have shown that as we increase the order, i.e. lengthen the time scale of equilibrium, we must impose increasingly severe constraints on f_o: however this process does <u>not</u> continue indefinitely as we shall now demonstrate.

In the third order, with $\partial f_o/\partial t = 0$, the initial form of the constraint condition is, as in previous orders,

$$\langle D\, f_2 \rangle = 0 \qquad \ldots (3.31)$$

or

$$\frac{\sigma q}{B}\frac{\partial g_2}{\partial s} = -L\, h_1 + \sigma L \int_{s_o}^{s} \frac{B ds}{q}\, L\, f_o - \left\langle D \int D \int D\, f_o \right\rangle \quad . \qquad \ldots (3.32)$$

This equation determines g_2 and leads, by precisely the same arguments as applied to equation (3.15) for g_1, to constraints similar to (3.21) or (3.22). All that is necessary to obtain these new constraints is to replace the operator L in (3.21) or (3.22) by the right hand side of (3.32). For particles which are not reflected at mirrors this yields;

$$\oint \frac{Bds}{q} \left\{ L\, h_1 - \sigma L \int \frac{Bds}{q}\, L\, f_o + \left\langle D \int D \int D\, f_o \right\rangle \right\} = 0 \qquad \ldots (3.33)$$

while for mirror trapped particles this must be summed over $\sigma = \pm 1$.

Now at this stage h_1 is a function of the same form as was f_o in equations (3.21) and (3.22) so that when $L\, h_1$ is evaluated and expressed in α, β coordinates, equation (3.33) becomes

$$\left(\frac{\partial h_1}{\partial \alpha} \frac{\partial J}{\partial \beta} - \frac{\partial h_1}{\partial \beta} \frac{\partial J}{\partial \alpha} \right) = \oint \frac{Bds}{q} \left\{ \sigma L \int \frac{Bds}{q}\, L\, f_o - \left\langle D \int D \int D\, f_o \right\rangle \right\} \equiv H\, f_o \qquad \ldots (3.34)$$

This is an equation of a type we have not met before. The terms in h_1 constitute the derivative along the direction J = constant so that

$$h_1 = k_1\,(\mu,\ \varepsilon,\ J) + \int_{J\,=\,\text{constant}} \frac{d\alpha}{\partial J/\partial \beta}\, H\, f_o \qquad \ldots (3.35)$$

which determines h_1, and hence f_1, up to an arbitrary function of μ, ε, J. At first sight it may appear that (3.35) does impose an additional constraint on f_o if the surfaces J = constant (which correspond to precessional drift surfaces) are closed. In this event h_1 can be single valued only if f_o satisfies

$$\oint_{J\,=\,\text{constant}} \frac{d\alpha}{\partial J/\partial \beta}\, H\, f_o = 0 \quad . \qquad \ldots (3.36)$$

However when the operator H (Appendix C) is evaluated in full, a lengthy calculation shows that this constraint is automatically satisfied by any function of the form $f_o \equiv f_o(\mu,\varepsilon,J)$; in the case of mirror trapped particles one finds that $H\, f_o$ is identically zero while for circulating particles it can be expressed in the form

$$H\, f_o = \frac{\partial J}{\partial \beta} \cdot \frac{\partial}{\partial \alpha} \left(P\, \frac{\partial f_o}{\partial J} \right) - \frac{\partial J}{\partial \alpha} \cdot \frac{\partial}{\partial \beta} \left(P\, \frac{\partial f_o}{\partial J} \right) \qquad \ldots (3.37)$$

(where P is defined in Appendix C). In either case, therefore, the loop integral (3.36) vanishes identically.

Another form for (3.37) is

$$H\, f_o = \left(\frac{\partial P}{\partial \alpha} \frac{\partial f_o}{\partial \beta} - \frac{\partial P}{\partial \beta} \frac{\partial f_o}{\partial \alpha} \right) \qquad \ldots (3.38)$$

which indicates that the velocities defined by

$$\bar{\dot{\alpha}}_2 = \left(\frac{m}{e}\right)^2 \frac{\partial P}{\partial \beta} \left(\frac{\partial J}{\partial \varepsilon}\right)^{-1} \qquad \bar{\dot{\beta}}_2 = -\left(\frac{m}{e}\right)^2 \frac{\partial P}{\partial \alpha} \left(\frac{\partial J}{\partial \varepsilon}\right)^{-1} \qquad \ldots (3.39)$$

must represent the second order drift velocity, averaged over the motion along the line of force, just as (3.30) represented the average of the first order drift velocity. The vanishing of (3.36) shows that the second order drifts produce no cumulative displacement from the first order drift surfaces.

However the most important feature of this section is that on carrying the expansion to a higher order we have not on this occasion needed to impose any new restriction on f_0. Instead we find that f_1 is now determined apart from a function of the same form as f_0 and which could be absorbed into f_0 if desired. We will later indicate why no further restrictions on f_0 are to be expected even if we were to calculate to still higher orders and will show that restricting f_0 to be of the form $f_0(\mu, \varepsilon, J)$ is sufficient for equilibrium to all orders. For the moment, however, we anticipate this result and turn our attention to the question of the fluid constraints.

4. THE FLUID EQUATIONS

It has been shown that if one requires equilibrium to persist for increasingly longer times then successively more stringent constraints must be imposed on f_0, culminating in the requirement that it be of the g.c. form (1.2); however, the fluid constraints (1.1) have not appeared in the intermediate time scales, as might have been expected. This is because we have so far been concerned only with the particle distribution function and have not considered the electromagnetic fields.

Electromagnetic fields

When the electric field is included a new physical time scale is introduced - by the plasma frequency ω_p - and the m/e expansion must be extended to incorporate this. If ω_p is comparable with ω_c this can be done most simply by formally regarding the m/e expansion as one in which $e \to \infty$ (m finite) for then both $\omega_p \to \infty$ and $\omega_c \to \infty$ but ω_p/ω_c remains finite. In a similar way the case $\omega_p \ll \omega_c$ can be dealt with by regarding the expansion as one in which $m \to 0$ (e finite), for then $\omega_p \to \infty$, $\omega_c \to \infty$ but $\omega_p/\omega_c \to 0$.

The condition for the electric fields to be stationary is

$$\frac{\partial \rho}{\partial t} = \sum_i e_i \int \frac{\partial f_i}{\partial t} \, d^3v = 0 \qquad \ldots (4.1)$$

and because e, but not m, appears in this equation there are differences between the theory with $\omega_p \sim \omega_c$ and that with $\omega_p \ll \omega_c$. In a situation where $\omega_p \ll \omega_c$ the charge e is treated as finite; then (4.1) shows that $\partial\rho/\partial t$ will vanish to any order so long as $\partial f/\partial t$ does so. Consequently there is nothing to be added to the discussion of equilibrium criteria and all the conclusions reached in Section 3 are unchanged. On the other hand, in a situation where $\omega_p \sim \omega_c$, the charge e must be treated as a large quantity; then equation (4.1) indicates that $\partial\rho/\partial t$ is one order lower than $\partial f/\partial t$. Consequently, the vanishing of $\partial f/\partial t$ to a given order only ensures that $\partial\rho/\partial t$ vanishes to one order lower and equilibrium can only be ensured by making both $\partial f/\partial t$ and $\partial\rho/\partial t$ vanish to the appropriate order.

In the case of equilibrium on the drift time scale, when f_o is already constrained to be of the g.c. form $f_o\ (\mu, \varepsilon, J)$, no new constraint is needed to ensure that $\partial\rho/\partial t = 0$, for it has been shown that if $f_o \equiv f_o\ (\mu, \varepsilon, J)$ the distribution is stationary not merely on the drift time scale but also to one order higher (indeed to all orders, as we shall see later). Consequently the criteria for equilibrium on the drift time scale are unaltered by the inclusion of electric fields.

However, in discussing equilibria on the intermediate time scale $v_{\parallel}/L$ an alteration <u>is</u> necessary; not only is it necessary that $f_o \equiv f_o\ (\mu, \varepsilon, \alpha, \beta)$, so making $\partial f/\partial t$ zero to order λ, but $\partial\rho/\partial t$ must also be zero to order λ. This leads to an extra constraint which is easily found by retaining the time dependence of f_o in equation (3.15) which then becomes

$$\frac{e}{mB}\frac{\partial f_o}{\partial t} + \frac{\sigma q}{B}\frac{\partial g_1}{\partial s} + \left\langle D \int D\, f_o \right\rangle = 0 \qquad \ldots (4.2)$$

The constraint $\partial\rho/\partial t = 0$ is therefore

$$\frac{\partial\psi}{\partial s} + \sum_{i,\sigma} m_i \int \frac{B\, d\mu\, d\varepsilon}{q} \left\langle D \int D\, f_o^i \right\rangle = 0 \qquad \ldots (4.3)$$

(where

$$\psi = \sum_{i,\sigma} \sigma \int m_i\, g_1^i\, d\mu \quad d\varepsilon \qquad \ldots (4.4)$$

is essentially the current parallel to B). The existence of a single valued ψ which vanishes in the vacuum surrounding the plasma requires the integral over s of the second term in (4.3) to vanish. When the appropriate form (3.23) is inserted for $\left\langle D \int D\, f_o \right\rangle$ the resulting constraint can be written entirely in terms of <u>macroscopic</u> quantities as

$$\int \frac{ds}{B^3}\, \nabla(p_\perp + p_\parallel) \cdot \underset{\sim}{B} \times \underset{\sim}{\rho} = 0 \qquad \ldots (4.5)$$

which will be recognised as an alternative expression for the fluid constraint (1.1). (This form (4.5) is applicable at finite pressure whereas (1.1) applies only in the low pressure limit.)

From this discussion we conclude that when the time dependence of electromagnetic fields is considered the guiding center constraint remains sufficient and necessary for equilibrium to all orders. However the weaker constraint $f_o = f_o(\mu, \varepsilon, \alpha, \beta)$, which was previously adequate for equilibrium on the time scale $L/V_{\parallel}$, now needs to be supplemented by the fluid constraint (1.1) unless the plasma density is so low that $\omega_p \ll \omega_c$.

PART II

5. ADIABATIC INVARIANTS AND EQUILIBRIA IN HIGHER ORDERS

In Part I it was shown that as the time scale of equilibrium was lengthened by going to higher orders in λ, the restrictions on f_o became increasingly severe until it was restricted to the guiding center form. However, once this point had been reached, an extension by another order imposed no extra restrictions on f_o; instead the restrictions affected f_1 which was thereby determined in terms of f_o (apart from an arbitrary function of μ, ε, J, which could be regarded as part of f_o). It was suggested that no matter how far the calculation was pursued no further restrictions on f_o would be found.

