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by

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A B S T R A C T

A new and simple method of finding an invariant, J, of a nearly periodic dynamical system is presented. The Hamiltonian is written as $H = p_1 + \varepsilon \Omega(q_i p_i)$, where Ω is periodic in q_1 and $\varepsilon \ll 1$. The first four terms of the invariant series are found explicitly in terms of Ω using Poisson Bracket and averaging operators. This invariant is related to the adiabatic invariant and to various constants of motion discussed in celestial mechanics, such as Whittaker's adelphic integral. J is shown to be an asymptotic constant by using the rigorous methods of Kruskal (1962) to calculate the adiabatic invariant K; it is found that $K/\tau = H - \varepsilon J$, where τ is the period in q_1. The adelphic integral has different functional forms depending on the presence of resonant denominators but is shown to be always a function of H and J. The present method provides a single functional form which is even applicable when Ω is only almost periodic in q_1. It is also much simpler than the methods of adiabatic invariant theory.

Many authors have found good general agreement between invariants and numerical orbit calculations. Some examples are discussed here where several terms of the series are required to achieve agreement near periodic orbits.

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CONTENTS

1. INTRODUCTION

This paper is primarily concerned with dynamical systems that are nearly periodic in the following sense. The Hamiltonian of the system is time independent and also a function of a small parameter ε. When $\varepsilon = 0$ all motions of the system execute closed orbits in phase space but for small values of ε the orbits are no longer exactly closed and slowly drift. One can show (see Section 2) that the Hamiltonian for such systems can be reduced to the form

$$H = p_1 + \varepsilon \Omega (q_i ; p_i) \qquad \ldots (1.1)$$

where $(q_i ; p_i)$ are some set of canonical coordinates and Ω is a periodic function of q_1. There is also a range of problems where Ω is only an almost-periodic function of q_1. Many of the results with Ω periodic can be extended to the almost-periodic case and so this class of motions will also be considered.

Being time independent, the Hamiltonian itself is one constant of the motion but unless it exhibits some other symmetry, like possessing an ignorable coordinate, there are no other obvious constants. However it is possible to find another integral of the motion, expressed as a power series in ε, associated with the fact that the system is nearly periodic. It is with this invariant that the paper is mainly concerned.

Invariants of this type have been discussed previously but in two quite different contexts, celestial mechanics and plasma physics. In celestial mechanics various authors (Birkhoff 1927, Cherry 1924, Contopoulos 1960, 1963b, Contopoulos and Moutsoulas 1965, Whittaker 1937) have studied the motion of a particle near a point of equilibrium and looked for another constant of the motion different from the Hamiltonian. For example, Whittaker (1937) obtains an invariant that he calls an adelphic integral, Φ. He solves Liouville's equation and find Φ as a series of terms of decreasing order of magnitude. In his method several difficulties arise. He encounters the problem of small or vanishing denominators which entails using different techniques in a variety of special cases. As a result the adelphic integral has not one but several different analytical forms. Another drawback is

that these methods do not allow any conclusion to be reached about the nature of the series, whether for example it is convergent or asymptotic. Applications of this invariant have been found in the three-body problem and the problem of the third integral of galactic motion.

In plasma physics the problem of particle containment has led to detailed study of adiabatic invariants. Originally they were regarded as quantities that remained virtually constant as the parameters of the system varied slowly in time. A more general invariant is obtained when the parameters are allowed to vary slowly in both space and time and a detailed discussion of this case has been given by Kruskal (1962). The invariant is found as an action integral and is evaluated as a series of terms which is shown to be an asymptotic series. Examples of the invariant are the three invariants of a charged particle in a magnetic field and the magnetic surfaces generated by nearly periodic magnetic field lines. Other examples arise in the motion of artificial satellites.

It is not immediately apparent and does not seem to have been remarked in the literature that the invariants obtained in these two situations are equivalent. The correspondence is most easily seen by considering the Hamiltonian in each case. The equations that Kruskal deals with are the equations of motion for a Hamiltonian of the form (1.1) but with Ω strictly periodic in q_1 with period τ. One can in general find an invariant as long as τ is a function of p_1 and a slowly varying function of $(q_2, q_3 \ldots q_n; p_2, p_3 \ldots p_n)$ and the time, t. The dynamical systems discussed by Whittaker also have Hamiltonians that can be reduced to the form (1.1). In his case Ω is the sum of several terms periodic in q_1 with periods τ_i $(i = 1,2 \ldots\ldots)$ and where the τ_i are constants, independent of the coordinates and time. In this paper we limit ourselves to a discussion of dynamical systems having this latter form with Ω either

(a) periodic in q_1 with period τ independent of space and time coordinates, or

(b) almost periodic in the sense that it is the sum of several such terms with periods τ_i that are incommensurable.

The methods used below can however be generalised to other cases where the periods are functions of the coordinates.

Once the Hamiltonian has been reduced to the standard form an invariant can be calculated explicitly in terms of Ω and this is done by two methods. In the first of these (Section 3), which is the most straightforward, Liouville's equation is expanded in powers of ε and solved order by order. It differs from Whittaker's method in several important respects and as a result the problem of vanishing denominators no longer arises. One formalism embraces all the cases that previously needed separate treatment and a single expression is obtained for the invariant. The operator techniques used also allow several terms in the series to be evaluated with comparative ease. The second method is to carry out explicitly the procedure described in general terms by Kruskal. This is more tedious to apply but has compensating advantages. It is systematic and so allows one, in principle, to obtain the series to any number of terms. It also gives information about the asymptotic nature of the invariant. Kruskal's method and its application are described in Section 4.

Although these two methods give essentially the same result it is not obvious that they are equivalent to the adelphic integral. The relation between Whittaker's invariant and the forms obtained in this paper are discussed in Section 5. In connection with the problem of the third integral of galactic motion there have been many calculations of particle orbits (Henon and Heiles 1964, Contopoulos and Moutsoulas 1965). In Section 6 the numerical data and the predictions of invariant theory are compared. Agreement is good for sufficiently small values of ε although careful treatment is sometimes necessary in the neighbourhood of periodic solutions of the perturbed motion.

2. REDUCTION OF THE HAMILTONIAN TO STANDARD FORM

The basic dynamical system considered in the following is n-dimensional and has a time independent Hamiltonian of the form

$$H = H_0 (q_i' , p_i') + \varepsilon H_1 (q_i' , p_i') \qquad \ldots (2.1)$$

where (q_i' , p_i') are canonical coordinates. It is assumed that the equations of motion for the zero-order Hamiltonian H_0 can be solved and that all the trajectories are closed curves in phase space, having the same periodic time τ. This being so one can use Hamilton-Jacobi theory (Goldstein 1959, p.280) to transform to new coordinates $(q_i ; p_i)$ such that H_0 is a function of the new momenta, p_i, only. In particular it is possible to choose $(q_i ; p_i)$ so that $H_0 = p_1$. This is done by solving the Hamilton-Jacobi equation

$$H_0 \left(q_i' , \frac{\partial W}{\partial q_i'}\right) = p_1 \qquad \ldots (2.2)$$

for Hamilton's characteristic function $W(q_i' , p_i)$. It is then found that

and

$$\left.\begin{aligned} q_1 &= t + \beta_1 \\ q_i &= \beta_i \ , \ i \neq 1 \end{aligned}\right\} \qquad \ldots (2.3)$$

where the β_i are constants. The new momenta are given by the equations

$$p_i' = \frac{\partial W}{\partial q_i'} \ (q_i' , p_i) \qquad \ldots (2.4)$$

During the motion described by H_0, therefore, q_1 is proportional to the time and $(q_2 \ldots q_n ; p_1 \ldots p_n)$ remain constant. The essential step in the following calculations is to transform to this coordinate system that displays the fundamental angle variable, q_1, of the zero order motion.

Because of the periodicity the point $(q_1 \ldots q_n ; p_1 \ldots p_n)$ is the same as the point $(q_1 + \tau , q_2 \ldots q_n ; p_1 \ldots p_n)$. Since the Hamiltonian is a constant of the motion and a single valued function of position in phase space it follows that

$$H(q_1) = H(q_1 + \tau)$$

The Hamiltonian can therefore be written in the required form

$$H = p_1 + \varepsilon \ \Omega \ (p_i , q_i) \qquad (1.1)$$

with Ω periodic in q_1 period τ.

