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THE EFFECT OF ION COLLISIONS AND OTHER TERMS ON THE RESISTIVE-G INSTABILITY IN PLASMAS

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THE EFFECT OF ION COLLISIONS AND OTHER TERMS ON THE RESISTIVE-G INSTABILITY IN PLASMAS

by

T.E. STRINGER

(Submitted for publication in Physics of Fluids)

A B S T R A C T

The treatment is based on guiding centre equations, which include collisional diffusion across the field lines. The eigenvalues of the resulting differential equation are found using phase-integral methods. Simple expressions are obtained for the conditions under which the resistive-g mode is stabilised. These show satisfactory agreement with a recent numerical solution of the same problem.

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CONTENTS

1. INTRODUCTION

Until recently theory predicted[1,2,3] that a plasma confined by a sheared magnetic field would be unstable to the resistive interchange mode if the magnetic field lines curved concavely towards the plasma, as is the case in all toroidal confinement devices. The instability could be slowed down, but never completely suppressed, by increasing shear in the magnetic field. As resistivity decreased, the maximum growth rate should decrease in proportion, but there was no critical value below which the instability completely disappeared.

Coppi and Rosenbluth[4] recently considered the effect on this mode of ion-ion collisions and other refinements. They start from the Vlasov equation with BGK collision term, and integrate along particle orbits. Although the details are not given, this is likely to involve rather lengthy analysis. In section 3 the same problem is treated using guiding centre equations for the motion perpendicular to the field lines, including finite Larmor radius (FLR) corrections, and using the generalised Ohms law for the current along field lines. Although the range of validity of these equations is not so wide as those used in reference 4, this method is analytically simpler and the equations can readily be interpreted physically. Apart from a few unimportant details, the final differential equation agrees with that of reference 4 when the wavelength is assumed larger than an ion Larmor radius.

Coppi and Rosenbluth[4] solved their equation numerically over a range of parameters and found that there is now a critical value of the collision frequency below which the mode is absolutely stable. Its analytical solution is difficult because of the number of new terms, each of which can be important over some part of the parameter range. In section 5 the effect of each of these new terms on the standard resistive-g theory is estimated. Parameter space can be divided into a number of regions, within each of which only one of the new effects need be included. Over most of the parameter range it is ion-ion collisions which first limits the validity of the standard theory. In section 6 the eigenvalues of the differential equation including ion-ion collisions is investigated analytically, using the phase-integral method described briefly in section 4. The computation of Coppi and Rosenbluth[4] is compared with the analytical marginal stability curves in section 7, where satisfactory agreement is found. The analytical results cover a very wide range of parameters, whereas prediction based on extrapolation of the computed stability curves can be highly misleading.

2. THE MODEL AND ASSUMPTIONS

Most of the important physical processes contributing to the resistive interchange mode in a plasma column can be simulated by the simpler model of a slab of plasma confined in one direction by a sheared magnetic field. This model is illustrated in Fig. 1. Density varies in the X-direction only. Relative to fixed cartesian co-ordinates (X,Y,Z) the confining magnetic field $\underline{B} = (0,\ sXB_0,\ B_0)$ is everywhere perpendicular to the X-direction, its orientation in the YZ plane varies with X to give shear. The shear parameter s will be assumed constant, and the shear assumed weak so that $sX \ll 1$ over the region of interest. It will sometimes be convenient to use a different co-ordinate system (x,y,z), which are curvilinear coordinates such that the x-axis coincides with the X axis, the z axis rotates with x so that it is always parallel to the local magnetic field, and the y axis is perpendicular to z and x.

A gravitational force, ρg, acting in the X direction simulates the effect of field line curvature in a confined cylindrical plasma. An estimate of the growth rates of the corresponding modes in a cylindrical plasma may be obtained by making the following substitutions

$$\frac{g}{n}\frac{dn}{dx} \rightarrow \frac{2}{R_c \rho}\frac{dp}{dx} \quad , \quad s = \frac{1}{L_s} \rightarrow \frac{r}{L}\frac{d\iota}{dr} \quad , \quad k_y \rightarrow \frac{m}{r} \quad , \qquad \ldots (1)$$

where R_c = radius of curvature of the field line, ι = rotational transform over the length L , m = azimuthal wave number. The shear length, L_s, will be used in place of s in the final expressions for conformity with the notation in reference 4.

Only low-β plasmas will be considered, which permits the variation of B_0 with X to be neglected. In the standard resistive-g theory[2,3] there are two sets of resistive-g modes, of which one set is approximately electrostatic. Only electrostatic modes will be considered here. It will be assumed that the mode frequency ω is much less than the ion cyclotron frequency (Ω_i) , and its wavelength much larger than the ion gyration radius, $a_i = [2\ T_i/M\Omega_i^2]^{1/2}$.

3. DERIVATION OF THE DIFFERENTIAL EQUATION

THE PERTURBED ION DENSITY

The last assumption permits the use of guiding centre equations to describe the motion perpendicular to field lines of both the ions and electrons. The unperturbed velocity of the ion guiding centres is the gravitational drift

$$\underline{V}_0 = -\frac{g}{\Omega_i}\ \underline{i}_y \qquad \ldots (2)$$

where $\underline{i}_y$ is a unit vector in the y direction. The mode produces a perturbation of the ion guiding centre velocity whose mean value, averaged over the ion distribution, is

$$v_{i\perp} = \left(1 + \frac{1}{4} a_i^2 \nabla_\perp^2\right)\left(\frac{\underline{E} \times \underline{B}}{B^2}\right) + \frac{1}{\Omega_i B} \frac{dE}{dt} - \frac{a_i^2 \nu_{ie}}{2N_o} \nabla_\perp N_1 + \frac{a_i^4 \nu_{ii}}{8} \nabla_\perp \left(\frac{1}{N_o} \nabla_\perp^2 N_1\right) - \frac{a_i^2 \nu_{ii}}{4\Omega_i B} \nabla_\perp^2 \underline{E}. \quad \ldots (3)$$

The first term is the electric drift, including the FLR correction which results from the averaging of the perturbation electric field ($\underline{E}$) over the ion gyration orbit[5]. The second term results from the finite ion inertia. The third and fourth terms are the mean diffusion rates resulting from ion-electron collisions and ion-ion collisions, and the last term is the ion mobility due to ion-ion collisions. ν_{ie} and ν_{ii} are the appropriate collision frequencies, N_o is the unperturbed density of ion guiding centres and N_1 the perturbation in this density.