That this conclusion is correct is indicated by the following; it is well known[5] that invariant quantities $\hat{\mu}$ and $\hat{J}$ exist which are constant <u>to all orders</u> in m/e and which are identical in lowest order with the μ, J <u>defined</u> in Part I. Therefore, within the framework of any m/e expansion scheme, a distribution such as

$$\psi = \psi\ (\hat{\mu}, \hat{J}, \varepsilon) \qquad \ldots (5.1)$$

can be regarded as an exact equilibrium, and if we put

$$\hat{\mu} = \mu_o + \lambda\, \mu_1 + \lambda^2\, \mu_2 \ldots. \qquad \ldots (5.2)$$

$$\hat{J} = J_o + \lambda\, J_1 + \lambda^2\, J_2 \ldots. \qquad \ldots (5.3)$$

(where for emphasis we now write μ_o, J_o for the zero order invariants μ, J) we see that an equilibrium correct to all orders can be expressed in the form

$$\psi\left(\mu_o, J_o, \varepsilon\right) + \lambda\left(\mu_1 \frac{\partial\psi}{\partial\mu_o} + J_1 \frac{\partial\psi}{\partial J_o}\right) + \lambda^2 \ldots. \qquad \ldots (5.4A)$$

This general equilibrium thus contains one arbitrary function of three variables and is completely defined once its lowest order form is given. However if ψ is regarded as explicitly dependent on λ (5.4A) can be written in the form

$$\psi_o\ (\mu_o, J_o, \varepsilon) + \lambda\ \left(\mu_1 \frac{\partial\psi_o}{\partial\mu_o} + J_1 \frac{\partial\psi_o}{\partial J_o} + \psi_1\ (\mu_o, \varepsilon, J_o)\right) + \lambda^2 \ldots \qquad \ldots (5.4B)$$

though this corresponds only to a relabelling of the equilibria and by summing $\psi_o + \lambda\, \psi_1 + \lambda^2\, \psi_2 \ldots$ (5.4B) can always be re-cast into the form (5.4A).

Clearly, the lowest order term in (5.4) must be identified with the f_o of Part I; it is then apparent that even if one demands equilibrium <u>to all orders</u> f_o remains an arbitrary function of μ_o, J_o, ε .

It is also clear the higher order corrections μ_1, J_1; μ_2, J_2; ... can be obtained by identifying higher order terms of the series $f_o + \lambda f_1 + \lambda^2 f_2 \ldots$ which have already been calculated in Part I, with the corresponding terms in (5.4). There are marked differences between the case of particles trapped between mirrors and those which circulate round a toroidal system and we first consider only particles trapped between mirrors.

Mirror-trapped particles

Collecting together results from Part I we have

$$f_1 = k_1 (\mu_o, J_o, \varepsilon) + \int Df_o - \sigma \int_{s_o}^{s} \frac{Bds}{q} Lf_o + \int_{J = \text{const}} \frac{d\alpha}{\partial J/\partial \beta} Hf_o \qquad \ldots (5.5)$$

where the various operators have been introduced in Part I and Appendix C. For convenience* we are taking s_o to be a turning point ($q = 0$) as this simplifies the evaluation of H, - in fact it then vanishes identically for trapped particles. When the operators D and L are explicitly introduced into (5.5) one finds, significantly, that f_1 depends only on derivatives $\partial f_o/\partial \mu_o$ and $\partial f_o/\partial J_o$ although higher derivatives appear in the individual operators. Consequently f_1 as given by (5.5) is, indeed, of the functional form indicated by (5.4) and can be completely identified with (5.4) by setting

$$\mu_1 = -\frac{1}{B}\left[\underset{\sim}{v}_\perp \cdot \underset{\sim}{W}_d + \frac{\underset{\sim}{v} \cdot \underset{\sim}{e}_1}{4} \left\{ \underset{\sim}{v}_\perp \cdot (\underset{\sim}{a} \cdot \underset{\sim}{\nabla})\, \underset{\sim}{e}_1 + \underset{\sim}{a} \cdot (\underset{\sim}{v}_\perp \cdot \underset{\sim}{\nabla})\, \underset{\sim}{e}_1 + 4\mu_o (\underset{\sim}{e}_1 \cdot \underset{\sim}{\nabla} \times \underset{\sim}{e}_1) \right\} \right] \qquad \ldots (5.6)$$

and

$$J_1 = \underset{\sim}{a} \cdot \nabla J_o + \mu_1 \left(\frac{\partial J_o}{\partial \mu_o} \right)_{\alpha\beta\varepsilon} - \sigma \int_{s_o}^{s} \frac{ds}{q} \underset{\sim}{W}_d \cdot \nabla J_o \qquad \ldots (5.7)$$

where

$$J_o = \oint [2(\varepsilon - \mu_o B)]^{\frac{1}{2}} ds \quad , \qquad \ldots (5.8)$$

$$\underset{\sim}{W}_d = \frac{\underset{\sim}{B}}{B^2} \times (q^2 \underset{\sim}{\rho} + \mu \nabla B) = \frac{e}{m} \underset{\sim}{v}_d \qquad \ldots (5.9)$$

$$\underset{\sim}{a} = \frac{\underset{\sim}{v}_\perp \times \underset{\sim}{B}}{B^2} \quad , \qquad \ldots (5.10)$$

$\underset{\sim}{e}_1$ is the unit vector along $\underset{\sim}{B}$ and $\underset{\sim}{\rho}$ is the curvature of the field. All quantities in (5.6) and (5.7) are referred to the position of the particle not to the guiding center which we have nowhere introduced, similarly the integrals in (5.7) and (5.8) are along the line of force through the instantaneous position of the particle.

* Any other choice leads, of course, to the same final result, any change in the explicit s integration being compensated by a corresponding change in the operator H which implicitly depends on s_o.

The expressions for μ_1 and J_1 agree with those given by Kruskal[7] and by Northrop, Liu and Kruskal[8] respectively. The second invariant J_1 can be cast into more convenient form by observing that the term $\underset{\sim}{a} \cdot \nabla J_o$ in (5.7) is simply the change in J_o which would be introduced if the path of integration in (5.8) were transferred to the field line through the instantaneous guiding center. Similarly $\mu_1 \, \partial J_o / \partial \mu_o$ is the change introduced if we replace μ_o by $(\mu_o + \mu_1)$ in the integrand of (5.8). Hence if we collect together zero and first order contributions to J and take all integrals along the field line through the guiding center, instead of through the particle, we can write

$$J_o + \frac{m}{e} J_1 = \oint \left[2(\varepsilon - (\mu_o + \frac{m}{e} \mu_1) B) \right]^{\frac{1}{2}} ds - \sigma \int_{s_o}^{s} \frac{ds}{q} \underset{\sim}{V}_d \cdot \nabla J_o \quad . \qquad \ldots (5.11)$$

This can be written in yet another form which is important for the later discussion of circulating particles. We introduce the <u>instantaneous</u> drift velocity in α, β space by defining

$$\dot{\alpha} \equiv \underset{\sim}{V}_d \cdot \nabla \alpha \quad , \quad \dot{\beta} \equiv \underset{\sim}{V}_d \cdot \nabla \beta \qquad \ldots (5.12)$$

and recall that the average drifts $\bar{\dot{\alpha}}$ and $\bar{\dot{\beta}}$ are related to $\partial J / \partial \beta$ and $\partial J / \partial \alpha$ by (3.30); then (5.11) can be written

$$(J_o + \frac{m}{e} J_1) = \oint \left[2(\varepsilon - (\mu_o + \frac{m}{e} \mu_1) B) \right]^{\frac{1}{2}} ds + \sigma \frac{m}{e} \oint \frac{ds''}{q(s'')} \int_{s_o}^{s} G(s', s'') \frac{ds'}{q(s')} \qquad \ldots (5.13)$$

where $G(s', s'')$ is a zero order quantity defined by

$$G(s', s'') \equiv \frac{e^2}{m^2} \left[\dot{\alpha}(s') \, \dot{\beta}(s'') - \dot{\alpha}(s'') \, \dot{\beta}(s') \right] . \qquad \ldots (5.14)$$

The factor σ which appears in the last term of (5.11) or (5.13) is due to our convention that ds is measured in the direction of $\underset{\sim}{e}_1$ irrespective of whether the particle is moving to left or right. If instead the integration is always taken in the direction of motion then the σ is unnecessary. Equation (5.11) can then be interpreted by observing that the last term represents the change in J_o since the particle left the turning point due to the drift of the guiding center and so is exactly the amount which must be added to J_o to ensure that $(J_o + \frac{m}{e} J_1)$ retains its original value as one follows the particle. (However note that $(J_o + \frac{m}{e} J_1)$ is entirely a local quantity which can be computed from the magnetic field and the instantaneous position of the particle; it is <u>not</u> necessary to calculate the orbit of particle or guiding center.)

In a simple axi-symmetric magnetic field the symmetry ensures that $\underset{\sim}{V}_d \cdot \nabla J_o$ is identically zero so that in such a field J is given correctly through first order by

$$J = \oint \left[2(\varepsilon - (\mu_o + \frac{m}{e}\mu_1) B) \right]^{\frac{1}{2}} ds \quad , \qquad \ldots (5.15)$$

the path of integration being along the field line through the guiding center.