There are dynamical systems where the lowest order motion is not strictly periodic with closed orbits but conditionally periodic, the orbits describing open

Lissajous figures. For example if the lowest order Hamiltonian is of the form

$$H = \sum_{i=1}^{n} \tfrac{1}{2}(p_i'^2 + \alpha_i^2 q_i'^2) \qquad \ldots (2.5)$$

the system represents n uncoupled simple harmonic oscillators. If all the ratios α_i/α_j are rational numbers the phase space trajectories are closed and the system periodic. If any ratio is an irrational number however then the motion is only conditionally periodic. In this case one can still transform to new co-ordinates such that the Hamiltonian takes the form (1.1) but one now finds that Ω is only almost-periodic in q_1 .

3. THE POISSON BRACKET METHOD

With the Hamiltonian in the reduced form a new constant of the motion J can now be found different from H. Considering firstly the case with Ω periodic in q_1 , an invariant is sought independent of time and periodic in q_1 with period τ . The constant J must satisfy Liouville's equation

$$\frac{dJ}{dt} \equiv \frac{\partial J}{\partial t} - [J,H] = 0 \qquad \ldots (3.1)$$

where $[J,H]$ is the Poisson bracket defined by

$$[J,H] \equiv \frac{\partial J}{\partial p_i}\frac{\partial H}{\partial q_i} - \frac{\partial J}{\partial q_i}\frac{\partial H}{\partial p_i} \quad . \qquad \ldots (3.2)$$

With $\frac{\partial J}{\partial t} = 0$ (3.1) reduces to the equation

$$[J,H] = 0 \quad . \qquad \ldots (3.3)$$

J is expanded as a power series in ε :

$$J = \sum_{n=0}^{\infty} \varepsilon^n J_n \quad . \qquad \ldots (3.4)$$

Substituting (1.1) and (3.4) in (3.3) and equating terms in ε^n one obtains

$$\frac{\partial J_o}{\partial q_1} = 0 \qquad \ldots (3.5)$$

and

$$\frac{\partial J_n}{\partial q_1} = [J_{n-1}, \Omega] \quad . \qquad \ldots (3.6)$$

Integration of (3.6) gives

$$J_n = \int [J_{n-1}, \Omega]\, dq_1 + G_n \qquad \ldots (3.7)$$

where G_n is independent of q_1. Since J_n is required to be periodic in q_1 one obtains the periodicity condition :

$$\int_0^\tau [J_{n-1}, \Omega]\, dq_1 = 0 \qquad \ldots (3.8)$$

To generate the terms in the series for J it is now convenient to introduce some new terminology. For a periodic function f of q_1 period τ, the functions $\bar{f}$ and $\hat{f}$ are defined as follows

$$\bar{f} = \frac{1}{\tau}\int_0^\tau f\, dq_1 \qquad \ldots (3.9)$$

and

$$\hat{f} = \int (f - \bar{f})\, dq_1 \qquad \ldots (3.10)$$

with the constant of integration chosen so that $\bar{\hat{f}} = 0$. Thus $\bar{f}$ is the mean value of f and $\hat{f}$ the indefinite integral of the oscillatory part of f. It follows from the definitions that

$$\bar{\hat{f}} = 0 \ , \qquad \ldots (3.11)$$

$$\frac{\partial}{\partial x}(\bar{f}) = (\overline{\frac{\partial f}{\partial x}}) \ , \qquad \ldots (3.12)$$

and

$$\frac{\partial}{\partial x}(\hat{f}) = (\widehat{\frac{\partial f}{\partial x}}) \qquad \ldots (3.13)$$

where x is any of the variables q_i, p_i. The following results can be proved using integration by parts :

$$\overline{f\hat{g}} + \overline{\hat{f}g} = 0 \qquad \ldots (3.14)$$

and

$$\widehat{fg} + \widehat{\hat{f}g} = \hat{f}\hat{g} - \overline{\hat{f}\hat{g}} + \overline{f}\hat{\hat{g}} + \hat{\hat{f}}\overline{g} \qquad \ldots (3.15)$$

where both f and g are periodic in q_1, period τ. Use will also be made of the Poisson bracket relation

$$[f,g] = -[g,f] \qquad \ldots (3.16)$$

and Jacobi's identity

$$[f, [g,h]] + [h, [f,g]] + [g, [h,f]] = 0 \quad . \qquad \ldots (3.17)$$

Finally for a function K, independent of q_1

$$\overline{[K,f]} = [K,\bar{f}] \qquad \ldots (3.18)$$

and

$$\widehat{[K,f]} = [K,\hat{f}]. \qquad \ldots (3.19)$$

The first few terms of J can now be calculated in terms of the operators $\bar{}$, $\hat{}$ and $[\ ]$. Equations (3.7) and (3.8) become

$$J_n = \widehat{[J_{n-1}, \Omega]} + G_n \qquad \ldots (3.20)$$

and

$$\overline{[J_{n-1}, \Omega]} = 0 \quad . \qquad \ldots (3.21)$$

J_0 is independent of q_1 from (3.5) and the lowest order periodicity condition is obtained from (3.21) :

$$\overline{[J_0, \Omega]} = 0 = [J_0, \bar{\Omega}] \quad . \qquad \ldots (3.22)$$

In generating a constant of the motion, J, one seeks only a particular solution of this equation and an obvious solution is

$$J_0 = J_0 (\bar{\Omega}) \quad .$$

Since any function of an invariant is also an invariant any functional form could be taken for J_0. For simplicity the choice made is that

$$J_0 = \bar{\Omega} \quad . \qquad \ldots (3.23)$$

(3.20) then gives

$$J_1 = [\bar{\Omega}\ \hat{\Omega}] + G_1$$

and the periodicity condition (3.21) gives

$$\overline{[\widehat{[\bar{\Omega},\Omega]}\ \Omega]} + [G_1, \bar{\Omega}] = 0 \quad . \qquad \ldots (3.24)$$

To solve this equation for G_1 the first term is rearranged as follows:

$$\overline{[[\bar{\Omega}, \hat{\Omega}], \Omega]} = - \overline{[[\Omega,\bar{\Omega}]\hat{\Omega}]} - \overline{[[\hat{\Omega},\Omega],\bar{\Omega}]} \qquad \text{using (3.17) and (3.18)}$$

$$= - \overline{[[\bar{\Omega},\hat{\Omega}],\Omega]} - \overline{[[\hat{\Omega},\Omega],\bar{\Omega}]} \qquad \text{using (3.14) and (3.19)} .$$

$$\therefore \quad \overline{[[\bar{\Omega}\ \hat{\Omega}]\Omega]} = - \tfrac{1}{2} \overline{[[\hat{\Omega}\ \Omega]\ \bar{\Omega}]} \quad . \qquad \ldots (3.25)$$

Equation (3.24) becomes

$$[G_1 - \tfrac{1}{2}\,\overline{[\hat{\Omega},\Omega]},\bar{\Omega}] = 0$$

With the equation rearranged in this way a particular integral is easily found:

$$G_1 = \tfrac{1}{2}\,\overline{[\hat{\Omega},\Omega]} + F(\bar{\Omega}) \qquad \ldots (3.26)$$

where F is any function of $\bar{\Omega}$. In generating the invariant series one chooses the particular case $F = 0$. Any other choice merely adds a function of the lower order invariant J_0 to J_1.

This process can be repeated, a new partial differential equation for G_n having to be solved in each order. The relations (3.11) to (3.19) allow these equations to be rearranged as above but the difficulties rapidly increase. By these methods it was possible to find G_2. For the particular case where $\bar{\Omega} = 0$ it also proved possible to find G_3. To sum up therefore

$$J = \sum_{n=0}^{\infty} \varepsilon^n J_n \qquad (3.4)$$

where

$$J_n = \overbrace{[J_{n-1},\Omega]} + G_n \qquad (3.7)$$

and

$$\overline{[G_n,\bar{\Omega}] + [\overbrace{[J_{n-1},\Omega]},\Omega]} = 0 \quad .$$

A solution is

$$J_0 = \bar{\Omega}$$

$$G_1 = \tfrac{1}{2}\overline{[\hat{\Omega},\Omega]} \qquad \ldots (3.27)$$

$$G_2 = \tfrac{1}{3}\overline{[\hat{\Omega},[\hat{\Omega},\Omega]]} + \tfrac{2}{3}\overline{[\hat{\Omega},[\hat{\Omega},\bar{\Omega}]]}$$

and when $\bar{\Omega} = 0$

$$G_3 = \tfrac{1}{8}\overline{[\hat{\Omega},[\hat{\Omega},[\hat{\Omega},\Omega]]]} + \tfrac{1}{8}\overline{[\overbrace{[\Omega,\hat{\Omega}]},[\Omega,\hat{\Omega}]]}$$

$$+ \tfrac{1}{4}\overline{[[[\hat{\Omega},\Omega],\hat{\Omega}],\hat{\Omega}]} \quad .$$

The distinction between Ω being periodic and almost periodic now appears to have been unnecessary. Instead of insisting that J be periodic in q_1, one could have applied the condition that J contain no terms that become indefinitely large as $q_1 \to \infty$. This is assured if

$$\lim_{\tau\to\infty} \frac{1}{\tau}\int_0^{\tau} [J_n, \Omega]dq_1 = 0 \quad . \qquad \ldots (3.28)$$

With a redefinition of $\bar{f}$ to be

$$\bar{f} = \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^{\tau} f\, dq_1 \qquad \ldots (3.29)$$

(which also includes the previous definition (3.9)) it is apparent that the calculation follows exactly as before. The results obtained in (3.27) therefore apply both for Ω periodic and almost periodic in q_1 with the averaging operator defined by (3.29).