The perturbed density of ion guiding centres may be found from the continuity equation for the guiding centres

$$\frac{\partial N_1}{\partial t} + (\underline{V}_o \cdot \nabla) N_1 + \nabla \cdot (N_o \underline{v}_1) = 0 \quad . \quad \ldots (4)$$

For the modes of interest the phase velocity along the field lines is much larger than the ion thermal velocity. In this case the contribution to $\nabla \cdot v_i$ from ion motion along field lines is negligible. The perturbation may be Fourier analysed in time and in the y direction, giving

$$\underline{E} = \underline{E}(x,z)\, e^{i(k_y y - \omega t)} \quad . \quad \ldots (5)$$

Combining equation (2), (3) and (4) then gives

$$i\left[\omega + \frac{k_y g}{\Omega_i} - \frac{i}{2} \nu_{ie} a_i^2 \nabla_\perp^2 + \frac{i}{8} \nu_{ii} a_i^4 (\nabla_\perp^2)^2\right] N_1$$

$$= \frac{dN_o}{dx}\left[(1 + \frac{1}{4} a_i^2 \nabla_\perp^2) \frac{E_y}{B} - (i\omega + \frac{1}{4} \nu_{ii} a_i^2 \nabla_\perp^2) \frac{E_x}{\Omega_i B}\right] \quad \ldots (6)$$

$$- \frac{N_o}{\Omega_i B}\left[i\omega + \frac{1}{4} \nu_{ii} a_i^2 \nabla_\perp^2\right] \nabla \cdot \underline{E}_\perp + O\left[\left(\frac{\omega}{\Omega_i}\right)^2, (k_\perp a_i)^4\right] .$$

The ion particle density (n), required later for substitution in Poisson's equation, is related to the guiding centre density by the equation[5]

$$n = \left(1 + \frac{a_i^2}{4} \nabla_\perp^2\right) N + O\left(a_i^2 \nabla^2\right)^2 N \quad . \quad \ldots (7)$$

Operating on both sides of equation (6) with the above FLR operator, and substituting $\underline{E} = - \nabla \varphi$, gives

$$i \left[\omega + \frac{k_y g}{\Omega_i} - \frac{i}{2} \nu_{ie} a_i^2 \nabla_\perp^2 + \frac{i}{8} \nu_{ii} a_i^4 (\nabla_\perp^2)^2 \right] n_i$$

$$= \frac{dn_o}{dx} \left[- \left(1 + \frac{1}{2} a_i^2 \nabla_\perp^2\right) \frac{ik_y\varphi}{B} - \frac{1}{\Omega_i B}\left(i\omega + \frac{1}{4} \nu_{ii} a_i^2 \nabla_\perp^2\right) \frac{\partial \varphi}{\partial x} \right] \qquad \ldots (8)$$

$$+ \frac{n_o}{\Omega_i B} \left[i\omega + \frac{1}{4} \nu_{ii} a_i^2 \nabla_\perp^2 \right] \nabla_\perp^2 \varphi + O\left[\left(\frac{\omega}{\Omega_i}\right)^2 , (k_\perp a_i)^4 \right] \frac{n_o k_y^2 \varphi}{B} .$$

THE PERTURBED ELECTRON DENSITY

Since the phase velocity along the field lines is generally less than the electron thermal speed, the parallel velocity must be retained in the electron continuity equation

$$i\omega\, n_{e1} = \frac{dn_o}{B} \frac{E_y}{B} - \frac{\nu_{ei} a_e^2}{2} \nabla_\perp^2 n_{e1} + \frac{\partial v_{e\parallel}}{\partial z} . \qquad \ldots (9)$$

The only finite electron Larmor radius effect which need be included is the electron-ion collisional diffusion (a_e = electron Larmor radius). For convenience, gradients along the field lines will be denoted by $\partial/\partial z$, rather than the more cumbersome $(\underline{B} \cdot \nabla)$ form, e.g.

$$\frac{\partial v_{e\parallel}}{\partial z} \equiv \frac{1}{B^2} (\underline{B} \cdot \nabla)(\underline{v}_e \cdot \underline{B}) . \qquad \ldots (10)$$

The parallel electron velocity may be determined from the component of the generalised Ohms law along the field lines

$$E_\parallel + \frac{1}{n_o e} \frac{\partial p_{e1}}{\partial z} - \eta j_\parallel = \frac{m}{ne^2} \frac{\partial j_\parallel}{\partial t} . \qquad \ldots (11)$$

The electron inertia term on the right hand side may conveniently be combined with the collisional resistance, when the latter is expressed in terms of the electron-ion collision frequency ν_{ei}

$$\eta j_\parallel + \frac{m}{ne^2} \frac{\partial j_\parallel}{\partial t} = \frac{m}{ne^2} (\nu_{ei} - i\omega)\, j_\parallel . \qquad \ldots (12)$$

If the phase velocity along the field lines is assumed less than the electron thermal velocity, it can be shown that the electrons behave isothermally. Thus p_{e1} can be eliminated using equation (9). Since the ion velocity along the field lines is negligible $j_\parallel = -n_o e v_{e\parallel}$, and equation (11) gives

$$\left[\nu_{ei} - i\omega - \frac{iT_e}{\omega m} \frac{\partial^2}{dz^2}\right] v_{e\parallel} = - \frac{e}{m} \left[E_\parallel - \frac{iT_e}{\omega e B n_o} \frac{dn_o}{dx} \frac{\partial E_y}{\partial z} \right] = \frac{e}{m} \left[1 - \frac{k_y U_e}{\omega} \right] \frac{\partial \varphi}{\partial z} , \qquad \ldots (13)$$

where $U_e = -\frac{T_e}{eB}\frac{1}{n_o}\frac{dn_o}{dx}$ = the electron diamagnetic velocity. Eliminating $v_{e\parallel}$ from equation (9) and (13) yields

$$\left[i\omega\,(\nu_{ei} - i\omega) + \frac{T_e}{m}\frac{\partial^2}{\partial z^2} \right]\left[1 - i\,\frac{\nu_{ei} a_e^2}{2\omega}\,\nabla_\perp^2 \right] n_{e1} = -\frac{ik_y}{B}\,(\nu_{ei} - i\omega)\,\frac{dn}{dx}\,\varphi + \frac{ne}{m}\frac{\partial^2\varphi}{\partial z^2} \qquad \ldots (14)$$

for brevity, $\nu_{ei} - i\omega$ will from now on be denoted by ν_e. Electron inertial effects will not be discussed explicity, although the following analysis is valid when these effects are important.

QUASI-NEUTRALITY CONDITION

The $\nabla \cdot \underline{E}$ term in Poisson's equation contributes terms of order $k^2\lambda_D^2$ (λ_D = electron Debye length) which can be neglected, giving the quasi-neutrality equation $n_{i1} = n_{e1}$. Eliminating n_1 from equation (8) and (14) yields the following equation for φ

$$\left[\omega - k_y U_e + \frac{k_y g}{\Omega_i} + \frac{i}{8}\,\nu_{ii}(1+\tau)\,a_i^4(\nabla_\perp^2)^2 \right]\frac{\partial^2\varphi}{\partial z^2} - \tau\left(\omega - k_y U_i\right)\frac{\partial^2}{\partial z^2}\left[\frac{a_i^2}{2}\,\nabla_\perp^2\,\varphi\right]$$

$$-\frac{i\nu_e}{\Omega_e\Omega_i}\left[k_y^2 g\,\frac{1}{n}\frac{dn}{dx}\,\varphi + \left(\omega - k_y U_i\right)\left(\omega - \frac{i}{4}\,\nu_{ii}\,a_i^2\,\nabla_\perp^2\right)\nabla_\perp^2\,\varphi \right] = 0 \;, \qquad \ldots (15)$$

where $$U_i = \frac{T_i}{eB}\frac{1}{n_o}\frac{dn_o}{dx} = \text{the ion diamagnetic velocity.}$$

The non-cummutability of the operators on the left hand sides of equations (8) and (14) has been corrected for, assuming that

$$\frac{k_y g}{\omega\Omega_i}\;,\quad \frac{\nu_{ie}a_i^2\nabla_\perp^2\varphi}{\omega\varphi}\;,\quad \frac{\nu_{ii}a_i^4(\nabla_\perp^2)^2\varphi}{\omega\,\varphi} \;\ll\; 1 \;.$$

Terms arising from the ion/electron collisional term on the left sides of equations (8) and (14) cancel, because $\nu_{ei}a_e^2 = \nu_{ie}a_i^2$. This is consistent with the quasi-static result that ion-electron collisions produce equal diffusion rates for each species, and hence do not affect the space charge. The mode amplitude is assumed to vary more rapidly with x than the unperturbed density, which allows the $\partial\varphi/\partial x$ term in equation (8) to be neglected.