Circulating particles

For circulating particles i.e. those not subject to mirror reflection, f_1 is again given by

$$f_1 = k_1 (\sigma, \mu_o, \varepsilon, J_o) + \int Df_o - \sigma \int_{s_o}^{s} \frac{Bds}{q} Lf_o + \int_{J = \text{const}} \frac{d\alpha}{\partial J/\partial\beta} Hf_o \qquad \ldots (5.16)$$

and

$$Hf_o = \oint \frac{Bds'}{q} \left\{ \sigma L \int_{s_o}^{s'} \frac{Bds}{q} Lf_o - \langle D \int D \int Df_o \rangle \right\} . \qquad \ldots (5.17)$$

but there is now no natural choice for the lower limit s_o and the operator H must be evaluated for an arbitrary s_o. As a result, H no longer vanishes; at first sight this appears to mean that f_1 involves an integral over the precessional drift surfaces J_o = constant. Fortunately, however, it is possible (see Appendix) to express the operator H as a total derivative along J = constant, i.e.

$$Hf_o = \frac{\partial J}{\partial\beta} \cdot \frac{\partial}{\partial\alpha} \left(P \frac{\partial f_o}{\partial J} \right) - \frac{\partial J}{\partial\alpha} \frac{\partial}{\partial\beta} \left(P \frac{\partial f_o}{\partial J} \right) \qquad \ldots (5.18)$$

where P is given by

$$P = \sigma \mu_o \oint ds' \left[\tau_1 + \tfrac{1}{2} (\tau_2 + \tau_3)\right] - \frac{\sigma}{2} \oint \frac{ds''}{q(s'')} \int_{s_o}^{s''} G(s',s'') \frac{ds'}{q(s')} \qquad \ldots (5.19)$$

the function G being again defined by (5.14). The coefficients τ_1, τ_2, τ_3 are related to the torsion of the field line and $\dot{\alpha}, \dot{\beta}$ are again the instantaneous drift velocities. Consequently

$$f_1 = k_1 + \int Df_o - \sigma \int_{s_o}^{s} \frac{Bds}{q} Lf_o + P \frac{\partial f_o}{\partial J} \qquad \ldots (5.20)$$

which once again involves only the first derivatives of f_o and so is of the form (5.4). Direct comparison with (5.4) now yields the invariants for circulating (non-reflected) particles as

$$\mu_1 = -\frac{1}{B} \left[\underset{\sim}{v}_\perp \cdot \underset{\sim}{W}_d + \frac{\underset{\sim}{e}_1 \cdot \underset{\sim}{v}}{4} \left\{ \underset{\sim}{v}_\perp \cdot (\underset{\sim}{a} \cdot \underset{\sim}{\nabla}) \underset{\sim}{e}_1 + \underset{\sim}{a} \cdot (\underset{\sim}{v}_\perp \cdot \underset{\sim}{\nabla}) e_1 + 4\mu_o \underset{\sim}{e}_1 \cdot \nabla \times \underset{\sim}{e}_1 \right\} \right] \qquad \ldots (5.21)$$

and

$$J_1 = \underset{\sim}{a} \cdot \nabla J_o + \mu_1 \frac{\partial J_o}{\partial \mu_o} + \sigma \mu_o \oint ds' \left[\tau_1 + \tfrac{1}{2}(\tau_2 + \tau_3)\right]$$
$$+ \sigma \oint \frac{ds''}{q(s'')} \int_{s_o}^{s} \frac{ds'}{q(s')} G(s', s'') - \frac{\sigma}{2} \oint \frac{ds''}{q(s'')} \int_{s_o}^{s''} \frac{ds'}{q(s')} G(s', s'') . \qquad \ldots (5.22)$$

The second invariant can again be simplified by changing the path of integration to the line of force through the guiding center, then

$$J = J_o + \frac{m}{e} J_1 = \oint \left[2(\varepsilon - (\mu_o + \frac{m}{e}\mu_1) B)\right]^{\frac{1}{2}} ds + \frac{\sigma m}{e} \mu_o \oint \left[\tau_1 + \tfrac{1}{2}(\tau_2 + \tau_3)\right] ds'$$

$$+ \frac{\sigma m}{e} \oint \frac{ds''}{q(s'')} \int_{s_o}^{s} \frac{ds'}{q(s')} G(s', s'') - \frac{\sigma}{2}\frac{m}{e} \oint \frac{ds''}{q(s'')} \int_{s_o}^{s''} \frac{ds'}{q(s')} G(s', s'') \quad \ldots (5.23)$$

The invariant μ_1 is identical with that for particles trapped between mirrors but the second invariant is of a different form*. Some difference was to be expected since the expression for $(J_o + \frac{m}{e} J_1)$ in the mirror case involved an integral whose lower limit s_o was a definite physical point - the mirror reflection point - which does not exist for circulating particles. If J had been given by the same expression therefore, it would now have involved an arbitrary value of s_o. The extra double integral in (5.23) rectifies this; both integrals of $G(s', s'')$ depend on the arbitrary point s_o but their sum is independent of s_o and can, indeed, be written as

$$\tfrac{1}{2}\sigma \oint \frac{ds''}{q(s'')} \int_{s''}^{s} \frac{ds'}{q(s')} G(s', s'') \quad \ldots (5.24)$$

in which there is no arbitrary quantity.

Another difference between J for circulating and oscillating particles - which was not foreseen - is the term involving the integral of the torsion around the closed line of force. As this term is one of the terms arising from the operator H it can be interpreted as one of the consequences of the second order drift velocities (3.39). It may seem surprising that one can relate part of a first order quantity J_1 to a second order drift, but just as

$$\int_{s_o}^{s} \underset{\sim}{V}_d \cdot \nabla J_o \frac{ds}{q} \quad \ldots (5.25)$$

can be regarded as the accumulated change in J_o due to the first order drifts over times of order L/q or $L/V_{\parallel}$, so can

$$\int_{J_o = \text{const.}} \underset{\sim}{V}_d^{(2)} \cdot \nabla J_o \frac{d\alpha}{\partial J/\partial \beta} \quad \ldots (5.26)$$

---

* Northrop, Liu and Kruskal[8] have calculated J_1 for mirror trapped particles and suggest that the value for circulating particles should be the same because the form (5.11) makes, dJ/dt of order $(m/e)^2$ whether the particle is trapped between mirrors or not. However this is also true of our form (5.23) and would be true of any function of the form $[J_1 + Q(\alpha, \beta, \mu, \varepsilon)]$ so that this argument is inconclusive. To be a first order invariant not only must dJ/dt be of order $(m/e)^2$ but its average must vanish to higher order so that the error in J remains of order $(m/e)^2$ as $t \to \infty$.

be regarded as the accumulated change in J_o due to <u>second</u> order drifts, over times of order $L/V_d^{(1)}$ or $\lambda^{-1}\ L/V_{\parallel}$. In (5.26) the drift is an order smaller than in (5.25) but the time for which it acts is an order longer so that (5.26) still yields a first order quantity. The second order drift changes sign with $V_{\parallel}$ so its accumulated value for oscillating particles is always zero. It must be emphasised again that this is merely an interpretation of the result of a rigorous calculation, it is certainly not necessary to invoke second order (or even first order) drifts in order to determine J_1 .

By the same token there is no necessity for the drift surface to be closed (i.e. for the drift motion to be periodic). The invariant $\hat{J}$ exists as a consequence of the periodicity of the motion <u>along</u> the lines of force and does not depend on any periodicity in the drift motion. If the drift motion <u>is</u> periodic it gives rise to a further invariant[3] - the flux invariant Φ (i.e. the total flux through a drift surface). In a static situation such as has been discussed thus far, this invariant is redundant since constancy of μ, J, ε inevitably ensures constancy of Φ. However Φ <u>can</u> be obtained by our procedure provided one includes appropriately slow variations of the magnetic field and this will be discussed in part 3. For the moment we return to $\hat{\mu}$ and $\hat{J}$ and consider the general n^{th} order term in their expansions.

6. HIGHER ORDER CORRECTIONS

A recursion formula for μ_n and J_n involving only the operators already introduced can be obtained as follows. When the equilibrium constraint $\langle Df_n \rangle$ was applied to f_o, f_1, f_2, we found that f_1 could be expressed in terms of f_o, for example

$$f_1 = k_1 + \int Df_o - \sigma \int_{s_o}^{s} \frac{Bds}{q} L\, f_o + \int \frac{d\alpha}{\partial J/\partial\beta} Hf_o \quad . \qquad \ldots (6.1)$$

If we carry out exactly the same calculation, but consider instead of f_o, f_1, f_2 the general consecutive terms f_n, f_{n+1}, f_{n+2} we obtain

$$f_{n+1} = k_{n+1} + \int Df_n - \sigma \int_{s_o}^{s} \frac{Bds}{q} L\, f_n + \int \frac{d\alpha}{\partial J/\partial\beta} H\, f_n \quad . \qquad \ldots (6.2)$$

Multiplying this by λ^{n+1} and summing we have

$$f = k + \lambda K f \qquad \ldots (6.3)$$

where K represents the sum of the three integral operators in (6.2) and $k(\mu,\varepsilon,J) \equiv \Sigma\, \lambda^n k_n$. Consequently f can be written as

$$f = (1 - \lambda K)^{-1}\, k(\mu, \varepsilon, J) \qquad \ldots (6.4)$$

which involves a single arbitrary function and generates f in the standard form (5.4A): it can therefore be directly identified with $\psi(\hat{\mu}, \hat{J}, \varepsilon)$. If now, we choose $\psi(\hat{\mu}, \hat{J}, \varepsilon) \equiv \hat{\mu}$ then the nth term in the expansion (5.4A) of ψ is just μ_n; similarly we may choose $\psi \equiv \hat{J}$ and generate a series whose n^{th} term is J_n. Comparing these with the solution generated by (6.4) allows one to write down a recursion formula for μ_n and J_n :-

$$\left.\begin{matrix} J_{n+1} \\ \mu_{n+1} \end{matrix}\right\} = \int D \left\{\begin{matrix} J_n \\ \mu_n \end{matrix}\right\} - \sigma \int_{s_0}^{s} \frac{B ds}{q} L \left\{\begin{matrix} J_n \\ \mu_n \end{matrix}\right\} + \int \frac{d\alpha}{\partial J/\partial \beta} H \left\{\begin{matrix} J_n \\ \mu_n \end{matrix}\right\} . \qquad \ldots (6.5)$$

Using this recursion formula we have determined μ_2 in an arbitrary magnetic field. This can be written

$$\mu_2 = c_0 + \sum_1^4 (c_n \cos n\varphi + s_n \sin n\varphi) \qquad \ldots (6.6)$$

where

$$\tan \varphi = \underset{\sim}{e}_3 \cdot \underset{\sim}{v} / \underset{\sim}{e}_2 \cdot \underset{\sim}{v} \quad . \qquad \ldots (6.7)$$

In a general magnetic field the coefficients c_n, s_n are very lengthy but for a vacuum magnetic field they simplify somewhat and putting $\eta \equiv (\rho_3 - \rho_2)$ and $\nu \equiv (\tau_3 - \tau_2)$ the coefficients for a vacuum magnetic field can be written* :

$$c_0 = -\frac{1}{2}\frac{q^4}{B^3}\rho^2 + \frac{q^2\mu B}{B^3}\left\{ \operatorname{div} \underset{\sim}{\rho} - \frac{1}{8}(\eta^2 + \nu^2)\right\} + \frac{(\mu B)^2}{B^3}\left\{\frac{3}{4}(\operatorname{div} \underset{\sim}{\rho} - \underset{\sim}{\rho}^2) + \frac{1}{8}(\nabla \cdot e)^2\right\}$$

$$c_4 = \frac{1}{16}(\eta^2 + \nu^2)\frac{(\mu B)^2}{B^3}$$

$$s_4 = -\frac{(\mu B)^2}{B^3}\frac{1}{8}\eta\nu$$

$$c_3 = \frac{q c_\perp \mu B}{12 B^3}\left\{ -\frac{\partial \nu}{\partial y} - \frac{\partial \eta}{\partial x} + \eta(2\sigma_3 - \rho_1) - \nu(2\sigma_2 - \sigma_1)\right\}$$