It must be stressed that this is merely a formal series solution of (3.3). The derivation allows no statement about the validity of the solution and whether the series is convergent or otherwise. However using Kruskal's method one is able to show that it is in fact an asymptotic series in the limit $\varepsilon \to 0$, at least for the case where Ω is periodic.

4. KRUSKAL'S AVERAGING METHOD

A large part of celestial mechanics is concerned with almost periodic systems and, since one can rarely obtain exact results, a particular aim has been to find approximate descriptions that are valid even for large values of the time. For this latter condition to be realised it is necessary for the solution to be free from secular terms and one of the achievements of the subject is to have produced perturbation techniques that accomplish this. They are all essentially what is now called "the averaging method" and associated with the names of Poincaré (1899), Bogoliubov (1961) and many others.

(a) General Description of the Method

A general formulation of the averaging method has been given by Kruskal (1962) and in several important respects he has extended its usefulness. He deals with a set of equations of the form

$$\underline{x}_t \equiv \frac{d\underline{x}}{dt} = F(\underline{x}, \varepsilon)$$

such that for $\varepsilon = 0$ the point $\underline{x}(t)$ traces out closed curves (loops) as t increases. (For this reason his method as it stands is only valid for the case where the lowest order motion is periodic and not for the conditionally periodic case.) It is then possible to transform to new variables $\underline{y}$ and an angle-like variable ν such that $\underline{y}$ is constant on a loop and ν varies around it. When the motions around the loops all have the same period τ the equations for $\underline{y}$ and ν become

$$\underline{y}_t = \varepsilon \, \underline{g} \, (\underline{y}, \nu) \qquad \ldots (4.1)$$

$$\nu_t = 1 + \varepsilon f(\underline{y}, \nu) \qquad \ldots (4.2)$$

where f and $\underline{g}$ are periodic in ν period τ. When ε is small these equations describe a slowly drifting oscillatory motion and the object of the method is to transform to new variables $\underline{z}$ and φ which separate the drift motion and the oscillation. It is required that $\underline{z}$ be periodic in ν and that φ be an angle-like variable i.e.

$$\underline{z}(\underline{y}, \nu) = \underline{z}(\underline{y}, \nu + \tau) \qquad \ldots (4.3)$$

$$\varphi(\underline{y}, \nu + \tau) = \varphi(\underline{y}, \nu) + \tau \qquad \ldots (4.4)$$

and also that the equations of the drift variables should not contain the angle variable i.e.

$$\underline{z}_t = \varepsilon \, \underline{h} \, (\underline{z}) \qquad \ldots (4.5)$$

$$\varphi_t = 1 + \varepsilon \, \omega \, (\underline{z}) \, . \qquad \ldots (4.6)$$

Kruskal has shown that it is possible to find systematically $\underline{z}$, φ, $\underline{h}$ and ω as power series in ε to any order required. It is also possible (and this is

essential in what follows) to find the inverse transformation as a power series in ε i.e.

$$\underline{y} = \underline{y}\ (\underline{z}, \varphi) = \sum_{n=0}^{\infty} \varepsilon^n\ \underline{y}_n$$

$$\nu = \nu\ (\underline{z}, \varphi) = \sum_{n=0}^{\infty} \varepsilon^n\ \nu_n$$

He is able to show that the solutions of the transformed equations (4.5) and (4.6) provide asymptotic solutions of (4.1) and (4.2) in the following sense.

If one writes the partial sums in forms like

$$\underline{h}^{[n]} = \sum_{\underline{h}=0}^{n} \varepsilon^n\ \underline{h}_n \quad , \qquad \ldots\ (4.7)$$

then $\underline{z}'$ and φ' are defined to be solutions of

$$d\underline{z}'/dt = \varepsilon\ \underline{h}^{[n]}\ (\underline{z}')$$

$$d\varphi'/dt = 1 + \varepsilon\ \omega^{[n]}\ (\underline{z}') \quad .$$

In terms of these one can write

$$\underline{y}' = \underline{y}^{[n]}\ (\underline{z}', \varphi')$$

and

$$\nu' = \nu^{[n]}\ (z', \varphi') \quad .$$

Kruskal proves that if $\underline{y}^*$ and ν^* are solutions of the original equations satisfying the same initial conditions as $\underline{y}'$ and ν' then

$$\underline{y}^* - \underline{y}' = O(\varepsilon^{n+1})$$

and

$$\nu^* - \nu' = O(\varepsilon^{n+1})$$

for t within a range of order $1/\varepsilon$. That is, solutions of (4.5) and (4.6) give asymptotically correct solutions of (4.1) and (4.2) as $\varepsilon \to 0$.

It is apparent that the equations of motion derived from (1.1) are equations of the form (4.1) and (4.2) with

$$(\nu, \underline{\chi}) \sim (\underline{q}, \underline{p})$$

and

$$(f, \underline{g}) \sim (\Omega_{\underline{p}}; -\Omega_{\underline{q}})$$

where

$$\Omega_{\underline{p}} = \frac{\partial \Omega}{\partial \underline{p}} \quad \text{etc.}$$

One could therefore find an asymptotic solution of the equations of motion to any order in ε by solving (4.5) and (4.6) for $\underline{z}$ and φ as functions of t and then substituting the results to obtain $\underline{q}(\underline{z}, \varphi)$ and $\underline{p}(\underline{z}, \varphi)$. However, if one seeks an invariant of the motion rather than the trajectories it is not necessary even to solve the differential equations for $\underline{z}$ and φ .

If C_o is any closed curve in phase space at time $t = 0$ and if at a later time, t, the points that comprised it now lie on a curve C_t , then it is well known that the action integral taken around C_t :-

$$\oint \underline{p} \cdot d\underline{q}$$

is an invariant of the dynamical system. In order to evaluate the integral it is necessary to know the general form of C_t which usually entails solving the equations of motion. However, one can see from (4.5) and (4.6) that the points comprising a φ-loop (a closed curve on which $\underline{z}$ is constant and φ varies) remain a φ-loop as the system develops in time. Choosing C_t to be a φ-loop one can therefore define an invariant K:

$$K(\underline{z}) = \oint \underline{p} \cdot \underline{dq} = \int_o^\tau p_i \frac{\partial q_i}{\partial \varphi} d\varphi \quad . \qquad \ldots (4.8)$$

The time rate of change of K is

$$K_t = \oint (\underline{p}_t \cdot \underline{dq} - \underline{q}_t \cdot d\underline{p}) \ ,$$

the second term having been integrated by parts i.e.

$$K_t = -\oint (H_{\underline{q}} \cdot \underline{dq} + H_{\underline{p}} \cdot d\underline{p})$$

$$= -\oint dH \quad .$$

If $\underline{p}$ and $\underline{q}$ were known exactly as functions of $\underline{z}$ and φ and used to evaluate this integral then K_t would be exactly zero. However $\underline{p}$ and $\underline{q}$ as functions of $\underline{z}$ and φ are only defined as power series in ε. If the invariant is calculated to order n using the partial sums $\underline{p}^{[n]}$ and $\underline{q}^{[n]}$ instead of $\underline{p}$ and $\underline{q}$ then the integral does not vanish exactly. One finds

$$K_t^{[n]} = O(\varepsilon^{n+1}) \quad .$$

The quantity K is therefore only asymptotically invariant.