To reduce equation (15) to an ordinary differential equation, a trial form must be assigned to either the x or z variation of φ. The differential equation can then be solved for the mode variation in the other direction. The trial form is valid if this complete solution can satisfy all boundary conditions. Two types of mode have previously been considered, whose equations relative to the fixed Cartesian co-ordinates (X,Y,Z) can be written

$$\varphi_1 = \varphi_1\,(X)\;e^{i[k_y Y + k_z Z - \omega t]} \;, \qquad \ldots (16)$$

$$\varphi_2 = \varphi_2\,(Z)\;e^{i[k_y\,(Y - sX\,Z) - \omega t]} \qquad \ldots (17)$$

φ_1 is the normal mode form. The wave number components k_y and k_z are constant and consequently the component along the magnetic field $[\underline{k} \cdot \underline{B}]/B \approx s\,(X - X_0)\,k_y$ varies with X due to the shear. φ_2 is the quasi-mode form in which the effective wave vector rotates with X so as to remain everywhere perpendicular to $B(X)$. The distinction between the mode types is discussed in more detail in references 6 and 7.

When the quasi-mode form is substituted, equation (15) becomes a second order differential equation for $\varphi_2(z)$. When instead the normal mode form is used, a fourth order differential equation for $\varphi_1(X)$ is obtained. Fourier transformation[8] reduces this to a second order differential equation for $\varphi(k_x)$, which has the same form as the equation for $\varphi_2(z)$. The boundary condition that $\varphi(z) \to 0$ as $z \to \pm\infty$ applied to the quasimode gives an eigenvalue equation for ω which is identical with that obtained for the normal mode under the condition $\varphi(X) \to 0$ as $X \to \pm\infty$. This analytic relationship is not surprising since, as was pointed out by Roberts and Taylor[6], φ_2 may be regarded as the superposition of a large number of normal modes, each highly localised about successive values of X, and having coherent phases. The quasi-mode form will be used in the following analysis, since this avoids the necessity to Fourier transform.

Substituting the quasi-mode form in equation (15) gives the following equation for $\varphi(z)$ when $s\,x \ll 1$ is assumed over the region of interest

$$\left[\omega_e + \frac{k_y g}{\Omega_i} + \frac{i}{8}(1+\tau)\,\nu_{ii}(k_y a_i)^4\,(1+s^2z^2)^2\right]\frac{\partial^2\varphi}{\partial z^2} + \frac{\tau}{2}\,\omega_i(k_y a_i)^2\,\frac{\partial^2}{\partial z^2}\left[(1+s^2z^2)\varphi\right]$$

$$+\frac{i\nu_e k_y^2}{\Omega_e\Omega_i}\left[\alpha\,g + \omega_i\left\{\omega + \frac{i}{4}\,\nu_{ii}k_y^2a_i^2\,(1+s^2z^2)\right\}(1+s^2z^2)\right]\varphi = 0 \quad , \qquad \dots (18)$$

where

$$\omega_e = \omega - k_y U_e \quad , \qquad \omega_i = \omega - k_y U_i \quad ,$$

$$\tau = \frac{T_e}{T_i} \quad , \qquad \alpha = -\frac{1}{n}\frac{dn}{dx} \quad .$$

When $k_\perp^2 a_i^2 < 1$ is assumed, equation (34) of reference (4) can be compared with the above equation. In Coppi and Rosenbluth's form, the terms containing ν_{ii} are reduced by a factor of 0.5. Apart from this, the equations differ only in a few unimportant details.

4. THE PHASE INTEGRAL CONDITION FOR LOCALISED SOLUTIONS

The problem has been reduced to the solution of a second order differential equation of the form:

$$\frac{d^2\varphi}{dz^2} - V(z)\,\varphi = 0 \quad , \qquad \dots (19)$$

subject to the boundary condition $\varphi(z) \to 0$ as $z \to \pm\infty$ along the real z-axis. This equation is reminiscent of the Schroedinger equation for a particle in a potential well. When $V(z)$ is a real function, which is negative for $z_1 < z < z_2$ and positive outside this range, then a solution exists which is oscillatory between z_1 and z_2 and falls off exponentially beyond these points, provided that V satisfies the phase integral condition

$$\int_{z_1}^{z_2} \sqrt{-V(z)}\, dz = (N + \tfrac{1}{2})\pi \quad , \qquad \ldots (20)$$

and that $V(z)$ does not vary too rapidly with z. When $V(z)$ is a complex function, the zeros of $V(z)$ will generally occur at complex z. If equation (20) is satisfied, solutions exist which are localised over some part of the complex plane, but this need not necessarily include the real z axis. The regions over which such solutions are bounded are defined by the anti-Stokes lines radiating from each turning point, z_1 and z_2.

The anti-Stokes lines from z_1 are the locii of z such that

$$\int_{z_1}^{z} \sqrt{-V(z')}\, dz' = \text{real quantity} \quad . \qquad \ldots (21)$$

The anti-Stokes lines for the case where $V(z)$ has two complex turning points, at $\pm z_0$, are illustrated by solid lines in Fig.2. From the theory of asymptotic solutions[9] we know that, when equation (20) is satisfied, $\varphi(z)$ is oscillatory between AB and falls off exponentially along AC and BD. Along any other line through the origin, $\varphi(z) \to 0$ as $|z| \to \infty$ provided the line goes to infinity within the sectors QOP and SOR. If the line lies outside these sectors, then $\varphi(z) \to \infty$ as $|z| \to \infty$ in one direction along the line. As the asymptotes SOP and QOR are approached, the decay at large z becomes progressively weaker, tending to a pure oscillatory solution on the anti-Stokes lines. The equation illustrated in Fig.2 has a solution localised along the real z axis, where amplitude increases slowly between O and E or F, and then decays as $z \to \pm\infty$.

5. ZONES OF INFLUENCE FOR EACH EFFECT

Even using phase integral methods, the solution of equation (18) is difficult because of the number of terms. Although each of these terms can be important over some part of the parameter range, they will not in general be simultaneously important. In this section the range of parameters over which each of the new terms becomes significant will be estimated by evaluating its effect on the standard resistive-g theory. This will provide the basis for the analytic investigations of these effects in the next section.