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* A special case of μ_2 - its value on the median plane of an axisymmetric vacuum field - was calculated some time ago by G. Gardner and is quoted by T. Northrop[7]. Our result does not agree with this formula but we understand from Dr. Gardner that there is an error in the formula of ref. 7 and his latest calculation agrees with ours.

$$s_3 = \frac{qc_\perp \mu B}{12\ B^3} \left\{ -\frac{\partial \eta}{\partial y} + \frac{\partial \nu}{\partial x} - \nu(2\sigma_3 - \rho_1) - \eta(2\sigma_2 \quad \sigma_1) \right\}$$

$$c_2 = +\frac{3q^2\mu B}{4B^3} \left\{ 2\nu\tau_1 - B\frac{\partial}{\partial s}\left(\frac{\eta}{B}\right)\right\} + \frac{(\mu B)^2}{2B^3} \left\{ -B\frac{\partial}{\partial s}\left(\frac{\eta}{B}\right) + 2\nu\tau_1 - \rho_1^2 + \sigma_1^2 - \frac{1}{2}\eta(\underset{\sim}{\nabla} \quad \underset{\sim}{e}_1)\right\}$$

$$s_2 = \frac{3q^2\mu B}{4B^3} \left\{ 2\eta\tau_1 + B\frac{\partial}{\partial s}\left(\frac{\nu}{B}\right)\right\} + \frac{(\mu B)^2}{2B^3} \left\{ B\frac{\partial}{\partial s}\left(\frac{\nu}{B}\right) + 2\eta\tau_1 + 2\rho_1\sigma_1 + \frac{1}{2}\nu\ (\underset{\sim}{\nabla}\cdot \underset{\sim}{e}\ )\right\}$$

$$c_1 = \frac{c_\perp q\ \mu B}{4B^3} \left\{ \frac{\partial \nu}{\partial y} - \frac{\partial \eta}{\partial x} + \nu(\sigma_1 - 2\sigma_2) + \eta(\rho_1 - 2\sigma_3)\right\}$$
$$- \frac{qc_\perp}{B^3}\ \sigma_1\tau_1\ (\mu B + q^2) + \frac{q}{B}\ \frac{\partial}{\partial s}\ \frac{c_\perp}{B^2}\ \rho_1(q^2 + \mu B)$$

$$s_1 = \frac{c_\perp q\mu B}{4B^3} \left\{ \frac{\partial \eta}{\partial y} + \frac{\partial \nu}{\partial x} + \eta(\sigma_1 - 2\sigma_2) - \nu(\rho_1 - 2\sigma_3)\right\}$$
$$- \frac{qc_\perp}{B^3}\ \rho_1\tau_1\ (q^2 + \mu B) - \frac{q}{B}\ \frac{\partial}{\partial s}\ \frac{c_\perp}{B^2}\sigma_1\ (q^2 + \mu B)$$

PART III

7. TIME DEPENDENT FIELDS, ELECTRIC FIELDS AND THE THIRD INVARIANT

In parts I and II only static magnetic fields were considered; this led to the two invariants μ, J which together with the energy ε form three "constants of the motion". In time dependent magnetic fields ε itself is no longer a constant but if the field variations are sufficiently slow it is known[3] that there is still a third invariant quantity, namely the flux Φ through a drift surface J = constant. We now consider how this third invariant arises from our present viewpoint and at the same time discuss the related effect of electric fields. These calculations are very similar to those of parts I and II so that we need give only an outline of the arguments involved.

To investigate these effects we first transform the Vlasov equation to a velocity frame moving with the field lines. For this we choose a velocity

$$\underset{\sim}{U} = \left\{ \frac{\partial \alpha}{\partial t} \nabla\beta - \frac{\partial \beta}{\partial t} \nabla\alpha \right\} \times \frac{\underset{\sim}{B}}{B^2} \qquad \ldots (7.1)$$

so that

$$\frac{\partial \alpha}{\partial t} + \underset{\sim}{U} \cdot \nabla\alpha = 0 \quad : \quad \frac{\partial \beta}{\partial t} + \underset{\sim}{U} \cdot \nabla\beta = 0 \ . \qquad \ldots (7.2)$$

This velocity $\underset{\sim}{U}$ is <u>not</u> the same as the $\underset{\sim}{E} \times \underset{\sim}{B}$ drift, in fact with

$$E = - \frac{\partial \underset{\sim}{A}}{\partial t} - \nabla\varphi \qquad \ldots (7.3)$$

and $\underset{\sim}{A} = \alpha \underset{\sim}{\nabla}\beta$ the velocity U is

$$\underset{\sim}{U} = \frac{(\underset{\sim}{E} + \underset{\sim}{\nabla}\psi) \times \underset{\sim}{B}}{B^2} \qquad \ldots (7.4)$$

where $\psi = (\alpha\ \partial\beta/\partial t + \varphi)$ and $\nabla\psi$ does not, in general, vanish.

After transforming to this frame of reference the variables μ, ε, φ are introduced as in parts I and II when the Vlasov equation can be written

$$\frac{\partial f}{\partial \varphi} = \lambda \left(\frac{1}{B} \frac{\partial f}{\partial t} + Df + Gf \right) \qquad \ldots (7.5)$$

where D is the operator introduced earlier and G is a new operator, given in Appendix C, which depends explicitly on $\underset{\sim}{U}$ and on $\nabla\psi$. In fact G may be split into two parts each depending only on either $\underset{\sim}{U}$ or $\nabla\psi$.

$$G \equiv \lambda^{-1} G^{\psi} + G^{U} \ . \qquad \ldots (7.6)$$

The calculation of f proceeds order by order just as in section 3, so that we can omit all details. The effects of electrostatic fields and time dependent magnetic fields

of various magnitudes are introduced by treating G as being of the appropriate order in λ. For example, to reveal the flux invariant the time dependence of the fields must be taken to be one order higher than that of the drifts V_d/L, that is

$$\frac{1}{a}\frac{\partial a}{\partial t} \sim \frac{U}{L} \sim \frac{a}{L}\cdot\frac{V_d}{L} , \qquad \ldots (7.7)$$

which makes the operators G^{ψ} and G^{U} of order λ^2. Then $\frac{1}{B}\frac{\partial f}{\partial t}$ and $G^{U}f$ are of the same order and $\frac{1}{B}\frac{\partial}{\partial t}$ can be absorbed into G^{U}.

The zero and first order calculations are unaffected by the additional operator G and, as before, constrain f_o to be independent of φ and s, respectively.

Second order

It is in second order that the operator G first affects the calculation and in place of (3.12) one now finds

$$\frac{\partial f_2}{\partial \varphi} = Df_1 + G^{\psi}f_o \qquad \ldots (7.8)$$

leading to

$$f_2 = g_2 + \int Dg_1 + \int D\int Df_o + \int G^{\psi}f_o \qquad \ldots (7.9)$$

and the associated constraint

$$\langle Dg_1 \rangle + \langle D\int Df_o \rangle + \langle G^{\psi}f_o \rangle = 0 . \qquad \ldots (7.10)$$

When the last term in this equation is evaluated it has a similar form to the first, with which it can be combined so that (7.10) can be written

$$\sigma\frac{q}{B}\frac{\partial}{\partial s}\left(g_1 - \psi\frac{\partial f_o}{\partial \varepsilon}\right) + \langle D\int Df_o \rangle = 0 . \qquad \ldots (7.11)$$

Consequently the constraint in this order is still just

$$\oint \frac{Bds}{q}\langle D\int Df_o \rangle = 0 \qquad \ldots (7.12)$$

indicating that f_o is of the form $f_o(\mu, \varepsilon, J, t)$. There is however a change in g_1 which is now given by

$$g_1 = -\sigma\int_{s_o}^{s}\frac{Bds}{q}Lf_o + \frac{\partial f_o}{\partial \varepsilon}\psi + h_1(\mu, \varepsilon, \alpha, \beta, \sigma) \qquad \ldots (7.13)$$

instead of by equation (3.16).

Third Order

In this order, both G^{ψ} and G^{U} enter the calculation, and we have

$$\frac{\partial f_3}{\partial \varphi} = Df_2 + G^{\psi}f_1 + G^{U}f_o \qquad \ldots (7.14)$$

leading to the constraint

$$\langle Df_2 \rangle + \langle G^{\psi} f_1 \rangle + \langle G^{U} f_o \rangle = 0 \quad . \qquad \ldots (7.15)$$

When f_2 and f_1 are introduced in terms of f_o one finds, as before, an equation for g_2, similar to equation (3.32). For brevity this will be written as

$$\sigma \frac{q}{B} \frac{\partial g_2}{\partial s} = - Lh_1 + \sigma L \int \frac{Bds}{q} Lf_o - \langle D \int D \int Df_o \rangle + \sigma \frac{q}{B} \frac{\partial}{\partial s} \left(\psi \frac{\partial h_1}{\partial \varepsilon} \right) + Mf_o \ldots (7.16)$$

where Mf_o represents all the several terms arising from G^{ψ} and G^{U} which appear in (7.15) and the operator M is given in Appendix C. Integration of equation (7.16) over s leads to a generalisation of (3.33) and (3.34), namely;

$$\left(\frac{\partial h_1}{\partial \alpha} \frac{\partial J}{\partial \beta} - \frac{\partial h_1}{\partial \beta} \frac{\partial J}{\partial \alpha} \right) = Hf_o + \oint \frac{Bds}{q} Mf_o \quad . \qquad \ldots (7.17)$$

Now, as we observed in section 3 the expression on the left of (7.17) is the derivative of h_1 along the precessional drift surfaces J = constant, and if these precessional surfaces are closed (7.17) may itself lead to a constraint on f_o namely;

$$\oint_{J = \text{constant}} \frac{d\alpha}{\partial J/\partial \beta} Hf_o + \oint_{J = \text{constant}} \frac{d\alpha}{\partial J/\partial \beta} \oint \frac{Bds}{q} Mf_o = 0 \quad . \qquad \ldots (7.18)$$

where the integrals are taken around the precessional drift surfaces.