(b) The Coordinate Transformation

As a preliminary to calculating K the transformations

$$(\underline{y}, \nu) \rightleftarrows (\underline{z}, \varphi)$$

are obtained. From (4.1) to (4.6) one finds

$$\underline{z}_t = \varepsilon \underline{h}(\underline{z}) = \varepsilon \underline{g} \cdot \underline{z}_{\underline{y}} + (1 + \varepsilon f)\underline{z}_\nu \qquad \ldots (4.9)$$

and

$$\varphi_t = 1 + \varepsilon\omega(\underline{z}) = \varepsilon \underline{g} \cdot \varphi_{\underline{y}} + (1 + \varepsilon f)\varphi_\nu \qquad \ldots (4.10)$$

where $\underline{z}_{\underline{y}} = \dfrac{\partial \underline{z}}{\partial \underline{y}}$ is the tensor with ij-th component $\dfrac{\partial z_j}{\partial y_i}$ etc.

On integrating over ν and applying the initial conditions

$$\underline{z} = \underline{y}$$

and

$$\varphi = 0 \qquad \ldots (4.11)$$

when

$$\nu = 0$$

(4.9) and (4.10) give

$$\underline{z} = \underline{y} + \varepsilon \int_0^\nu d\nu \; (\underline{h}(\underline{z}) - \underline{g} \cdot \underline{z}_{\underline{y}} - f\underline{z}_\nu) \qquad \ldots (4.12)$$

and

$$\varphi = \nu + \varepsilon \int_0^\nu d\nu \; (\omega(\underline{z}) - \underline{g} \cdot \varphi_{\underline{y}} - f\varphi_\nu) \quad . \qquad \ldots (4.13)$$

There is some freedom in choosing the initial conditions of the transformation. In the work of Bogoliubov and Mitropolsky (1961) the integration constants are chosen to eliminate the contribution of the lower limits to the integrals in (4.12) and (4.13). This makes the transformation formulae somewhat simpler but

the only convenient choice, if the inverse transformation is to be obtained also, is that given here.

If one now writes

$$\underline{Z} = (\varphi, \underline{z}),\ \underline{H}(\underline{z}) = (\omega, \underline{h}) ,$$
$$\underline{Y} = (\nu, \underline{y}) \quad \text{and} \quad G(\underline{Y}) = (f, \underline{g}) \qquad \ldots (4.14)$$

equations (4.12) and (4.13) become

$$\underline{Z} = \underline{Y} + \varepsilon \int_0^{\nu} d\nu\ (\underline{H}(\underline{z}) - \underline{G} \cdot \underline{Z}_{\underline{Y}}) \quad . \qquad \ldots (4.15)$$

The conditions that $\underline{z}$ be periodic in ν, and that φ be an angle-like variable with period τ can be written

$$\int_0^{\tau} d\nu\ (\underline{H}(\underline{z}) - \underline{G} \cdot \underline{Z}_{\underline{Y}}) = 0$$

or

$$\underline{H}(\underline{y}) = \frac{1}{\tau} \int_0^{\tau} d\nu\ (\underline{G} \cdot \underline{Z}_{\underline{Y}} + \underline{H}(\underline{y}) - \underline{H}(\underline{z}))$$

$$= \overline{\underline{G} \cdot \underline{Z}_{\underline{Y}}} + \overline{\underline{H}(\underline{y})} - \overline{\underline{H}(\underline{z})} \quad . \qquad \ldots (4.16)$$

Equations (4.15) and (4.16) can now be solved iteratively to give a power series expansion of $\underline{Z} = \sum_0^{\infty} \varepsilon^n \underline{Z}_n$. $\underline{H}$ must also be expanded as follows

$$\underline{H}(\underline{z}) = \underline{H}_0(\underline{z}) + \varepsilon \underline{H}_1(\underline{z}) + \varepsilon^2 \underline{H}_2(\underline{z}) + \ldots$$

$$= \underline{H}_0(\underline{z}_0) + \varepsilon(\underline{H}_1(\underline{z}_0) + \underline{z}_1 \frac{\partial}{\partial \underline{z}} \underline{H}_0(\underline{z}_0))$$

$$+ \varepsilon^2(\underline{H}_2(\underline{z}_0) + \underline{z}_1 \frac{\partial}{\partial \underline{z}} \underline{H}_1(\underline{z}_0) + \frac{1}{2}(\underline{z}_1 \frac{\partial}{\partial \underline{z}})^2 \underline{H}_0(\underline{z}_0)) \qquad \ldots (4.17)$$

It is convenient at this stage to introduce the $\sim$ operator defined by

$$\tilde{f} = \int_0^{\nu} (f - \bar{f})\, d\nu' \qquad \ldots (4.18)$$

where f is any periodic function of ν . Comparison of (3.10) and (4.18) shows that $\hat{f}$ and $\tilde{f}$ differ only by an integration constant. The $\sim$ operator has

slightly more complicated properties than the $\hat{}$ operator as follows:

$$\tilde{\bar{f}} = 0, \quad \bar{\tilde{f}} \neq 0 \qquad \ldots (4.19)$$

and

$$\frac{\partial}{\partial \nu}(\bar{f}) = (\overline{\frac{\partial f}{\partial \nu}}) = 0 \quad . \qquad \ldots (4.20)$$

However

$$\frac{\partial}{\partial \nu}(\tilde{f}) = f - \bar{f} \qquad \ldots (4.21)$$

and

$$(\widetilde{\frac{\partial f}{\partial \nu}}) = f(\nu) - f(0) \qquad \ldots (4.22)$$

where $f(0)$ is f evaluated at $\nu = 0$. One can also show that

$$\overline{\tilde{f}g} + \overline{f\tilde{g}} = \overline{\tilde{f}\bar{g}} + \overline{\bar{f}\tilde{g}} \qquad \ldots (4.23)$$

and

$$\widetilde{\tilde{f}g} + \widetilde{f\tilde{g}} = \tilde{\tilde{f}}\tilde{g} + \tilde{\bar{f}}\bar{g} + \bar{\tilde{f}}\tilde{g} \quad . \qquad \ldots (4.24)$$

We can now proceed to calculate the required transformation, using (4.15) to (4.24);

$$\underline{Z}_0 = \underline{Y}$$

$$\underline{H}_0 = \bar{\underline{G}}$$

$$\underline{Z}_1 = \int_0^\nu d\nu (\bar{\underline{G}} - \underline{G}) = - \tilde{\underline{G}}$$

$$\underline{H}_1 = \frac{1}{\tau} \int_0^\tau d\nu \; (- \underline{G} \cdot \tilde{\underline{G}}_{\underline{Y}} + \tilde{\underline{g}} \cdot \bar{\underline{G}}_{\underline{Y}})$$

Adding $\widetilde{\bar{f}\underline{G}}_\nu$, which is zero by (4.20), to the integrand we can write $\underline{H}_1$ as

$$\underline{H}_1 = - \overline{\underline{G} \cdot \tilde{\underline{G}}_{\underline{Y}}} + \overline{\tilde{\underline{G}} \cdot \bar{\underline{G}}_{\underline{Y}}} = - \overline{\underline{G} \cdot \tilde{\underline{G}}_{\underline{Y}}} + \bar{\tilde{\underline{G}}} \cdot \bar{\underline{G}}_{\underline{Y}}$$

and similarly

$$\underline{Z}_2 = \widetilde{\underline{G} \cdot \tilde{\underline{G}}_{\underline{Y}}} - \tilde{\bar{\underline{G}}} \cdot \bar{\underline{G}}_{\underline{Y}} \quad .$$

Evidently one can find $\underline{Z}$ and $\underline{H}$ to any desired order in ε. To find the inverse transformation we start with the equivalent of equation (4.9) for $\underline{y}$,

$$\underline{y}_t = \varepsilon \underline{g}(\underline{z}, \varphi) = \varepsilon \underline{h} \cdot \underline{y}_{\underline{z}} + (1 + \varepsilon\omega)\, \underline{y}_\varphi$$

and follow exactly the same procedure. As shown by Kruskal the series and their inverses are unique and the choice of integration constant shows that the series

are equal at $\nu = 0$, $\varphi = 0$. To order ε^2, therefore, the transformations are

$$\underline{Z} = \underline{Y} - \varepsilon\widetilde{\underline{G}} + \varepsilon^2 \left(\widetilde{\underline{G} \cdot \underline{\widetilde{G}}_{\underline{Y}}} - \underline{\widetilde{G}} \cdot \underline{\overline{G}}_{\underline{Y}}\right) + O(\varepsilon^3) \qquad \ldots (4.25)$$

$$\underline{Y} = \underline{Z} + \varepsilon\widetilde{\underline{G}} + \varepsilon^2 \left(\widetilde{\underline{\widetilde{G}} \cdot \underline{G}_{\underline{Z}}} - \underline{\overline{G}} \cdot \underline{\widetilde{\overline{G}}}_{\underline{Z}}\right) + O(\varepsilon^3) \qquad \ldots (4.26)$$