THE STANDARD RESISTIVE-G THEORY

This theory neglects ion-ion collisions and the electron acoustic term due to electron motion along the field line [the second bracket in equation (18)]. Equation (18) then reduces to

$$\frac{\partial^2\varphi}{\partial z^2} + \frac{i\nu_e k_y^2}{\Omega_e\Omega_i\omega_e}\left[\alpha g + \omega\omega_i(1 + s^2z^2)\right]\varphi = 0 \quad . \qquad \ldots (22)$$

The phase integral condition, equation (20), applied to this equation gives the eigenvalue equation

$$\omega(\omega - k_yU_i)(\omega - k_yU_e) = -\frac{i\nu_e k_y^2}{\Omega_e\Omega_i s^2}\frac{[\alpha g + \omega(\omega - k_y U_i)]^2}{(2N + 1)^2} \qquad \ldots (23)$$

The mode number N will be put zero, since the lowest order mode is the most unstable. The equation for the corresponding mode in a cylindrical plasma may be obtained using the equivalences set out in equation (1). This equation can be expressed in terms of the non-dimensional parameters used in reference (4).

$$\left(\frac{\omega}{k_yU}\right)\left(\frac{\omega}{k_yU} - 1\right)\left(\frac{\omega}{k_yU} + 1\right) = -\frac{16iC}{S}\left[\psi + \frac{b_o}{4}\frac{\omega}{k_yU}\left(\frac{\omega}{k_yU} - 1\right)\right]^2 , \qquad \ldots (24)$$

where

$$\psi = \frac{r}{R_c} \;, \quad S = \frac{M}{m}\frac{r^2}{L_s^2} \;, \quad C = \frac{\nu_e}{k_yU} \;, \quad b_o = \frac{1}{2}k_y^2a_i^2 \;.$$

$T_i = T_e$ has been assumed, and so $U_i = -U_e = U$. $L_s = 1/s$ is the shear length, and $r \equiv n/(dn/dx)$ is a characteristic radial dimension.

The second term in the bracket on the right of equation (24) is usually neglected, although in certain conditions it can influence the instability(4). This term will be referred to as the ion inertia correction term. Equation (24) is then a cubic in ω, and one root is purely imaginary, $\omega = i\Upsilon$, corresponding to a pure growing mode. When $S \ll 16C\psi^2$, then $\Upsilon \gg k_yU$ and is given approximately by

$$\frac{\Upsilon}{k_yU} = \left[\frac{16C\psi^2}{S}\right]^{1/3} \qquad \ldots (25)$$

This condition will be referred to as the magnetohydrodynamic (MHD) condition, since FLR effects are then unimportant. When $S \gg 16C\psi^2$, then $\Upsilon \ll k_yU$ and is given approximately by

$$\frac{\Upsilon}{k_yU} = \frac{16C\psi^2}{S} \quad . \qquad \ldots (26)$$

This condition will be referred to as the FLR condition. In the MHD condition the turning points lie on the real z axis, and so the solution decays rapidly beyond the turning points. In the FLR condition, the turning points are almost mid-way between the real and imaginary axis, as illustrated in Fig.2, and the anti-Stokes lines lie close to the real z axis, indicating that the solution at large z decays relatively slowly. The phase integral approach to the solution for the standard resistive modes is discussed in more detail in reference 10.

ESTIMATED MAGNITUDE OF THE NEW TERMS

We now return to the full differential equation and estimate the magnitude of each term neglected in the above treatment. The zones of parameter space within which the various effects become important are illustrated in Fig.3. The eigenvalue depends on four parameters, ψ, S, C and b_0 which are measures of the destabilising curvature, shear, collision frequency, and mode number respectively. To reduce the parameter space to two dimensions some special relation between these parameters must be assumed. The curves in Fig.3 have been drawn assuming ψ = constant. With a different assumption these curves may differ in shape, but their relative positions should be much the same. The relative magnitudes of most of the new terms do not depend explicitly on b_0. O B D is the line $S = 16C\psi^2$, below which is the MHD regime and above which is the FLR regime.

(a) Electron Acoustic Term

The mathematical and physical origin of this effect, which gives rise to the second bracket in equation (18), is discussed fully in reference (10). It can be neglected if

$$\left(\frac{\omega - kU}{\omega + kU}\right) b_0\, s^2 z^2 < 1 \quad . \qquad \ldots (27)$$

This inequality must be satisfied at least out to the turning points

$$s^2 z^2 = -\frac{\alpha g}{\omega\omega_i} \equiv \frac{4\psi}{b_0}\frac{(kU)^2}{\omega\omega_i} \quad , \qquad \ldots (28)$$

at which the coefficient of φ in equation (22) changes sign. Substituting the FLR expression for ω , condition (27) becomes $(S/4C\psi) < 1$ which is satisfied below line OEFG in Fig.3. Since the field lines are assumed to be only weakly curved ($\psi \ll 1$) the above condition is automatically satisfied over the MHD region where $(S/4C\psi) < \psi$.

(b) Breakdown in the FLR Approximation

The FLR approximation is valid only if $\frac{1}{2} a_i^2 \nabla_\perp^2 \varphi < \varphi$ i.e. $b_0 s^2 z^2 < 1$. In the FLR regime this condition is practically the same as inequality (27) above. Hence the condition for validity of the FLR approximation is again $(S/4C\psi) < 1$.

(c) Ion Collisions

The ion collision term in equation (18) can be neglected if

$$\frac{\nu_{ii}\ b_o s^2 z^2}{4\omega} < 1 \quad . \qquad \ldots (29)$$

Substituting equation (28) for z and equation (26) for ω gives the condition for neglect of ion collisions within the FLR regime

$$\frac{S}{4C\psi} < 4\left(\frac{\psi}{C}\right)^{\frac{1}{2}}\left(\frac{\nu_e}{\nu_{ii}}\right)^{\frac{1}{2}} = 4\left(\frac{\psi}{C}\right)^{\frac{1}{2}}\left(\frac{M}{m}\right)^{\frac{1}{4}} \quad . \qquad \ldots (30)$$

Within the MHD regime, equation (25) must be used for ω giving $(S/\psi) < 16(M/m)^{\frac{1}{2}}$. Thus ion collisions should not significantly affect the mode if the representative point lies below curve OHFBJ.

(d) Ion Inertia Correction

The condition that the second term in brackets on the right of equation (24) shall not significantly affect the unstable root when the parameters lie within the FLR regime is $S/4C\psi > b_o$. If $b_o < 4\psi$ this condition is satisfied over the entire FLR region. Within the MHD region, equation (25) must be used and the condition for neglect of the ion inertia correction term is $(S^2/4C^2\psi) > b_o^3$. In the following sections, $b_o < 4\psi$ is assumed, so the ion inertia correction is unimportant over most of the parameter range. The case of shorter wavelengths, $b_o > 4\psi$, where this term may become dominant, is discussed in the Appendix.

Within the hatched regions of Fig.3 the standard resistive-g solution is inapplicable because one or other of the neglected terms becomes important. It does not necessarily follow that the mode is then completely stabilised, although the following physical arguments suggest that effects (b)-(d) should have a stabilising effect. The driving mechanism for the mode is the same as for the MHD interchange mode. If the plasma is given an initial neutral density perturbation, the gravitational drift (or the centrifugal drift in the cylindrical plasma) pulls apart the ion and electron density perturbations. This gives rise to a space charge field, and the resulting $\underline{E} \times \underline{B}$ motion of both ions and electrons produces an additional neutral density modulation, and so on. In a sheared magnetic field the phase or amplitude of the space charge field will vary along each field line. The instability owes its existence to resistivity, which limits the ability of electrons to neutralise any space charge by flowing along field lines. The electron pressure gradient also affects the neutralising flow, giving rise to the acoustic term on the left hand side of equation (13). It is not obvious how this will affect the growth. Ion-ion collisions, which allow the ion space charge to leak away across the field lines, will obviously reduce the growth rate. If the wavelength becomes less than an ion Larmor radius, the magnetic field cannot prevent the ions from neutralising the electron density perturbation. Ion inertia is likely to slow down the growth rate.