It was noted in part II that the first term is automatically zero, so that (7.18) leads to no new constraints in the time independent case. However in the present situation there <u>is</u> a constraint. To find this we must evaluate M; this is a tedious calculation given elsewhere[9] and we content ourselves here with the result which is

$$\oint_{J = \text{const}} \frac{d\alpha}{\partial J/\partial \beta} \left[\left(\frac{\partial f_o}{\partial t} \right) \left(\frac{\partial J}{\partial \varepsilon} \right)_{\alpha\beta} - \left(\frac{\partial f_o}{\partial \varepsilon} \right) \left(\frac{\partial J}{\partial t} \right)_{\alpha\beta} \right] = 0 \quad . \qquad \ldots (7.19)$$

[Here, $\partial J/\partial t$ at constant α, β means the rate of change following the field line labelled by the <u>numbers</u> (α, β) (i.e. the Lagranian derivative moving with the velocity U,) the <u>functions</u> $\alpha(x)$ and $\beta(x)$ must of course vary if the magnetic field is to change.]

Equation (7.19) can be written

$$\left(\frac{\partial \Phi}{\partial t} \frac{\partial f_o}{\partial \varepsilon} - \frac{\partial \Phi}{\partial \varepsilon} \frac{\partial f_o}{\partial t} \right) = 0 \qquad \ldots (7.20)$$

where

$$\Phi \equiv \oint_{J = \text{const.}} \alpha \, d \, \beta \qquad \ldots (7.21)$$

is the flux contained within the precessional drift surface J = constant. Equation (7.20) shows that f_o is of the form $f_o(\mu, J, \Phi)$. Hence in time varying fields, although the energy ε is no longer a constant it is replaced as an invariant by the flux Φ and there are still three "constants of motion" μ, J, Φ.

PART IV

TOROIDAL SYSTEMS WITH SMALL ROTATIONAL TRANSFORM

8. MAGNETIC SURFACES

In our discussion of adiabatic invariants in part II, two different cases were distinguished, that of mirror-trapped (i.e. oscillating) particles and that of particles circulating round a closed line of force in a toroidal field. Closure of the lines is a very special circumstance and more usually in a toroidal system the lines of force are not closed. Instead, as e.g. in a stellarator with small rotational transform, the field lines generate toroidal magnetic surfaces. The structure of such fields has been considered in detail by Kruskal and Kulsrud[10] whose notation will be closely followed.

A magnetic field possessing magnetic surfaces can be represented by

$$\underset{\sim}{B} = \underset{\sim}{\nabla}\psi \times \underset{\sim}{\nabla}\nu \qquad \ldots(8.1)$$

where ψ is a single valued function which is constant on each toroidal magnetic surface and ν is a multiple-valued function. By a suitable choice of scale, ψ can be made equal to the longitudinal magnetic flux inside the magnetic surface ψ. Then ν is an angle like variable which increases by unity during one loop encircling the magnetic axis and increases by $\iota/2\pi$ during one circuit around the torus.

For the moment we ignore any complexity introduced by the multivalued nature of ν e.g. by introducing appropriate "cuts" across the torus. Then we can use ψ, ν in exactly the same way that we used α, β in parts I and II, and the equilibrium constraints can be determined by the same procedure. We consider only the $\sigma = +1$ stream, the changes necessary for $\sigma = -1$ are obvious.

In zero order the constraint is again that f_o be independent of φ and in first order that it satisfy

$$\langle Df_o \rangle = \frac{q}{B}\frac{\partial f_o}{\partial s} = 0 \ . \qquad \ldots (8.2)$$

This implies that f_o is constant along a line of force and as each line generates its magnetic surface this is usually interpreted to mean that f_o must be constant over a magnetic surface. This is also indicated directly by the alternative form of (8.2), namely

$$\langle Df_o \rangle = \frac{q}{B^2}(B \cdot \nabla f_o) = 0 \ . \qquad \ldots (8.3)$$

For the present we adopt this interpretation and examine whether any further constraints appear in next order.

Second Order

The second order equation is

$$\frac{\partial f_2}{\partial \varphi} = Df_1 = Dg_1 + D\int Df_0 \qquad \ldots (8.4)$$

leading to the usual equation for g_1,

$$\frac{q}{B}\frac{\partial g_1}{\partial s} + \left\langle D\int Df_0 \right\rangle = 0 \ , \qquad \ldots (8.5)$$

but because the field lines are no longer assumed to close on themselves this no longer leads directly to the constraint (3.17) which was obtained by integrating around a closed field line. Nevertheless there <u>is</u> a constraint on f_0 implied in (8.5) as can be seen by writing it as

$$\underset{\sim}{B} \cdot \nabla g_1 + \frac{B^2}{q} \left\langle D\int Df_0 \right\rangle = 0 \qquad \ldots (8.6)$$

and then annihilating the first term by multiplying by $|\nabla\psi|^{-1}$ and integrating over a magnetic surface. Then the first term vanishes identically and f_0 must satisfy

$$\iint \frac{dS}{|\nabla\psi|} \frac{B^2}{q} \left\langle D\int Df_0 \right\rangle = 0 \ . \qquad \ldots (8.7)$$

From this it may be deduced[10] that

$$f_0 \equiv f_0 \ (J^*, \mu, \varepsilon) \qquad \ldots (8.8)$$

where J^* is a surface adiabatic invariant

$$J^* = \iint \frac{Bq \ dS}{|\nabla\psi|} \qquad \ldots (8.9)$$

which is an obvious generalisation of the simple invariant J obtained by replacing

$$\lim_{s\to\infty} \int ds \rightarrow \int \frac{BdS}{|\nabla\psi|} \qquad \ldots (8.10)$$

as is appropriate if the line of force covers a ψ surface ergodically. Since J^* is by definition constant over a magnetic surface (8.9) merely confirms what had been concluded from the first order calculation, namely that f_0 is constant over a magnetic surface. Under certain circumstances however (8.8) remains true in the time dependent situation[10].

Although these results appear to be the natural extension of those in part II they may not be really appropriate, particularly when we recall our observations in part I about the relation of the constraints on f_0 to the time scale of the corresponding equilibrium. In particular it is clear that J can only be a relevant quantity on time scales much longer than $L/V_{\parallel}$, (where L is of the order of the circumference of the torus). Similarly J^*, or even the magnetic surface ψ itself can only be relevant on

time scales long compared to the time taken for a particle to "sample" the whole of a magnetic surface which, especially in the case of small rotational transform, may be very long compared to $L/V_{\parallel}$. Consequently the following argument would seem more suitable.

9. SMALL ROTATIONAL TRANSFORM

In the case of small rotational transform one should introduce the transform itself as another small parameter and include it in the expansion (or ordering) procedure. This can be simply done if one regards the field as composed of two parts; a large field which possesses closed field lines and a smaller additional field which produces the rotational transform and the ergodic behaviour of the field lines. (This is, in fact, the conventional way of treating stellarator fields.) Thus we can write

$$\underset{\sim}{B} = \nabla\psi \times \nabla\nu_0 + \nabla\psi \times \nabla\nu_1 \qquad \ldots (9.1)$$

where (ψ, ν_0) label the closed field lines of the dominant field and ν_1 corresponds to the small rotational transform. The final results do not depend on the exact way in which ν is split into its component parts ν_0 and ν_1. Corresponding to the splitting of the magnetic field we can formally regard D as split into a dominant part D_0 and a small D_1, however, as will be seen there is no need to explicitly determine D_1. The calculation proceeds like all its predecessors. In zero order there is no change; in first order there is the constraint $\langle D_0 f_0 \rangle = 0$ indicating that f_0 is a function only of $(\psi, \nu_0, \varepsilon, \mu)$, and f_1 is given by

$$f_1 = g_1(\psi, \nu_0, \varepsilon, \mu, s) + \int Df_0 \ . \qquad \ldots (9.2)$$

In second order there is an essential change from section 8. The second order equation is now

$$\frac{\partial f_2}{\partial \varphi} = D_0 f_1 + \lambda^{-1} D_1 f_0 \qquad \ldots (9.3)$$

so providing the constraint

$$\langle D_0 g_1 \rangle + \langle D_0 \int D_0 f_0 \rangle + \lambda^{-1} \langle D_1 f_0 \rangle = 0 \ . \qquad \ldots (9.4)$$

The first two terms are very familiar; it is only necessary to note that they refer entirely to the field corresponding to ν_0 which has closed field lines. The last term is easily found without the need to explicitly determine D_1 by observing that, since both $\langle D_0 f_0 \rangle$ and $B_0 \cdot \nabla f_0$ are zero,

$$\langle D_1 f_0 \rangle = \langle D f_0 \rangle = \frac{q}{B^2} (\underset{\sim}{B} \cdot \nabla f_0) = \frac{q}{B^2} (\underset{\sim}{B}_1 \cdot \nabla f_0) \ . \qquad \ldots (9.5)$$

Hence equation (9.4) can be written

$$\frac{q}{B}\left(\frac{\partial g_1}{\partial s}\right)_o + \left\langle D_o \int D_o\, f_o \right\rangle + \lambda^{-1}\, \frac{q}{B^2}\,(\underset{\sim}{B}\cdot \nabla f_o) = 0 \qquad \ldots (9.6)$$

where $(\partial g_1/\partial s)_o$ must be taken along the direction of $\underset{\sim}{B}_o$. The constraint which arises from (9.6) and the requirement that g_1 be single valued is

$$\oint \frac{B ds}{q} \left\langle D \int D\, f_o \right\rangle + \lambda^{-1} \oint \frac{\underset{\sim}{B}\cdot \nabla f_o}{B_o}\, ds = 0 \ . \qquad \ldots (9.7)$$

The familiar first term in (9.7) becomes

$$\left(\frac{\partial f_o}{\partial \psi}\,\frac{\partial J}{\partial \nu} - \frac{\partial f_o}{\partial \nu}\,\frac{\partial J}{\partial \psi}\right) \qquad \ldots (9.8)$$

where J is defined by the line integral

$$J \equiv \oint \left[2(\varepsilon - \mu B)\right]^{\frac{1}{2}}\, ds \qquad \ldots (9.9)$$

around the closed line corresponding to the dominant part of the field. The second term in (9.7) is unfamiliar but can be reduced to a more transparent form by introducing $\underset{\sim}{B} = (\nabla\psi \times \nabla\nu)$ and

$$\nabla f_o = \nabla\psi\,\frac{\partial f_o}{\partial \psi} + \nabla\nu_o\,\frac{\partial f_o}{\partial \nu_o} \qquad \ldots (9.10)$$

then

$$\oint \frac{\underset{\sim}{B}\cdot \nabla f_o}{B_o}\, ds = -\frac{\partial f_o}{\partial \nu_o} \oint \frac{(\underset{\sim}{B}_o \cdot \underset{\sim}{\nabla\nu})}{B_o}\, ds \equiv -\frac{\partial f_o}{\partial \nu_o} \oint \underset{\sim}{\nabla\nu}\cdot d\underset{\sim}{s} \qquad \ldots (9.11)$$

the last identity following because the path of integration is along the closed field line $\underset{\sim}{B}_o$. Now recalling that the vector potential $\underset{\sim}{A}$ can be written $\underset{\sim}{A} = \psi\nabla\nu$ some further manipulation allows us to write

$$\oint \underset{\sim}{\nabla\nu}\cdot d\underset{\sim}{s} = \frac{\partial}{\partial \psi} \oint \underset{\sim}{A}\cdot d\underset{\sim}{s} \qquad \ldots (9.12)$$

and to show that

$$\frac{\partial}{\partial \nu} \oint \underset{\sim}{A}\cdot d\underset{\sim}{s} = 0 \ . \qquad \ldots (9.13)$$