The equation of motion for $\underline{Z}$, $\underline{Z}_t = \varepsilon\underline{H}$, looks different according to which transformation, $\underline{Z} \to \underline{Y}$ or $\underline{Y} \to \underline{Z}$, one calculates but with the aid of equation (4.24) they can both be shown to be

$$\underline{Z}_t = \varepsilon\underline{\overline{G}}(\underline{Z}) - \varepsilon^2 \left(\overline{\underline{G} \cdot \underline{\widetilde{G}}_{\underline{Z}}} - \underline{\overline{\widetilde{G}}} \cdot \underline{\overline{G}}_{\underline{Z}}\right) . \qquad \ldots (4.27)$$

For the Hamiltonian system (1.1), $\underline{Y}$ and $\underline{G}$ can be chosen as

$$\underline{Y} = (\underline{q} \; ; \; \underline{p})$$

$$\underline{G} = (\Omega\underline{p} \; ; \; -\Omega\underline{q}) .$$

The averaged coordinates $\underline{Z}$ are denoted by

$$\underline{Z} = (\underline{Q} \; ; \; \underline{P})$$

For any function $S(\underline{q} \; ; \; \underline{p})$ the dyadic products of the transformation formulae can be conveniently written in terms of Poisson brackets,

$$\underline{G} \cdot S_{\underline{Y}} \equiv -\Omega_{\underline{q}} \cdot S_{\underline{p}} + \Omega_{\underline{p}} \cdot S_{\underline{q}} \equiv [\Omega, S] .$$

The formulae (4.25) and (4.26) can then be written

$$\underline{P} = \underline{p} + \varepsilon\widetilde{\Omega}_{\underline{q}} - \varepsilon^2 \left[\Omega, \widetilde{\Omega}_{\underline{q}}\right]\widetilde{\;} + \varepsilon^2 \left[\widetilde{\Omega}, \overline{\Omega}_{\underline{q}}\right] + O(\varepsilon^3) \qquad \ldots (4.28)$$

$$\underline{Q} = \underline{q} - \varepsilon\widetilde{\Omega}_{\underline{p}} + \varepsilon^2 \left[\Omega, \widetilde{\Omega}_{\underline{p}}\right]\widetilde{\;} - \varepsilon^2 \left[\widetilde{\Omega}, \overline{\Omega}_{\underline{p}}\right] + O(\varepsilon^3) \qquad \ldots (4.29)$$

$$\underline{p} = \underline{P} - \varepsilon\widetilde{\Omega}_{\underline{Q}} - \varepsilon^2 \left[\widetilde{\Omega}, \Omega_{\underline{Q}}\right]\widetilde{\;} + \varepsilon^2 \left[\overline{\Omega}, \widetilde{\overline{\Omega}}_{\underline{Q}}\right] + O(\varepsilon^3) \qquad \ldots (4.30)$$

$$\underline{q} = \underline{Q} + \varepsilon\widetilde{\Omega}_{\underline{P}} + \varepsilon^2 \left[\widetilde{\Omega}, \Omega_{\underline{P}}\right]\widetilde{\;} - \varepsilon^2 \left[\overline{\Omega}, \widetilde{\overline{\Omega}}_{\underline{P}}\right] + O(\varepsilon^3) \qquad \ldots (4.31)$$

The equations of motion of the averaged variables $\underline{P}$ and $\underline{Q}$ are

$$\underline{P}_t = -\varepsilon\overline{\Omega}_{\underline{Q}} - \varepsilon^2 \overline{\left[\widetilde{\Omega}, \Omega_{\underline{Q}}\right]} + \varepsilon^2 \left[\overline{\Omega}, \widetilde{\Omega}_{\underline{Q}}\right] + O(\varepsilon^3) \qquad \ldots (4.32)$$

$$\underline{Q}_t = \varepsilon\overline{\Omega}_{\underline{P}} + \varepsilon^2 \overline{\left[\widetilde{\Omega}, \Omega_{\underline{P}}\right]} - \varepsilon^2 \left[\overline{\Omega}, \overline{\widetilde{\Omega}}_{\underline{P}}\right] + O(\varepsilon^3) \qquad \ldots (4.33)$$

(c) The Action Integral

It is now possible to evaluate the action integral

$$K = \int_0^\tau p_i \frac{\partial q_i}{\partial Q_1} \, dQ_1 = \tau \overline{p_i \frac{\partial q_i}{\partial Q_1}} \qquad (4.8)$$

The leading term comes from

$$\overline{p_1 \frac{\partial q_1}{\partial Q_1}} = \frac{1}{\tau} \int_0^\tau dQ_1 \left\{ P_1 - \varepsilon \tilde{\Omega}_{Q_1} - \varepsilon^2 \left[\widetilde{\Omega, \Omega_{Q_1}} \right] + \varepsilon^2 \left[\bar{\Omega}, \tilde{\tilde{\Omega}}_{Q_1} \right] + O(\varepsilon^3) \right\}$$

$$\times \left\{ 1 + \varepsilon (\Omega_{P_1} - \bar{\Omega}_{P_1}) + \varepsilon^2 (F - \bar{F}) + O(\varepsilon^3) \right\}$$

$$= P_1 - \varepsilon \bar{\tilde{\Omega}}_{Q_1} - \varepsilon^2 \overline{\tilde{\Omega}_{Q_1} (\Omega_{P_1} - \bar{\Omega}_{P_1})} - \varepsilon^2 \left[\widetilde{\Omega, \Omega_{Q_1}} \right]$$

$$+ \varepsilon^2 \left[\bar{\Omega} \, \bar{\tilde{\tilde{\Omega}}}_{Q_1} \right] + O(\varepsilon^3) \, .$$

The second order term in $\frac{\partial q_1}{\partial Q_1}$ denoted by $(F - \bar{F})$ has zero average and does not affect the result to this order.

The remainder of $\overline{p \cdot dq}$ gives

$$\overline{p_i \frac{\partial q_i}{\partial Q_1}} = \frac{1}{\tau} \int_0^\tau dQ_1 \left\{ P_i - \varepsilon \tilde{\Omega}_{Q_i} + O(\varepsilon^2) \right\} \left\{ \varepsilon (\Omega_{P_i} - \bar{\Omega}_{P_i}) + \varepsilon^2 (F - \bar{F}) + O(\varepsilon^3) \right\}$$

$$= - \varepsilon^2 \overline{\tilde{\Omega}_{Q_i} (\Omega_{P_i} - \bar{\Omega}_{P_i})} + O(\varepsilon^3)$$

Now

$$\overline{\tilde{\Omega}_{Q_i} (\Omega_{P_i} - \bar{\Omega}_{P_i})} = \overline{(\tilde{\Omega}_{Q_i} - \bar{\tilde{\Omega}}_{Q_i}) (\Omega_{P_i} - \bar{\Omega}_{P_i})} \quad \text{from (4.19)}$$

$$= - \overline{(\Omega_{Q_i} - \bar{\Omega}_{Q_i}) (\tilde{\Omega}_{P_i} - \bar{\tilde{\Omega}}_{P_i})} \quad \text{using (4.23)}$$

$$= \frac{1}{2} \overline{\left[\widetilde{(\Omega - \bar{\Omega})}, \, (\Omega - \bar{\Omega}) \right]}$$

$$= \frac{1}{2} \overline{[\tilde{\Omega}, \, \Omega]} - \frac{1}{2} [\bar{\tilde{\Omega}}, \bar{\Omega}]$$

Also

$$\overline{[\widetilde{\Omega}, \Omega_{Q_i}]} - [\Omega, \bar{\widetilde{\Omega}}_{Q_i}] = - \overline{[\Omega, \widetilde{\Omega}_{Q_i}]} - \overline{[\widetilde{\Omega}, \widetilde{\Omega}_{Q_i}]} \quad \text{using (4.24)}$$

$$= \overline{[\Omega, \Omega - \Omega(0)]} - \overline{[\widetilde{\Omega}, \Omega - \Omega(0)]}$$

$$= [\bar{\widetilde{\Omega}}, \Omega(0)] - \overline{[\widetilde{\Omega}, \Omega]} + \overline{[\widetilde{\Omega}, \Omega(0)]}$$

$$= - \overline{[\widetilde{\Omega}, \Omega]}$$

As a result

$$\overline{p_i \frac{\partial q_i}{\partial Q_1}} = P_1 - \varepsilon \bar{\widetilde{\Omega}}_{Q_1} + \frac{\varepsilon^2}{2} [\bar{\Omega}, \widetilde{\Omega}] + \frac{\varepsilon^2}{2} \overline{[\Omega, \widetilde{\Omega}]} + O(\varepsilon^3)$$