6. SOLUTION OF THE DIFFERENTIAL EQUATION

Equation (18) will now be investigated over the range of parameters where ion collisions first limit the validity of the standard resistive-g theory, i.e. above the point F in Fig.3. Dropping the electron acoustic term, equation (18) can be written in terms of the non-dimensional parameters defined in equation (24)

$$\frac{\partial^2\varphi}{\partial\zeta^2} + i\,\frac{C}{S}\left(\frac{k_yU}{\omega_e}\right)\left[4\psi + b_o\left(\frac{\omega_i}{k_yU}\right)\left(\frac{\omega}{k_yU} + \frac{i}{2}\sqrt{\frac{m}{M}}\,Cb_o\right) + \frac{\omega_i}{k_yU}\left\{\left(\frac{\omega}{k_yU} + i\sqrt{\frac{m}{M}}\,Cb_o\right) + \frac{i}{2}\sqrt{\frac{m}{M}}\,C\zeta^2\right\}\zeta^2\right]\varphi = 0 \;, \quad \ldots (31)$$

where $\sqrt{2}\,\zeta = k_ya_i\,sz$, and $T_i = T_e$ is again assumed. The eigenvalues for ω which give solutions localised in z will now be found using the phase-integral method.

EFFECT OF ION COLLISIONS WHEN $(S/4\psi) > 2(M/m)^{\frac{1}{2}}$

Let us first assume that over most of the localisation region viscosity predominates over inertia i.e. the term in ζ^2 is negligible in equation (31). The approximate phase-integral condition is then

$$\left[i4\psi\,\frac{C}{S}\,\frac{k_yU}{\omega_e}\right]^{\frac{1}{2}}\int_0^{\Delta}\left[1 - \zeta^4/\Delta^4\right]^{\frac{1}{2}} d\zeta = (N + \tfrac{1}{2})^{\pi}/2 \;, \quad \ldots (32)$$

where $\Delta^4 = (8i\psi/C)\sqrt{M/m}\,(k_yU/\omega_i)$. Substituting $t = (\zeta/\Delta)^4$ transforms the integral into a Beta function

$$\int_0^1 (1 - z^4)^{1/2}\,dz = \frac{\Gamma(5/4)\,\Gamma(3/2)}{\Gamma(7/4)} = .875 \;.$$

Hence the eigenvalue equation for bounded solutions is

$$\frac{\omega_e^2\omega_i}{(k_yU)^3} = -\,\frac{16iC\psi^2}{S}\left(8\,\frac{\psi}{S}\sqrt{\frac{M}{m}}\right)\left[\frac{3.5}{(2N+1)\,\pi}\right]^4 \;. \quad \ldots (33)$$

This result is consistent with the neglect of the inertia terms in equation (31) if $(S/4\psi) > 2(m/M)^{1/2}$ i.e. above BJ in Fig.3.

The parameter range over which FLR effects are negligible should now be revised using this revised eigenvalue equation. Equation (33) predicts that $|\omega| > k_y U$ so long as $S^2 < 2C(4\psi)^3 (M/m)^{1/2}$. Thus within the region JBQ, equation (33) predicts a pure growing mode, $\omega = i\Upsilon$, where

$$\left(\frac{\Upsilon}{k_y U}\right)^3 = \frac{16C\psi^2}{S}\left(8\frac{\psi}{S}\sqrt{\frac{M}{m}}\right)\left[\frac{3.5}{(2N+1)\pi}\right]^4 . \qquad \dots (34)$$

Comparing equations (25) and (34) it may be seen that within region JBQ the growth rate is reduced compared to the standard resistive-g expression. Equation (34) agrees with the expression deduced in reference (1) for the non-electrostatic normal mode in the MHD limit when ion viscosity is dominant. This confirms the supposition(11) that the eigenvalues for the electrostatic and non-electrostatic modes will still be the same, even when ion collisions are included.

When the representative point in parameter space lies above BQ, the unstable solution of equation (33) is given approximately by

$$\frac{\omega}{k_y U} = -1 + \left[i\,\frac{(4\psi)^3 C}{S^2}\sqrt{\frac{M}{m}}\right]^{1/2}\left(\frac{3.5}{\pi}\right)^2 \qquad \dots (35)$$

The mode now has a phase velocity equal to the electron diamagnetic drift velocity. It still grows with time. If the collision frequency falls below the value where $4C\psi = (M/m)^{1/2}$ (i.e. to the left of point B), the ζ^2 term becomes important and equation (33) is no longer valid. The correct eigenvalue equation for this range, deduced later in this section, shows the mode to be stable.

In pinch experiments, such as Zeta, $\psi \sim 1$ and $S \sim M/m$, i.e. the representative point lies above BJ. Thus the resistive interchange mode will be slowed down by ion collisions if $4C\psi > (M/m)^{1/2}$, and completely stabilised at lower collision frequencies.

EFFECT OF ION COLLISIONS WHEN $4\psi\,(M/m)^{1/2} < (S/4\psi) < 2(M/m)^{1/2}$

The neglect of the ζ^2 term, which leads to the phase integral condition of equation (33), is no longer valid below the line BJ. The turning points of the equation (the zeros of the coefficient of φ in equation (31)) now occur at $\pm\zeta_1$, $\pm\zeta_2$, where ζ_1 and ζ_2 are generally of widely different magnitude. The phase integral condition between either pair of turning points can be evaluated using the following result*

$$\int_0^1 (1-y^2)^{1/2}(\kappa'^2+\kappa^2 y^2)^{1/2}\,dy = \frac{1}{3\kappa^2}\Big[(1-2\kappa'^2)\,E(\kappa) + \kappa'^2 K(\kappa)\Big] ,$$

---

* I am indebted to R.J.P. WHIPPLE for this evaluation.

where $K(\kappa)$ and $E(\kappa)$ are the complete elliptic integrals of the first and second kind, and $\kappa'^2 = 1-\kappa^2$. The phase integral condition, equation (20), between turning points $\pm \zeta_1$ is

$$\frac{A\zeta_1}{3\kappa'\kappa^2}\left[(1-2\kappa'^2)\,E(\kappa) + \kappa'^2\,K(\kappa)\right] = (N+\tfrac{1}{2})\,\frac{\pi}{2}\ , \qquad \ldots (36)$$

where

$$\frac{\kappa^2}{\kappa'^2} = -\frac{\zeta_1^2}{\zeta_2^2} = -\frac{iC}{8\psi}\left(\frac{\omega_i}{kU}\right)\left(\frac{m}{M}\right)^{\frac{1}{2}}\zeta_1^4\ , \quad A^2 = \frac{4iC\psi}{S}\left(\frac{k_yU}{\omega_e}\right)\ .$$

When $\zeta_1^2 \ll \zeta_2^2$, and hence $\kappa^2 \ll 1$, $E(\kappa)$ and $K(\kappa)$ can be expanded in powers of κ^2 and this equation takes the simple form

$$A^2\zeta_1^2 = (1 - \tfrac{1}{4}\kappa^2)\,(2N+1)^2\ . \qquad \ldots (37)$$

This expression can then be substituted into the quadratic for ζ^2, to give the following eigenvalues equation for ω for the $N = 0$ mode

$$\frac{\omega_e\omega_i}{(k_yU)^2}\left[\frac{\omega}{k_yU} + \frac{1}{2}\sqrt{\frac{m}{M}}\,\frac{S}{4\psi}\,\frac{\omega_e}{k_yU} + i\sqrt{\frac{m}{M}}\,Cb_o\right](1 - \frac{1}{4}\kappa^2)$$

$$= -\frac{iC}{S}\left[4\psi + b_o\left(\frac{\omega}{k_yU}\right)\left(\frac{\omega}{k_yU} + \frac{i}{2}\sqrt{\frac{m}{M}}\,Cb_o\right)\right]^2\ . \qquad \ldots (38)$$