Collecting these results together the final form of the constraint (9.7) can be written;

$$\left(\frac{\partial f_o}{\partial \psi}\,\frac{\partial J^{**}}{\partial \nu} - \frac{\partial f_o}{\partial \nu}\,\frac{\partial J^{**}}{\partial \psi}\right) = 0 \qquad \ldots (9.14)$$

so that

$$f_o \equiv f_o\,(\mu,\ \varepsilon,\ J^{**}) \qquad \ldots (9.15)$$

where J^{**} is a new form of invariant defined by

$$J^{**} = \oint \left(\underset{\sim}{q} + \frac{e}{m}\,\underset{\sim}{A}\right)\cdot d\underset{\sim}{s} \ . \qquad \ldots (9.16)$$

This is a result which could have been anticipated from the form of the canonical angular momentum in a magnetic field. However the value and simplicity of introducing J^{**} does not seem to have been appreciated. By its use one automatically incorporates the effect of both rotational transform and that of g.c. drifts, whereas previous calculations[11] have achieved this only by a direct calculation. For example the average motion of a particle under the combined effect of transform and drift is given by

$$\bar{\dot{\psi}} = \left(\frac{\partial J^{**}}{\partial \nu}\right)\left(\frac{\partial J}{\partial \varepsilon}\right)^{-1} \quad \bar{\dot{\nu}} = -\left(\frac{\partial J^{**}}{\partial \psi}\right)\left(\frac{\partial J}{\partial \varepsilon}\right)^{-1} \qquad \ldots (9.17)$$

and the particle remains on a surface of constant J^{**}, not on a magnetic surface of constant ψ. Similarly the equilibrium distribution f_o is not constant over a magnetic surface but over a J^{**} surface. This is because the actual path of a particle is due to a combination of the drifts, which divert the particle from the magnetic surface and the rotational transform which generates the surface. Both these effects are included in J^{**}; $\oint q\, ds$ represents the effect of drifts and $\oint A\, ds$ that of the rotational transform. When the latter is the dominant effect (small drift during one circuit compared to $\iota/2\pi$) then

$$J^{**} \rightarrow \frac{e}{m} \oint A\, ds$$

which <u>is</u> constant over a magnetic surface. Hence the results of section 8 can be recovered in the appropriate limit.

The invariant J^{**} is different according to whether the particle is moving parallel or antiparallel to the field. This asymmetry arises because the rotational transform and drifts are additive in the one case and subtractive in the other, this asymmetry in J^{**} has, therefore, no connection with the asymmetry in J_1 (part II) which is a rather subtle consequence of the dynamics of a charged particle in a magnetic field.

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11. REFERENCES

1. TAYLOR, J.B., Phys. Fluids, 6, 1529, (1963).

2. TAYLOR, J.B., Phys. Fluids, 7, 767, (1964).

3. NORTHROP, T.G. and TELLER, E., Phys. Rev. 117, 215, (1960).

4. CHANDRASEKHAR, S., KAUFMAN, A.N. and WATSON, K.M., Ann. Phys. 5, 1, (1958).

5. KRUSKAL, M.D., J. Math. Phys. 3, 806, (1962).
BERKOWITZ, J. and GARDNER, C. Comm. on Pure and Applied Math. 12, 501, (1959).
GARDNER, C. Phys. Rev. 115, 791, (1959).

6. BenDANIEL, D.L. Phys. Rev. 8, 1567, (1965).

7. KRUSKAL, M.D. Princeton Univ. Report PM-S-33 (1958).

8. NORTHROP, T.G., LIU, C.S. and KRUSKAL, M.D. Private Communication. Lawrence Radiation Laboratory Report UCRL-16610.

9. HASTIE, R.J., HAAS, F.A. and TAYLOR, J.B. Culham Laboratory Report in preparation.

10. KULSRUD, R.M. Phys. Fluids, 4, 403 (1961).

11. MOROZOV, A.I. and SOLOCEV, L.S. Zh. Tech. Fiz. 30, 271, (1960-
English Transl. Soviet Phys. Tech. Phys. 5, 250 (1960).

APPENDIX A

As an example of a guiding center distribution which is not of the equilibrium form $F(\mu, \varepsilon, J)$ but which nevertheless leads to anisotropic pressures satisfying the fluid equilibrium constraints, one may take the separable distribution (for each species).

$$F_i\ (\mu, \varepsilon, \alpha, \beta) = H_i\ (\mu, \varepsilon)\ Q(\alpha, \beta)\ . \qquad \ldots \text{A.1}$$

Clearly (A.1) is not in general a guiding center equilibrium. The pressure resulting from (A.1) can be expressed in the form

$$\begin{aligned} p_\perp &= R_\perp(B)\ \ Q(\alpha, \beta) \\ p_\parallel &= R_\parallel(B)\ \ Q(\alpha, \beta) \end{aligned} \qquad \ldots \text{A.2}$$

where

$$R_\perp\ (B) = \sum_i m_i \int \frac{\mu B^2}{q} H_i\ (\mu, \varepsilon)\ d\mu\ d\varepsilon \qquad \ldots \text{A.3}$$

$$R_\parallel\ (B) = \sum_i m_i \int q\ B\ H_i\ (\mu, \varepsilon)\ d\mu\ d\varepsilon\ . \qquad \ldots \text{A.4}$$

It can be seen by direct substitution that A.2 satisfies the first fluid constraint

$$\frac{\partial p_\parallel}{\partial s} + \frac{(p_\perp - p_\parallel)}{B} \frac{\partial B}{\partial s} = 0\ . \qquad \ldots \text{A.5}$$

When (A.2) is substituted into the second fluid constraint

$$\int \nabla\ (p_\perp + p_\parallel)\ \frac{(\underset{\sim}{B} \times \nabla\ B)}{B^4}\ ds = 0 \qquad \ldots \text{A.6}$$

the result can be written as

$$\int d\chi\ \frac{(R_\parallel(B) + R_\perp(B))}{B^3} \left\{ \frac{\partial Q}{\partial \alpha} \frac{\partial B}{\partial \beta} - \frac{\partial Q}{\partial \beta} \frac{\partial B}{\partial \alpha} \right\} = 0 \qquad \ldots \text{A.7}$$

However, on differentiating (A.4),

$$\frac{\partial}{\partial \beta} \left(\frac{R_\parallel}{B^2} \right) = - \frac{(R_\parallel(B) + R_\perp(B))}{B^3} \frac{\partial B}{\partial \beta} \qquad \ldots \text{A.8}$$

so that (A.7) becomes

$$\frac{\partial Q}{\partial \alpha} \frac{\partial}{\partial \beta} \left(\int R_\parallel(B) \frac{d\chi}{B^2} \right) - \frac{\partial Q}{\partial \beta} \frac{\partial}{\partial \alpha} \left(\int R_\parallel(B) \frac{d\chi}{B^2} \right) = 0 \qquad \ldots \text{A.9}$$

The second equilibrium condition (A.6) can therefore be satisfied by making $Q(\alpha,\beta)$ a function of $\int R_\parallel(B) \frac{ds}{B}$. Another form for this, obtained from (A.4) is

$$\int R_\parallel(B) \frac{ds}{B} = \sum_i m_i \int H_i\ (\mu, \varepsilon)\ J(\mu, \varepsilon, \alpha, \beta)\ d\mu\ d\varepsilon \equiv \langle J \rangle \qquad \ldots \text{A.10}$$

so the distribution (A.1) can be expressed as

$$F_i\ (\mu, \varepsilon, \alpha, \beta) = H_i\ (\mu, \varepsilon)\ \ Q\ (\langle J \rangle)\ . \qquad \ldots \text{A.11}$$

Although this is not a g.c. equilibrium, it nevertheless satisfies both of the fluid equilibrium conditions (A.5), (A.6).

APPENDIX B

Coordinate Systems

The operators introduced in part I and discussed in the following Appendix are most conveniently evaluated in one of the following coordinate systems.

The first is a generalisation of that discussed by Chandrasekhar, Kaufman and Watson[4]. Three orthogonal unit vectors $\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3$ are chosen with $\underset{\sim}{e}_1 \equiv \underset{\sim}{B}/B$ but unlike C.K.W. we do not necessarily choose $\underset{\sim}{e}_2$ in the direction of the principal normal; for some purposes other choices are more useful. Then if ds, dx, dy denote elements of arc length in the direction $\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3$ respectively.

$$\nabla = \underset{\sim}{e}_1 \frac{\partial}{\partial s} + \underset{\sim}{e}_2 \frac{\partial}{\partial x} + \underset{\sim}{e}_3 \frac{\partial}{\partial y} \qquad \text{... B.1}$$

and

$$\begin{aligned} \frac{\partial \underset{\sim}{e}_1}{\partial s} &= \rho_1 \underset{\sim}{e}_2 - \sigma_1 \underset{\sim}{e}_3 \\ \frac{\partial \underset{\sim}{e}_2}{\partial s} &= -(\tau_1 \underset{\sim}{e}_3 + \rho_1 \underset{\sim}{e}_1) \\ \frac{\partial \underset{\sim}{e}_3}{\partial s} &= \tau_1 \underset{\sim}{e}_2 + \sigma_1 \underset{\sim}{e}_1 \quad . \end{aligned} \qquad \text{... B.2}$$

Here ρ_1 σ_1 define the curvature of the line of force and τ_1 is related to the torsion. [If we choose $\underset{\sim}{e}_2$ along the principal normal τ_1 <u>is</u> the torsion, otherwise the torsion equals $\tau_1 + \frac{d}{ds}(\tan^{-1} \sigma_1/\rho_1)$.] The other derivatives are

$$\begin{aligned} \frac{\partial \underset{\sim}{e}_1}{\partial x} &= \rho_2 \underset{\sim}{e}_2 - \tau_2 \underset{\sim}{e}_3 & \frac{\partial \underset{\sim}{e}_1}{\partial y} &= \tau_3 \underset{\sim}{e}_2 + \rho_3 \underset{\sim}{e}_3 \\ \frac{\partial \underset{\sim}{e}_2}{\partial x} &= \sigma_2 \underset{\sim}{e}_3 - \rho_2 \underset{\sim}{e}_1 & \frac{\partial \underset{\sim}{e}_2}{\partial y} &= \sigma_3 \underset{\sim}{e}_3 - \tau_3 \underset{\sim}{e}_1 \\ \frac{\partial \underset{\sim}{e}_3}{\partial x} &= \tau_2 \underset{\sim}{e}_1 - \sigma_2 \underset{\sim}{e}_2 & \frac{\partial \underset{\sim}{e}_3}{\partial y} &= -\rho_3 \underset{\sim}{e}_1 - \sigma_3 \underset{\sim}{e}_2 \quad . \end{aligned} \qquad \text{... B.3}$$