Transforming back to coordinates $\underline{p}$ and $\underline{q}$ using (4.28) and (4.29) and transforming from $\sim$ to $\hat{}$ operators using

$$\widetilde{\Omega} = \hat{\Omega} - \hat{\Omega}(0)$$

one obtains finally

$$\overline{p_i \frac{\partial q_i}{\partial Q_1}} = p_1 + \varepsilon(\Omega - \bar{\Omega}) - \varepsilon^2 [\bar{\Omega}, \hat{\Omega}] + \frac{\varepsilon^2}{2} \overline{[\Omega, \hat{\Omega}]} + O(\varepsilon^3) \qquad \ldots (4.34)$$

One then has

$$K = \tau \overline{p_i \frac{\partial q_i}{\partial Q_1}}$$

$$= \tau H - \varepsilon \tau \bar{\Omega} - \varepsilon^2 \tau \left([\bar{\Omega}, \hat{\Omega}] - \frac{1}{2} \overline{[\Omega, \hat{\Omega}]} \right) \quad \ldots.$$

This is obviously related to the invariant J calculated from the Poisson bracket method, by

$$K = \tau H - \varepsilon \tau J \qquad \ldots (4.35)$$

at least to order ε^2 .

The advantages of the Poisson bracket method is that the first few terms of the invariant can easily be found. The advantages of Kruskal's method are that it is systematic (although the task of calculating higher order terms would be

arduous) and that one shows that the invariant is an asymptotic invariant. The correspondence of the two methods to terms of order ε^2 suggests that the Poisson bracket method generates a true asymptotic series. Another advantage of Kruskal's method is that one obtains some physical understanding of what the invariant is. In a periodic dynamical system with an ignorable angle variable the corresponding action is a constant of the motion. In the present system which is nearly periodic there is an ignorable (in an asymptotic sense) coordinate φ. The invariant K is the action corresponding to this almost ignorable angle variable.

(d) Reduction to a System of Lower Dimensions

Having found a constant of the motion it is well known that new canonical coordinates can be found in which the constant is one of the new momenta. In fact Kruskal shows that new coordinates can be found of which K and Q_1 are a conjugate pair. For the particular systems considered in this paper such a set of coordinates is easily obtained. Although $(\underline{Q}; \underline{P})$ are not canonical coordinates the set of quantities with K replacing P_1 are. This can be demonstrated directly by transforming the Hamiltonian to obtain

$$H = K + \varepsilon \bar{\Omega} (Q_2, \ \ldots \ Q_n \ ; \ K, \ P_2, \ \ldots \ P_n) + O(\varepsilon^2) \qquad \ldots \ (4.36)$$

The equations of motion, (4.32) and (4.33), are of Hamiltonian form with the above Hamiltonian.

Another method of finding new canonical coordinates $(Q' \ ; \ \underline{P}')$ is to solve the Poisson bracket relations

$$[P_i', \ P_j'] = [Q_i', \ Q_j'] = 0$$

$$[P_i', \ Q_j'] = \delta_{ij} \qquad \ldots \ (4.37)$$

where P_1' is chosen equal to K. The equations (4.37) have one set of solutions

$$P_i' = p_i + \varepsilon(\hat{\Omega}_{q_i} + \frac{\partial S}{\partial q_i}) + O(\varepsilon^2), \ i \neq 1$$

$$Q_i' = q_i - \varepsilon(\hat{\Omega}_{p_i} + \frac{\partial S}{\partial p_i}) + O(\varepsilon^2)$$

where S is any function independent of q_1. There exists therefore considerable freedom in finding new canonical coordinates. If the particular choice

$$S = - \hat{\Omega}(q_1 = 0)$$

is made, the coordinates are identical with the set mentioned in the previous paragraph since

$$\tilde{\Omega} = \hat{\Omega} - \hat{\Omega}\,(q_1 = 0) \quad .$$

As in Section 3(b), the only one of these transformations that can be readily inverted is the one obtained from Kruskal's averaging method.

In the reduced Hamiltonian (4.37) K appears merely as a parameter and the system has been effectively reduced to $n-1$ dimensions. If this Hamiltonian contains a further small parameter δ, such that for $\delta = 0$ the motion is once more periodic, then another adiabatic invariant exists and can be calculated by the methods of Section 3 and Section 4. It is worth remarking that these other periodicities may not appear in the original Hamiltonian, only being created by the construction of the first invariant and the subsequent transformation of the Hamiltonian. This behaviour occurs in plasma physics where firstly the magnetic moment can be found and subsequently the longitudinal and flux invariants. A recent derivation of these from Liouville's equation but not using canonical formalism has been given by Hastie, Haas and Taylor (1966).

5. WHITTAKER'S ADELPHIC INTEGRAL

A dynamical system of the type under consideration is the motion of a particle near a point of stable equilibrium. The Hamiltonian can be written (Whittaker 1937 Ch. XVI) in the form

$$H = \sum_{n=2}^{\infty} \varepsilon^n H_n \qquad \ldots (5.1)$$

where H_n is a homogeneous polynomial of a degree n in the canonical coordinates $(q_i' ; p_i')$ with $i = 1, \ldots N$. In particular H_2 is of the form (2.5) so that to

lowest order the system can be either periodic or conditionally periodic. Invariants of such systems have been studied extensively (Whittaker 1937, Cherry, Birkhoff 1927, Contopoulos 1960, 1963b) and Contopoulas (1963a) has shown that the various treatments are equivalent. Whittaker's method is to change to new variables q_i'' , p_i'' that reduce the Hamiltonian to

$$H = \sum_{r=1}^{N} (\alpha_r p_r'') + H_3 + H_4 \ldots\ldots \qquad \ldots (5.2)$$

where the α_r are the periods of the lowest order simple harmonic motions. H_n is an homogeneous polynomial of degree n in $(p_i)^{\frac{1}{2}}$ and also a function of sin q_i'' and cos q_i'' . One now looks for a time independent constant of the motion, Φ and for simplicity only a two dimensional system is considered. Φ satisfies the equation

$$[\Phi, H] = 0 \qquad \ldots (5.3)$$

which when expanded gives successively

$$\alpha_1 \frac{\partial \Phi_2}{\partial q_1''} + \alpha_2 \frac{\partial \Phi_2}{\partial q_2''} = 0 , \qquad \ldots (5.4)$$

$$\alpha_1 \frac{\partial \Phi_3}{\partial q_1''} + \alpha_2 \frac{\partial \Phi_3}{\partial q_2''} = [\Phi_2, H_3] \text{ etc.} \qquad \ldots (5.5)$$

To find a solution Φ different from H one can take Φ_2 to be any function of p_1'' and p_2'' independent of H_2 (for example Whittaker took $\Phi_2 = \alpha_1 p_1'' - \alpha_2 p_2''$) and the Φ_n can be found as particular integrals of the successive equations.

A feature of this solution is that in the series one finds coefficients of the form

$$(\alpha_1 \ell \pm \alpha_2 m)^{-1}$$

where ℓ and m are integers, the range of possible values of which increases as the order of the term n increases. This means that, with α_1/α_2 irrational and for sufficiently large n , there are in the series coefficients that are arbitrarily large. When α_1/α_2 is rational the problem is even more acute since in the formal series for Φ there appear some coefficients with vanishing denominators. Whittaker overcame this difficulty using two techniques. In solving for Φ_n he

had selected only a particular integral but could equally well have added an arbitrary amount of the solution of the homogeneous equation. In some cases it proves possible to avoid zero denominators by a suitable choice of these constants in the lower order terms. For others however it is necessary to renormalise the series solution for Φ by, effectively, multiplying by the zero denominator, the new form for Φ now starting with the term that previously had the troublesome coefficient.

The problem with these methods is that it is not clear at any stage whether further difficulties will arise in higher order terms. There is also the curious point that the formal expression for the adelphic integral is different depending on whether α_1/α_2 is rational or irrational and also whether terms of a particular type appear in the Hamiltonian.