When the ion inertia correction term on the right hand side of this equation is neglected, the unstable root is given by

$$\frac{\omega}{k_yU} = i\left\{\frac{16C\psi^2}{S} - \sqrt{\frac{m}{M}}\ Cb_o\right\} + \frac{3}{8}\sqrt{\frac{m}{M}}\,\frac{S}{4\psi} \qquad \ldots (39)$$

This is a valid solution of equation (36) (i.e. consistent with the assumption $\kappa^2 \ll 1$) only if the parameters lie outside the ion collision dominated region (to the right of curve FB on Fig.3). From condition (d) of Section 5 ($S/4C\psi > b_o$) it follows that the magnitude of the second term is less than the third, which in turn is smaller than the first. The influence of viscosity can be seen by comparing equations (38) and (39) with equations (24) and (26). The main effect on the eigenvalue is the introduction of a real part, corresponding to a phase velocity in the direction of the electron diamagnetic drift. The reduction in the growth rate due to viscous damping is numerically less. When this eigenvalue is substituted into the expressions for the turning points and anti-Stokes lines, the diagram sketched in Fig.4 is obtained. The positions of the nearer turning points is not substantially altered by ion collisions, and the anti-Stokes lines emanating from these points still intersect the real axis. The solution will thus decay slowly along the real axis, outside the range EF.

As the representative point in parameter space approaches curve FG of Fig.3, the turning points A and B in Fig.4 move outwards, and $|\zeta_1^2/\zeta_2^2| \to 1$. The modulus of the elliptic integrals in equation (36) are now complex functions, of order unity, and so the integrals cannot be approximated analytically. As the representative point moves beyond FG, into the ion collision dominated region, the two roots again separate in magnitude. The phase integral condition between the closer of the two pairs of turning points is again given by equation (38). However, equation (39) is no longer a permissible solution because it leads to $|\kappa^2| = |\zeta_1^2/\zeta_2^2| > 1$. The other two roots of equation (38), $\omega \approx k_yU$ and $-k_yU$, both have small imaginary parts corresponding to damped modes.

For completeness, the possibility that the phase integral condition may be satisfied between the more distant turning points should also be investigated. The phase integral condition is still given by equation (36), but now $\kappa^2 \approx 1$ and $\kappa'^2 \ll 1$. K and E may now be expanded in powers of κ'^2. Retaining only lowest order terms, equation (36) becomes

$$A^2\zeta_1^2 = \left(\frac{3\pi}{4}\right)^2 (2N+1)^2 \kappa'^2 \approx \frac{i8\psi}{C} \left(\frac{kU}{\omega_i}\right) \frac{1}{\zeta_1^4} \left(\frac{M}{m}\right)^{\frac{1}{2}} \left(\frac{3\pi}{4}\right)^2 (2N+1)^2 \quad , \qquad \ldots (40)$$

giving the eigenvalue equation for ω

$$\left(\frac{\omega}{k_yU}\right)^3 \frac{\omega_i}{\omega_e} = iSC\frac{m}{M}\left(\frac{3\pi}{8}\right)^2 (2N+1)^2 \quad . \qquad \ldots (41)$$

Since the right hand side is less than unity throughout the ion collision dominated region, this equation has an imaginary root $\omega = i\gamma < k_yU$. However, the corresponding turning points lie on the imaginary ζ axis, and hence the solution cannot be bounded along the real ζ axis, i.e. it does not satisfy the condition $\varphi(z) \to 0$ as $z \to \pm\infty$. Equation (41) has a second unstable root, $\omega = k_yU + i\varepsilon$. For this root, however, $\zeta_1^2 \approx \zeta_2^2$ and so it lies outside the range of validity of this equation.

This completes the investigation of all possible solutions of the differential equation (31), satisfying the condition $\varphi(\zeta) \to 0$ as $\zeta \to \pm\infty$ along the real axis. It has demonstrated that to the right of curve FB in Fig.3, the growth rate of the resistive-g mode is not significantly affected by ion collisions, whereas to the left of FB there are no solutions which grow in time. Close to the line FB the phase integral condition of equation (36) cannot be solved analytically. However, the marginal stability curve, beyond which the resistive-g mode is damped by collisions, must lie close to the line FB.

7. COMPARISON WITH NUMERICAL SOLUTION

Coppi and Rosenbluth solved numerically their general differential equation over a range of parameters. The values of the parameters at which they found marginal stability are compared in Fig.5 with the criteria deduced in Sections 5 and 6. The solid line is the computed curve, taken from Fig.8 of reference 4 which plots S vs C for constant ψ. Above the dashed line OFG, $S > 4C\psi$ and the electron acoustic term in equation (18) becomes dominant. This region has not been investigated here, because the FLR approximation is no longer valid and more general equations, such as those derived by Coppi and Rosenbluth[4], should be used. However, from physical arguments the mode would be expected to become stable above this line. OHFB is the curve $S^2 = 4(4\psi)^3\, C(M/m)^{\frac{1}{2}}$ above which ion collisions become dominant. It has been shown in Section 6 that the marginal stability curve must lie close to FB. The computed curve is higher by a factor of 2, which is quite a reasonable agreement.

When the empirical best fit to the computed parameters for marginal stability, equation (36) of reference 4, is plotted in Fig.5 it coincides with the line OEFG. As may be seen from Fig.2 and 5, this best fit equation would be highly misleading if extrapolated into the region $(S/4\psi) \gg 4\psi\ (M/m)^{\frac{1}{2}}$ where ion collisions are the dominant stabilising mechanism.

8. CONCLUSIONS

The results can be summarised in terms of the non-dimensional parameters defined in equation (24).

(a) If $S/(4\psi)^2 < (M/m)^{\frac{1}{2}}$, the standard theory for the resistive-g mode ceases to be valid when the normalised collision frequency C falls below the value $S/4\psi$. On the basis of an earlier numerical solution[4] and physical arguments, it appears that the mode is no longer unstable when C is below this value.

(b) If $4\psi\ (M/m)^{\frac{1}{2}} < (S/4\psi) < 2(M/m)^{\frac{1}{2}}$, the resistive-g mode is stabilised by ion collisions when $2C < D(m/M)^{\frac{1}{2}}\, S^2/(4\psi)^3$, where D is a numerical factor of order unity (computation suggests $D = 1/4$).

(c) If $(S/4\psi) > (M/m)^{\frac{1}{2}}$, ion collisional effects are important at all values of the collision frequency, and reduce this growth rate to that given by equations (34) and (35). However, the mode is completely stabilised only if $4C\psi < (M/m)^{\frac{1}{2}}$.