Because s, x, y are not true curvilinear coordinates the derivatives are non-commuting; (in fact the operation $\partial/\partial x$ must only be regarded as short hand for $\underset{\sim}{e}_2 \cdot \nabla$ etc.). The commutators, which are required extensively in the evaluation of the operators in Appendix C, are:

$$\begin{aligned} \frac{\partial^2 Q}{\partial x\, \partial y} - \frac{\partial^2 Q}{\partial y\, \partial x} &= (\tau_2 + \tau_3) \frac{\partial Q}{\partial s} - \sigma_2 \frac{\partial Q}{\partial x} - \sigma_3 \frac{\partial Q}{\partial y} \\ \frac{\partial^2 Q}{\partial s\, \partial x} - \frac{\partial^2 Q}{\partial x\, \partial s} &= (\tau_2 - \tau_1) \frac{\partial Q}{\partial y} - \rho_2 \frac{\partial Q}{\partial x} - \rho_1 \frac{\partial Q}{\partial s} \\ \frac{\partial^2 Q}{\partial s\, \partial y} - \frac{\partial^2 Q}{\partial y\, \partial s} &= (\tau_1 - \tau_3) \frac{\partial Q}{\partial x} - \rho_3 \frac{\partial Q}{\partial y} + \sigma_1 \frac{\partial Q}{\partial s} \quad . \end{aligned} \qquad \text{... B.4}$$

Other useful identities include

$$\text{div } \underset{\sim}{e}_1 = \rho_2 + \rho_3 = -\frac{1}{B}\frac{\partial B}{\partial s} \qquad \text{... B.5}$$

$$\text{curl } \underset{\sim}{e}_1 = \rho_1 \underset{\sim}{e}_3 + \sigma_1 \underset{\sim}{e}_2 - (\tau_2 + \tau_3)\, \underset{\sim}{e}_1 \qquad \text{... B.6}$$

and

$$\frac{j_{\parallel}}{B} = \underset{\sim}{e}_1 \cdot \text{curl } \underset{\sim}{e}_1 = -(\tau_2 + \tau_3) \quad . \qquad \text{... B.7}$$

A second coordinate system which is frequently convenient is one in which the lines of force are labelled by α, β, where $\underset{\sim}{B} = \nabla\alpha \times \nabla\beta$. Then α, β form two coordinates and as the third coordinate one may use the distance s along the line of force. Then (α,β,s) form a true curvilinear coordinate system with

$$\underset{\sim}{\nabla} = \nabla\alpha \frac{\partial}{\partial\alpha} + \nabla\beta \frac{\partial}{\partial\beta} + \nabla s \frac{\partial}{\partial s} \quad .$$

However this is still not an orthogonal coordinate system and the fact that ∇s is not in the direction of $\underset{\sim}{B}$ causes much complication which can be avoided in the special case of a vacuum magnetic field ($\underset{\sim}{B} = \nabla\chi$) or in one in which $\underset{\sim}{j} \cdot \underset{\sim}{B} \equiv 0$, ($\underset{\sim}{B} = \psi\nabla\chi$). For then one can use (α, β, χ) as curvilinear coordinates with

$$\nabla = \nabla\alpha \frac{\partial}{\partial\alpha} + \nabla\beta \frac{\partial}{\partial\beta} + \nabla\chi \frac{\partial}{\partial\chi} \quad .$$

In this case $\nabla\chi$ <u>is</u> the direction of $\underset{\sim}{B}$ and this affords an enormous simplification of the algebraic steps in the evaluation of the various operators.

APPENDIX C

In the body of this paper we have shown that the equilibrium conditions and the adiabatic invariants are determined by operators D, L, H and their integrals. For convenience we now collect the operators together and outline their origins; the full calculations, especially of the later operators, are too lengthy to be included here and will be published elsewhere[9].

(a) The basic operator D

This is merely the operator $\frac{1}{B}\,\underset{\sim}{v}\cdot\underset{\sim}{\nabla}$ expressed in terms of μ, ε, φ, σ and using $\underset{\sim}{e}_1$, $\underset{\sim}{e}_2$, $\underset{\sim}{e}_3$ as base vectors. Then

$$\begin{aligned}\frac{1}{B}\,\underset{\sim}{v}\cdot\underset{\sim}{\nabla} \rightarrow D \equiv \frac{1}{B}\Big\{ & \sigma q \frac{\partial}{\partial s} + c_\perp \cos\varphi \left(\frac{\partial}{\partial x} - V_x \frac{\partial}{\partial \mu}\right) + c_\perp \sin\varphi \left(\frac{\partial}{\partial y} - V_y \frac{\partial}{\partial \mu}\right) \\ & + \mu\sigma q\,(\rho_3 - \rho_2)\cos 2\varphi \frac{\partial}{\partial \mu} + \mu\sigma q\,(\tau_2 - \tau_3)\sin 2\varphi \frac{\partial}{\partial \mu}\Big\} \\ & + \frac{1}{B}\Big\{\sigma q\left[\tau_1 + \tfrac{1}{2}(\tau_2 + \tau_3)\right] + \cos\varphi\left[\frac{q^2}{c_\perp}\sigma_1 - c_\perp \sigma_2\right] + \sin\varphi\left[\frac{q^2}{c_\perp}\rho_1 - c_\perp \sigma_3\right] \\ & + \tfrac{1}{2}\sigma q\,(\tau_2 - \tau_3)\cos 2\varphi + \tfrac{1}{2}\sigma q\,(\rho_2 - \rho_3)\sin 2\varphi\Big\}\frac{\partial}{\partial \varphi} \qquad \ldots \text{C.1}\end{aligned}$$

where

$$q \equiv \left[2\,(\varepsilon - \mu B)\right]^{\frac{1}{2}} \quad ; \quad c_\perp \equiv (2\mu B)^{\frac{1}{2}}$$

$$V_x = \frac{\mu}{B}\frac{\partial B}{\partial x} + \frac{q^2}{B}\rho_1 \quad ; \quad V_y = \frac{\mu}{B}\frac{\partial B}{\partial y} - \frac{q^2}{B}\sigma_1 \qquad \ldots \text{C.2}$$

The significance of σ has been described in section 3 and $\partial/\partial x \equiv \underset{\sim}{e}_2\cdot\nabla$ etc.

All other operators are formed by repeated operation with D. Fortunately we do not need these operators as applied to a general function but only to functions which are independent of φ, or even of both φ and s. First, therefore we record that for an f_o independent of φ;

$$\langle Df_o \rangle = \frac{\sigma q}{B}\frac{\partial f_o}{\partial s} \qquad \ldots \text{C.3} = (3.10)$$

and for an f_o independent of φ and s;

$$\begin{aligned}\int Df_o = \frac{1}{B}\Big\{ & c_\perp \sin\varphi\left(\frac{\partial f_o}{\partial x} - V_x \frac{\partial f_o}{\partial \mu}\right) - c_\perp \cos\varphi\left(\frac{\partial f_o}{\partial y} - V_y \frac{\partial f_o}{\partial \mu}\right) \\ & + \frac{\mu\sigma q}{2}\frac{\partial f_o}{\partial \mu}\left[(\rho_3 - \rho_2)\sin 2\varphi + (\tau_3 - \tau_2)\cos 2\varphi\right]\Big\} \qquad \ldots \text{C.4}\end{aligned}$$

(b) The operator $Lf_o \equiv \langle D \int Df_o \rangle$

For a function f_o independent of φ and s this is found by operating on (C.4) with D and using the commutator relations of Appendix B. It can then be expressed as

$$Lf_o = \frac{\underset{\sim}{W}_d\cdot\nabla f_o}{B} + \mu\,\frac{\partial f_o}{\partial \mu}\,\frac{q}{B}\,\frac{\partial}{\partial s}\,\frac{q\,(\underset{\sim}{e}_1\cdot \text{curl}\,\underset{\sim}{e}_1)}{B} \qquad \ldots \text{C.5}$$

where

$$\underset{\sim}{W}_d = \frac{e}{m}\underset{\sim}{V}_d = \underset{\sim}{e}_1 \times (q^2 \underset{\sim}{\rho} + \mu \underset{\sim}{\nabla} B)/B \quad .$$

We also need

$$\int_{s_0}^{s} \frac{Bds}{q} Lf_o = \int_{s_0}^{s} \frac{ds}{q} \underset{\sim}{W}_d \cdot \nabla f_o + \mu \frac{\partial f_o}{\partial \mu} \; q \; \frac{\underset{\sim}{e}_1 \cdot (\underset{\sim}{\nabla} \times \underset{\sim}{e}_1)}{B} \qquad \ldots \text{C.6}$$

and

$$\oint \frac{Bds}{q} Lf_o = \oint \frac{ds}{q} \underset{\sim}{W}_d \cdot \nabla f_o \quad . \qquad \ldots \text{C.7}$$

This last integral can be expressed in a far more convenient form by transforming to the (α, β, s) coordinate system when

$$\oint \frac{Bds}{q} Lf_o \equiv \frac{\partial J}{\partial \beta} \frac{\partial f_o}{\partial \alpha} - \frac{\partial J}{\partial \alpha} \frac{\partial f_o}{\partial \beta} \qquad \ldots \text{C.8}$$

(c) <u>The operator Hf_o</u>

This is defined in terms of the operators D and L by

$$Hf_o = \oint \frac{Bds}{q} \left\{ \sigma L \int_{s_0}^{s} \frac{Bds}{q} Lf_o - \left\langle D \int D \int Df_o \right\rangle \right\} \qquad \ldots \text{C.9}$$

and is required only for a function f_o which is not merely independent of φ and s but is restricted to the form $f_o(\mu, \varepsilon, J)$.

For the case of mirror reflection (oscillating particle), Hf_o can be found fairly easily: it will be recalled that our definition of the loop integral for oscillating particles implies summation over $\sigma = \pm 1$ and that f_o is even in σ. The operator L is even in σ and $\langle D \int D \int D \rangle$ can be shown to be odd so that the whole integrand (in C.9) is odd in σ and the loop integral vanishes identically.