The Hamiltonian (5.2) can be written in standard form (1.1) using the transformation

$$\begin{aligned} p_1 &= \alpha_1 p_1'' + \alpha_2 p_2'' \qquad & q_1 &= \tfrac{1}{2}(q_1''/\alpha_1 + q_2''/\alpha_2) \\ p_2 &= \alpha_1 p_1'' - \alpha_2 p_2'' \qquad & q_2 &= \tfrac{1}{2}(q_1''/\alpha_1 - q_2''/\alpha_2) \end{aligned} \qquad \ldots (5.6)$$

and the invariant J of Section 3 calculated. The question arises as to whether the adelphic integral and J are equivalent or not. The main disparity is that the lowest order term of the adelphic integral is a function only of p_1 and p_2 (except in some special circumstances associated with rational values of α_1/α_2) whereas the lowest order term in J is apparently only independent of q_1. However one is able to show that the first term in the series for J is in general a function of p_1 and p_2 only (depending on q_2 only when α_1/α_2 is rational). The way in which this occurs is as follows; Ω consists of a number of terms like

$$f_{mn} \begin{matrix}\cos \\ \sin\end{matrix} \{(n\alpha_1 + m\alpha_2)q_1 + (n\alpha_1 - m\alpha_2)q_2\} \qquad \ldots (5.7)$$

where f_{mn} is a function of p_1 and p_2 only and m and n take a range of integral values. The lowest order term in J is $\bar{\Omega}$ which is only non-zero if there are values of m and n such that

$$\frac{\alpha_1}{\alpha_2} = \frac{-m}{n}$$

and then

$$\bar{\Omega} = f_{mn} \cos \{ (n\alpha_1 - m\alpha_2)q_2 \} \qquad \ldots (5.8)$$

Except under these special circumstances $\bar{\Omega} = 0$ and the lowest order term is $J_1 = \frac{1}{2} \overline{(\Omega, \hat{\Omega})}$. This quantity consists of terms that are averages over q_1 of products of two expressions like (5.7) and leads to integrals of the form

$$I = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} dq_1 f \cos \{ ((n-k)\alpha_1 + (m-\ell)\alpha_2)q_1 + ((n-k)\alpha_1 - (m-\ell)\alpha_2)q_2 \} \qquad \ldots (5.9)$$

where f is a function of p_1 and p_2 only and n, m, k and ℓ take various integral values. When $n = k$ and $m = \ell$ this integral is equal to f and there always occur some terms of this form in J_1. The only way for I to be non-zero and to depend on q_2 is for

$$\frac{\ell - m}{n - k} = \frac{\alpha_1}{\alpha_2}$$

(i.e. for α_1/α_2 to have a particular rational value).

So, apart from these special rational cases, both H and J have lowest order terms that are functions of p_1 and p_2 only. It is possible therefore to construct another invariant (a function of J and H) that has as its lowest order term any function of p_1 and p_2 whatsoever. This in fact is what the adelphic integral is. In the three examples quoted by Whittaker (1937 Ch. XVI) one can show that the adelphic integral is a function of H and J. Examples worked out by Contopoulos and Moutsoulas (1965) are also equivalent to J in the same way.

The advantage of the methods of Section 3 and Section 4 is that they overcome all but one of the difficulties encountered by Whittaker. The problem of vanishing denominators does not arise at all. A single expression is obtained for the invariant irrespective of the value of α_1/α_2 and, for the rational case at least, one can show that the series is asymptotically invariant. The problem of small denominators when α_1/α_2 is irrational still occurs however and it is this fact that prevents one proving simply that the formal series is asymptotically invariant for all values of α_1/α_2.

6. COMPARISON WITH NUMERICAL DATA

A point mentioned by none of the methods is just how small ε should be for the analysis to be valid. There is in fact a useful range of values of ε for which the theory applies as can be seen by comparing the invariant theory and the results of numerically integrating the equations of motion. It had been intended in this section to make a detailed comparison between calculation and theory for a simple example. Recently, however, Contopoulos and Moutsoulas (1965) have presented results of computations for the dynamical system with Hamiltonian

$$H = \frac{1}{2} (\dot{x}^2 + \dot{y}^2 + x^2 + y^2 + 2x^2y) \quad . \qquad \ldots (6.1)$$

They have also applied the methods outlined by Whittaker and found the adelphic integral. The comparison between the two was close even taking only the first term of the invariant series and improved as the second and third terms were included. The detailed work of these authors gives ample evidence of the validity of the invariant theory even for values of ε where the drift of the orbits around the invariant surfaces is substantial.

Hénon and Heiles (1964) have also described orbit calculations for the system with the Hamiltonian

$$H = \frac{1}{2} (\dot{x}^2 + \dot{y}^2 + x^2 + y^2 + 2x^2y - \tfrac{2}{3} y^3) \quad . \qquad \ldots (6.2)$$

This represents motion in a potential well with a potential

$$U = \frac{1}{2} (x^2 + y^2 + 2x^2y - \tfrac{2}{3} y^3) \quad .$$

The equipotential lines are illustrated in Fig.1. There is an additional feature that arises in the study of this dynamical system that is worthy of mention. The topological structure of invariant surfaces, even for small ε, can be complex and the lowest order term of the invariant may not represent the surfaces adequately. To explain how this occurs this example will be considered in more detail. The invariant surfaces for a two dimensional system lie in the four dimensional phase space $(x, y ; \dot{x}, \dot{y})$. If particles with a constant energy H_o are considered, the Hamiltonian can be used to eliminate one of the coordinates, $\dot{x}$ say. The surfaces for given H_o now lie in a three dimensional space and some knowledge of their

structure can be obtained from their cross section in some plane, $x = 0$ say. Fig.2 shows the section of the surfaces for $H_0 = 0.01$ in the plane $x = 0$ obtained from numerical orbit calculations. (An orbit was integrated and its intersections with the plane $x = 0$ plotted. As the distance along the orbit increased the intersections gradually traced out the cross section of the invariant surface.) Henon and Heiles (1964) presented diagrams of this kind but only for large values of H_0 in order to show how the invariant surfaces disintegrated. The present authors also made orbit calculations for the same potential for smaller values of H_0 and Fig.2 is reproduced from this data. From the symmetry of the potential one can see that, for all values of the energy, there exist periodic orbits lying along the lines RZ, QY and PX. These particular orbits transform in Fig.2 into the points A and B and the bounding curve of the diagram.

The invariant J can be calculated as follows. Using the canonical transformations

$$x = \sqrt{2p_1}\,\sin q_1 \quad , \quad y = \sqrt{2p_2}\,\sin q_2$$

$$\dot{x} = \sqrt{2p_1}\,\cos q_1 \quad , \quad \dot{y} = \sqrt{2p_2}\,\cos q_2$$

followed by

$$P_1 = p_1 + p_2 \qquad Q_1 = \frac{1}{2}(q_1 + q_2)$$

$$P_2 = p_1 - p_2 \qquad Q_2 = \frac{1}{2}(q_1 - q_2)$$

the Hamiltonian (6.2) reduces to the standard form with $\varepsilon = \sqrt{H}$. It is found that $\bar{\Omega}$ is zero so that the lowest order invariant is

$$J_1 = \frac{1}{2}\,\overline{[\Omega\ \hat{\Omega}]}$$

which when evaluated gives

$$J_1 = \frac{1}{12}\left\{(5p_1^2 - 4p_1p_2 + 5p_2^2 + 14p_1p_2\cos 2(q_1 - q_2)\right\} \qquad \ldots (6.3)$$

Transforming back to $(x,y\ ;\ \dot{x},\dot{y})$ one finds that for $x = 0$ and for particles with energy H_0 the invariant becomes

$$J_1 = \frac{5}{12}H_0 + 28\,y^2(y^2 + \dot{y}^2 - 2H_0) \qquad \ldots (6.4)$$

Fig.3 shows the surfaces of constant J_1 and it is apparent that although they are similar to Fig.2 sufficiently far from A, B and the bounding curve they

differ substantially near these periodic orbits. It can also be seen that the function (6.4) is topologically unstable in the neighbourhood of the line $y = 0$. Further small terms in the series could alter the topology of the function drastically. The next term in the series J_2 was therefore evaluated and Fig.4 shows the contours of $(J_1 + \varepsilon J_2)$. Although the topology has been altered it is still different from the orbit calculations; the invariant predicts that the periodic orbits are unstable whereas the numerical data shows that they are stable. However, yet another term in the invariant series is required in order to determine the topology near the periodic orbits. Near A the invariant is of the form

$$J_1 + \varepsilon J_2 = y^2 - 2\varepsilon\alpha\, y\, \dot{y}$$

where $(y\ \dot{y})$ are cartesian coordinates with origin at A and α is some positive number. This results in the J-contours near A being hyperbolic. One can see that the addition of a term $\varepsilon^2 \beta^2 \dot{y}^2$ with $\beta > \alpha$ would change the surfaces from hyperbolic to elliptical form and such a term could appear in J_3.