9. ACKNOWLEDGEMENTS

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10. REFERENCES

1. FURTH, H., KILLEEN, J. and ROSENBLUTH, M.N. Phys. Fluids 6, 459 (1963).

2. JOHNSON, J.L., GREENE, J.M. and COPPI, B. Phys. Fluids 6, 1169 (1963).

3. COPPI, B. Phys. Fluids 7, 1501 (1964).

4. COPPI, B. and ROSENBLUTH, M.N. I.A.E.A. Conference Paper CN-21/105 Culham (1965).

5. SCHMIDT, G. Physics of High Temperature Plasmas, Academic Press (1966).

6. ROBERTS, K.V. and TAYLOR, J.B. Phys. Fluids 8, 315 (1965).

7. STRINGER, T.E. I.A.E.A. Conference Paper CN-21/40, Culham (1965).

8. FRIEMAN, E., WEIMER, K. and RUTHERFORD, P. I.A.E.A. Conference Paper CN-21/118.

9. HEADING, J. Introduction to Phase-Integral Methods. Methuen Monograph (1962).

10. STRINGER, T.E. Princeton University Report MATT-320 (1965).

11. COPPI. B. Private Communication.

APPENDIX

THE GRAVITATIONAL WAVE

At shorter wavelengths, when $b_o > 4\psi$, the ion inertia correction term on the right hand sides of equations (24) or (38) can no longer be neglected over the whole FLR parameter range (see condition (d) of Section 5). As was pointed out in reference 4, the eigenvalue equation must then be solved as a quartic in ω. When $(S/4\psi) < b_o$ the left hand side of equation (38) may be treated as small. The lowest order solution is obtained by equating the right hand side to zero, and the effect of the left hand side obtained by iteration. One root has a positive imaginary part, given approximately by

$$\frac{\omega}{k_y U} = \frac{4\psi}{b_o} + \frac{i}{b_o}\left[i\,\frac{4\psi S}{C b_o} \right]^{\frac{1}{2}} - \frac{i}{2}\sqrt{\frac{m}{M}}\, C b_o \qquad \text{... (A1)}$$

This mode is the FLR form of the fast interchange instability of reference 2. The last term in equation (A1), which comes from ion-ion collisions, damps the mode when

$$S < \frac{m}{M}\,\frac{C^3 b_o^5}{16\psi} \qquad \text{... (A2)}$$

The stability zones for the short wavelength case $(b_o > 4\psi)$ are illustrated in Fig.6. This figure is drawn assuming b_o and ψ are constant, and S and ψ varying, and is on a different scale from Fig.3. The range of unstable parameters is now bounded both below and above, and shrinks rapidly as $4\psi/b_o$ gets smaller.

At longer wavelengths, where $b_o < 4\psi$, the ion inertia correction term becomes important over that part of the MHD region where $S^2 < 4\psi\, C^2 b_o^3$. The unstable solution of the eigenvalue equation is significantly affected by ion vicosity only when the dimensionless collision frequency exceeds

$$C\psi = \frac{1}{2}\left(\frac{M}{m}\right)^{\frac{1}{2}}\left(\frac{4\psi}{b_o}\right)^{3/2} \qquad \text{... (A3)}$$

The gravitational wave is then slowed down, but not completely stabilised, by ion viscosity, its growth being given by

$$\frac{\omega}{k_y U} = 2i\left(\frac{M}{m}\right)^{\frac{1}{2}} \frac{4\psi}{C b_o^2} \qquad \text{... (A4)}$$

Thus to obtain the full stability diagram for the case $b_o < 4\psi$ a vertical line, defined by equation (A3), must be inserted in Fig.3. This line lies to the right of point B. To the right of this line the instability, now in the form of a gravitational wave, is slowed down by ion viscosity.

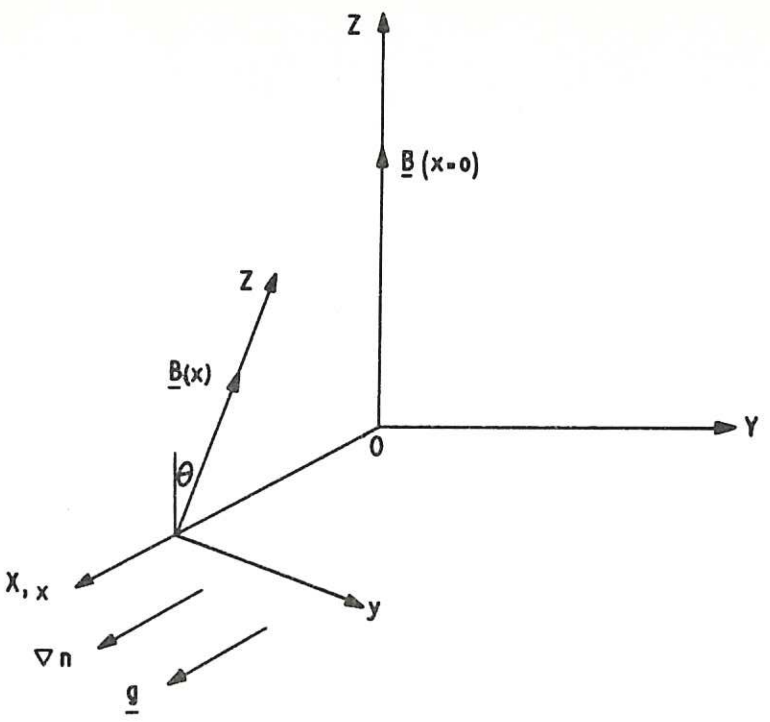

Fig. 1 The slab model (CLM-P 110)

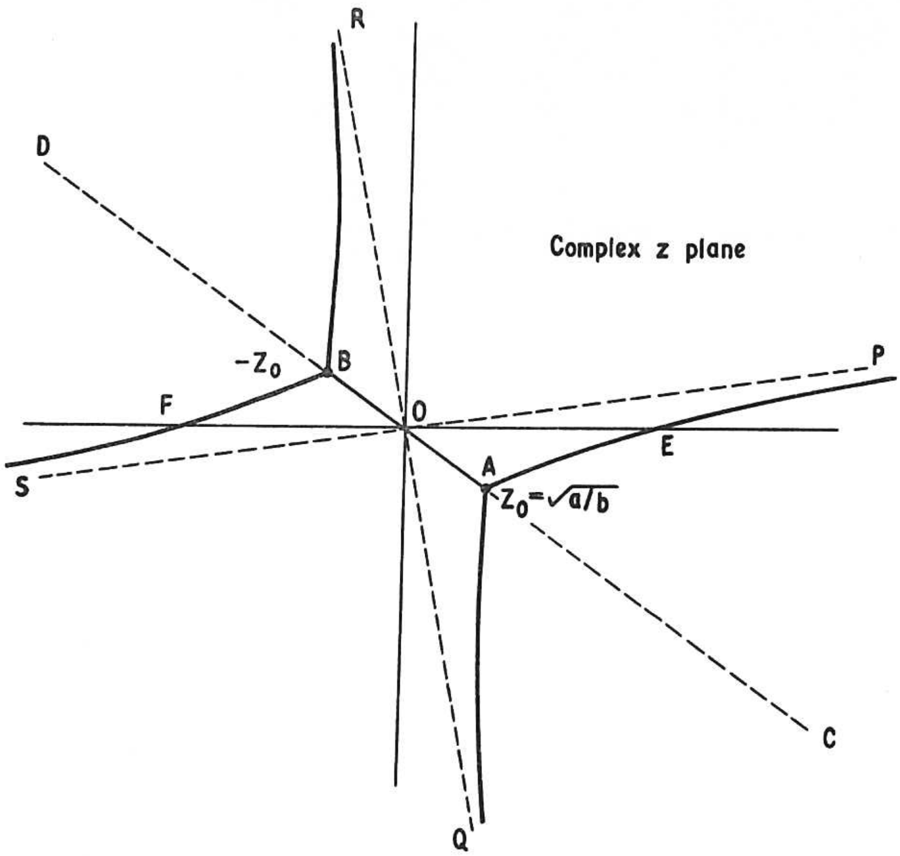

Fig. 2 Anti-Stokes lines when $V(z) = a - bz^2$ (CLM-P 110)