Unfortunately no corresponding simplification occurs for particles circulating round a closed field line, Hf_o, is then non-zero and must be evaluated in full. The calculation in a general magnetic field is extremely lengthy, but is considerably simplified in the special case of a vacuum magnetic field. In general H may be reduced to the form

$$Hf_o = \sigma \oint \frac{ds}{q} \underset{\sim}{W}_d \cdot \underset{\sim}{\nabla} \int_{s_0}^{s} \frac{ds'}{q(s')} \underset{\sim}{W}_d \cdot \underset{\sim}{\nabla} f_o - \mu\sigma \oint \frac{ds}{B} (\nabla \times \underset{\sim}{Q}) \cdot \underset{\sim}{\nabla} f_o \qquad \ldots \text{C.10}$$

where

$$Q = [\tau_1 + \tfrac{1}{2}(\tau_2 + \tau_3)] \underset{\sim}{e}_1 - \sigma_2 \underset{\sim}{e}_2 - \sigma_3 \underset{\sim}{e}_3 \quad . \qquad \ldots \text{C.11}$$

A further crucial step is the reduction of C.10 to the form of a complete derivative along J = constant, since only then can the invariants of part II be obtained in local form.

This reduction is achieved by introducing the (α, β, χ) or the (α, β, s) coordinates. Then the first term of Hf_o can be reduced to

$$\frac{\partial f_o}{\partial J}\left(\frac{\partial J}{\partial \beta}\frac{\partial P_1}{\partial \alpha} - \frac{\partial J}{\partial \alpha}\frac{\partial P_1}{\partial \beta}\right) \qquad \ldots \text{C.12}$$

with

$$P_1 = -\frac{\sigma}{2}\oint \frac{ds''}{q(s'')}\int_{s_o}^{s''}\frac{ds'}{q(s')}\,G(s', s'') \qquad \ldots \text{C.13}$$

where $G(s', s'')$ is a zero order quantity defined by

$$G(s', s'') \equiv \left(\frac{e}{m}\right)^2 [\dot{\alpha}(s')\,\dot{\beta}(s'') - \dot{\alpha}(s'')\,\dot{\beta}(s')] \qquad \ldots \text{C.14}$$

and $\dot{\alpha}$, $\dot{\beta}$ are the first order drifts $\underset{\sim}{V}_d \cdot \nabla\alpha$, $\underset{\sim}{V}_d \cdot \nabla\beta$.

The second part of Hf_o can be reduced to a similar form to C.12, that is

$$\mu\sigma\oint \frac{ds}{B}(\nabla \times \underset{\sim}{Q})\cdot\nabla f_o = \frac{\partial f_o}{\partial J}\left(\frac{\partial J}{\partial \beta}\frac{\partial P_2}{\partial \alpha} - \frac{\partial J}{\partial \alpha}\frac{\partial P_2}{\partial \beta}\right) \qquad \ldots \text{C.15}$$

with

$$P_2 = \mu\sigma\oint [\tau_1 + \tfrac{1}{2}(\tau_2 + \tau_3)]\,ds' \;. \qquad \ldots \text{C.16}$$

So that finally

$$Hf_o = \frac{\partial f_o}{\partial J}\left(\frac{\partial J}{\partial \beta}\frac{\partial P}{\partial \alpha} - \frac{\partial J}{\partial \alpha}\frac{\partial P}{\partial \beta}\right) \qquad \ldots \text{C.17}$$

with

$$P = -\frac{\sigma}{2}\oint \frac{ds''}{q(s'')}\int_{s_o}^{s}\frac{ds'}{q(s')}\,G(s', s'') + \mu\sigma\oint [\tau_1 + \tfrac{1}{2}(\tau_2 + \tau_3)]\,ds' \qquad \ldots \text{C.18}$$

(d) <u>Operators G for time dependent case</u>

In part III the time dependent problem was investigated by transforming the Vlasov equation to a coordinate frame moving with the flux preserving velocity $U = \frac{(\underset{\sim}{E} + \underset{\sim}{\nabla}\psi) \times \underset{\sim}{B}}{B^2}$, and then introducing μ, ε, φ variables to replace $(\underset{\sim}{v} - \underset{\sim}{U})$. Equation (3.1) of part I is then replaced by (7.5), namely

$$\frac{\partial f}{\partial \varphi} = \lambda\left(\frac{1}{B}\frac{\partial f}{\partial t} + Df + Gf\right) \qquad \ldots \text{C.19}$$

where D is the same operator as in parts I and II (C.1), and the operator G depends explicitly on $\underset{\sim}{U}$ and on $\psi = \varphi + \alpha\frac{\partial \beta}{\partial t}$, and can be separated into operators G^{ψ}, G^{U} which involve respectively ψ and $\underset{\sim}{U}$ only.

Thus $G = \lambda^{-1}G^{\psi} + G^{U}$, where

$$G^{\psi} = -\frac{1}{B}\left[\sigma q\frac{\partial \psi}{\partial s}\frac{\partial}{\partial \varepsilon} + \underset{\sim}{\mathcal{E}}_{\perp}\cdot\nabla\psi\left(\frac{1}{B}\frac{\partial}{\partial \mu} + \frac{\partial}{\partial \varepsilon}\right) + \frac{\underset{\sim}{e}_1 \times \underset{\sim}{\mathcal{E}}_{\perp}}{2\mu B}\cdot\nabla\psi\frac{\partial}{\partial \varphi}\right] \qquad \ldots \text{C.20}$$

and

$$\frac{1}{B}\frac{\partial}{\partial t} + G^U = \frac{1}{B}\left\{\frac{d}{dt} + \left(q^2\, \underset{\sim}{U}\cdot\underset{\sim}{e} + \mu B\frac{dB}{dt} + \sigma q\frac{\partial}{\partial s}\,\tfrac{1}{2}U^2\right)\frac{\partial}{\partial\varepsilon}\right.$$

$$-\left[\underset{\sim}{e}_\perp\cdot\frac{d\underset{\sim}{U}}{dt} + \sigma q\,\underset{\sim}{e}_\perp\cdot\frac{\partial\underset{\sim}{U}}{\partial s} + \mu B\cos 2\varphi\left(\underset{\sim}{e}_2\cdot\frac{\partial\underset{\sim}{U}}{\partial x} - \underset{\sim}{e}_3\cdot\frac{\partial\underset{\sim}{U}}{\partial y}\right)\right.$$

$$\left.+\,\mu B\sin 2\varphi\left(\underset{\sim}{e}_2\cdot\frac{\partial\underset{\sim}{U}}{\partial y} + \underset{\sim}{e}_3\cdot\frac{\partial\underset{\sim}{U}}{\partial x}\right)\right]\left(\frac{1}{B}\frac{\partial}{\partial\mu} + \frac{\partial}{\partial\varepsilon}\right) - \frac{\sigma q}{B}\,\underset{\sim}{e}_\perp\cdot\frac{\partial\underset{\sim}{U}}{\partial s}\frac{\partial}{\partial\mu}$$

$$\left.-\,\frac{1}{2\mu B}\,\underset{\sim}{e}_1\times\underset{\sim}{e}_\perp\cdot\left[\frac{d\underset{\sim}{U}}{dt} + 2\sigma q\frac{\partial\underset{\sim}{U}}{\partial s} + (\underset{\sim}{e}_\perp\cdot\underset{\sim}{\nabla})\,\underset{\sim}{U}\right]\frac{\partial}{\partial\varphi}\right\} \qquad \ldots \text{C.21}$$

where $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \underset{\sim}{U}\cdot\underset{\sim}{\nabla}$ and $\underset{\sim}{e}_\perp = (2\mu B)^{\frac{1}{2}}\,(\underset{\sim}{e}_2\cos\varphi + \underset{\sim}{e}_3\sin\varphi)$.

(e) <u>The operator M</u>

The operator M (equation 7.16) is obtained from the operators D and G and is defined by

$$Mf_o = \tfrac{1}{2}\frac{\sigma q}{B}\frac{\partial}{\partial s}\left(\psi^2\frac{\partial^2 f_o}{\partial\varepsilon^2}\right) - \left\langle D\int G^\psi f_o\right\rangle - \left\langle G^\psi\int Df_o\right\rangle$$

$$-\,L\left(\frac{\partial f_o}{\partial\varepsilon}\psi\right) + \sigma\left\langle G^\psi\int_{s_o}^{s}\frac{Bds}{q}Lf_o\right\rangle - \left\langle G^U f_o\right\rangle\ . \qquad \ldots \text{C.22}$$

The operator obtained by integrating (C.22) over a field line is also needed. This is

$$\oint\frac{Bds}{q}Mf_o = -\oint\frac{Bds}{q}\left\langle G^U f_o\right\rangle$$

$$-\oint\frac{Bds}{q}\left\{\left\langle D\int G^\psi f_o\right\rangle + \left\langle G^\psi\int Df_o\right\rangle - L\left(\psi\frac{\partial f_o}{\partial\varepsilon}\right)\right.$$

$$\left.+\,\sigma\left\langle G^\psi\int_{s_o}^{s}\frac{Bds}{q}Lf_o\right\rangle\right\}\ . \qquad \ldots \text{C.23}$$

Using the explicit form for the operator G, it can be shown that the first term in (C.23) is

$$\left(\frac{\partial f_o}{\partial\varepsilon}\frac{\partial J}{\partial t} - \frac{\partial f_o}{\partial t}\frac{\partial J}{\partial\varepsilon}\right)\ . \qquad \ldots \text{C.24}$$

The remaining parts of (C.23) can also be reduced, after a tedious calculation to the form

$$\left(\frac{\partial f_o}{\partial\alpha}\frac{\partial\bar{\psi}}{\partial\beta} - \frac{\partial f_o}{\partial\beta}\frac{\partial\bar{\psi}}{\partial\alpha}\right) \qquad \ldots \text{C.25}$$

where

$$\bar{\psi} = \oint\psi\,\frac{ds}{q}\ . \qquad \ldots \text{C.26}$$

Finally, therefore, (C.23) becomes

$$\oint\frac{Bds}{q}Mf_o = \left(\frac{\partial f_o}{\partial\varepsilon}\frac{\partial J}{\partial t} - \frac{\partial f_o}{\partial t}\frac{\partial J}{\partial\varepsilon}\right) + \left(\frac{\partial f_o}{\partial\alpha}\frac{\partial\bar{\psi}}{\partial\beta} - \frac{\partial f_o}{\partial\beta}\frac{\partial\bar{\psi}}{\partial\alpha}\right) \qquad \ldots \text{C.27}$$

When this last expression is integrated around a closed drift surface J = constant the second part contributes nothing to the integral and leaves us with the result quoted in equation (7.20).