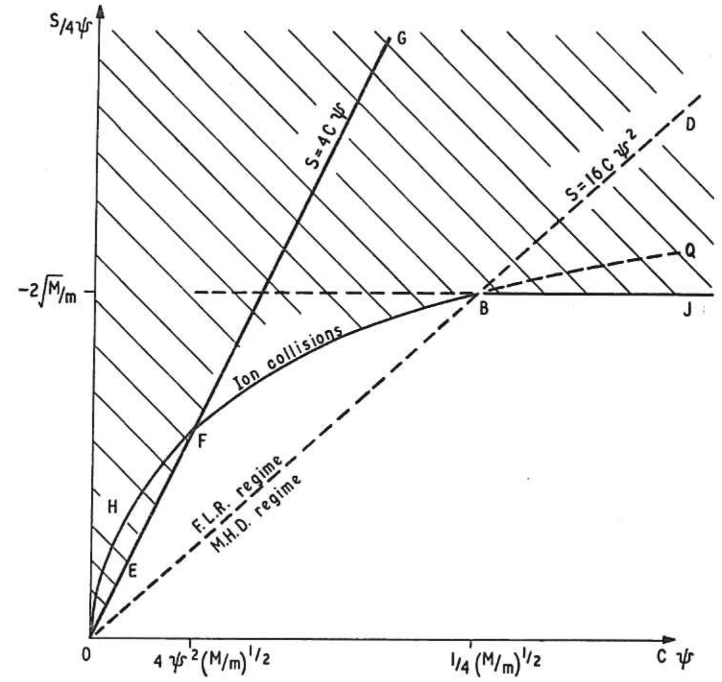

Fig. 3 (CLM-P 110)
Zones in which different effects become important (ψ = constant)

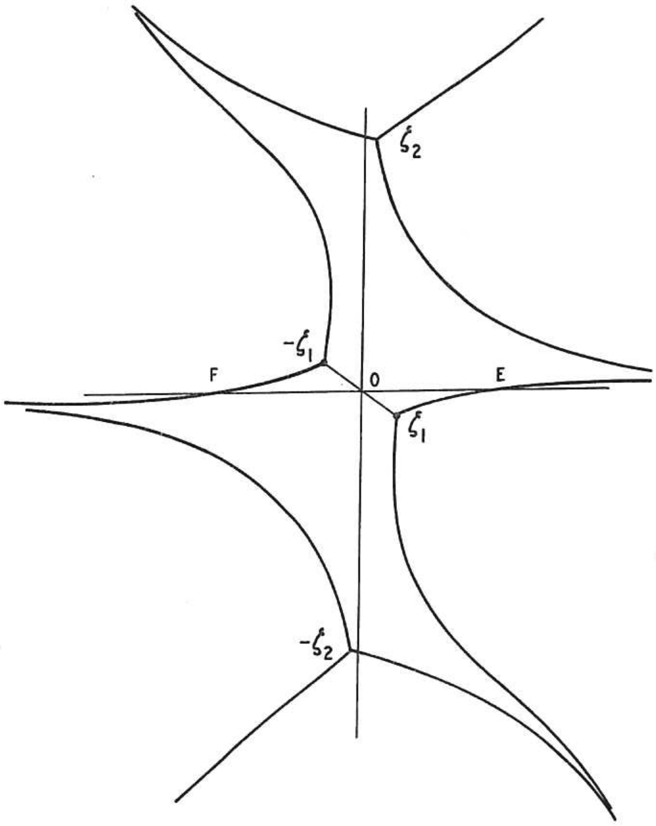

Fig. 4 (CLM-P 110)
Anti-Stokes lines for eq. (31) when ω satisfies eq. (39)

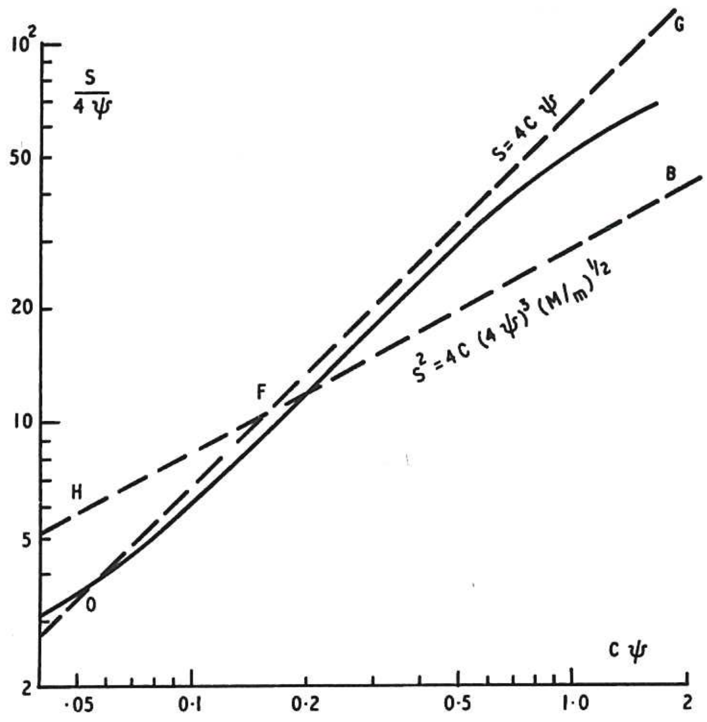

Fig. 5 (CLM-P 110)
Comparison between analytical and computed stability curves $\psi = 1.5 \times 10^{-2}$

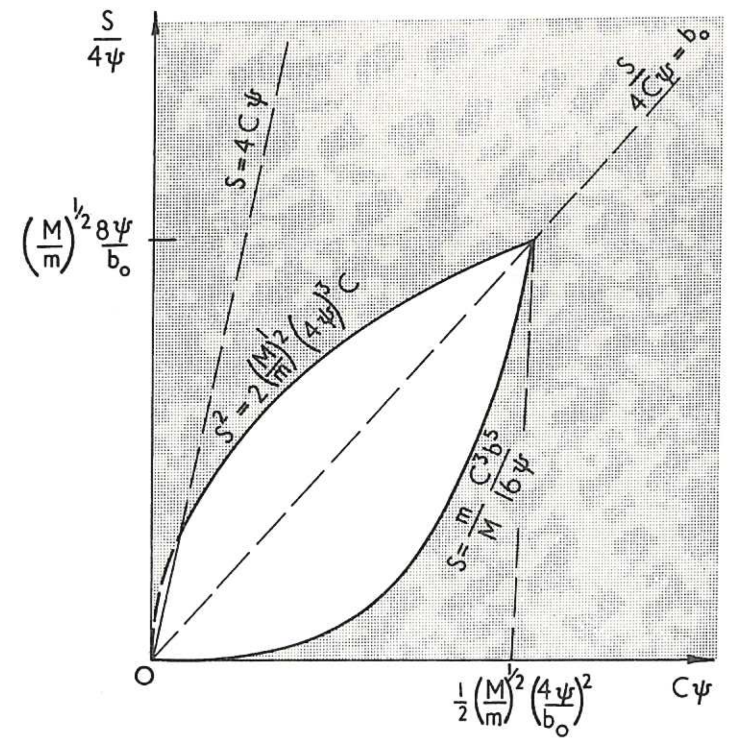

Fig. 6 (CLM-P 110)
Stability zones at short wavelengths, $b_o > 4\psi$