Another example of this behaviour occurs in the study of magnetic surfaces generated by almost periodic magnetic field lines (Aleksin 1962). When small helically symmetric components of a particular form are superimposed on a uniform magnetic field the lowest order invariant surfaces are concentric cylinders. However in the neighbourhood of the axis the magnetic surface topology is only adequately described by calculating the first three terms of the series. Another example, so far unexplained that may have a similar explanation concerns the magnetic moment of a charged particle. Numerical orbit calculations (Garren et al 1958) show that the particle orbits do lie in surfaces for sufficiently small values of the Larmor radius. However the topology of the surfaces is more complex than the first three terms of the adiabatic invariant predict.

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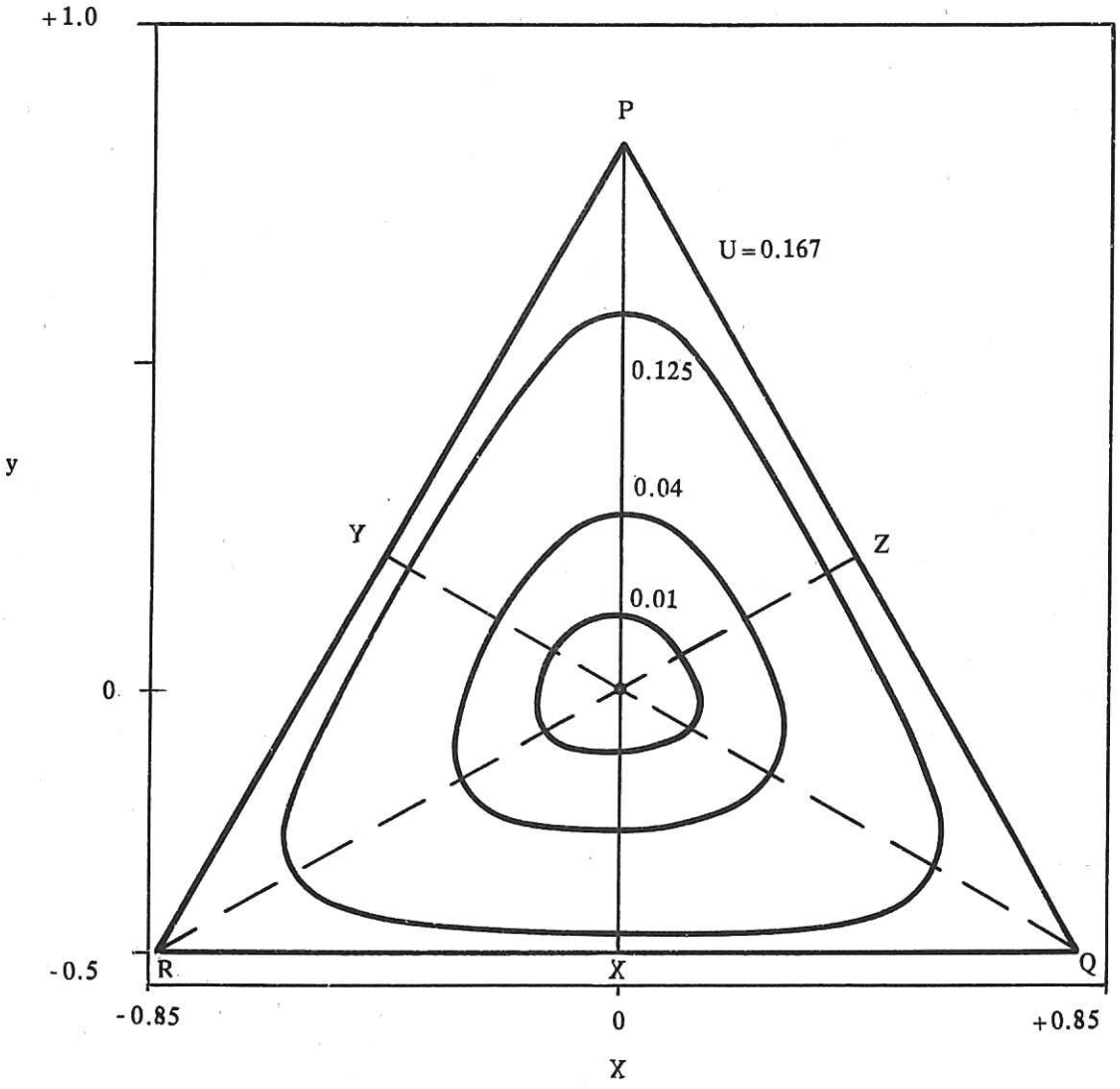

Fig. 1 The potential well : lines of constant U (CLM-P 111)

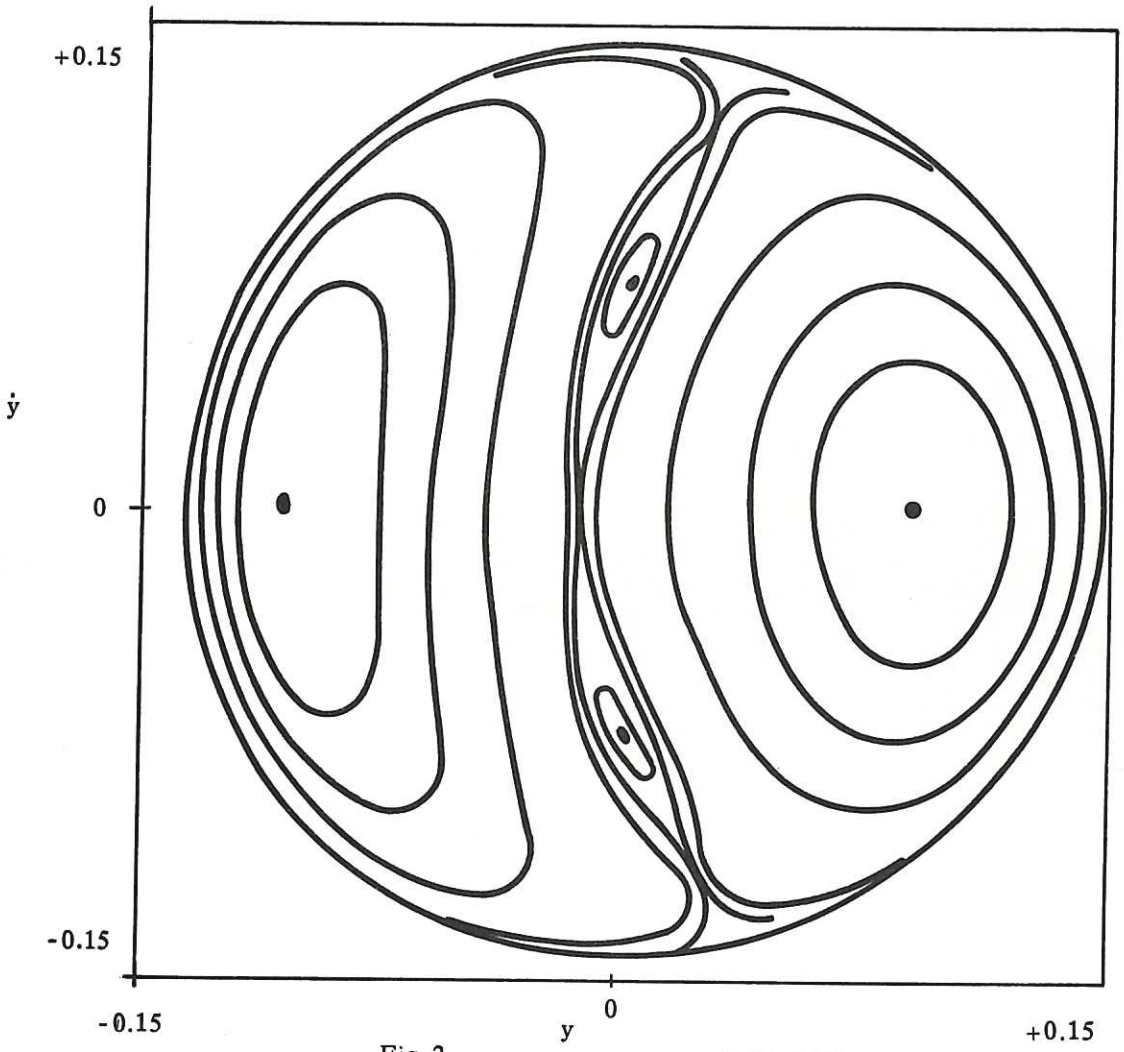

Fig. 2 (CLM- P 111)

Cross section of invariant surfaces from orbit calculations : H = 0.01

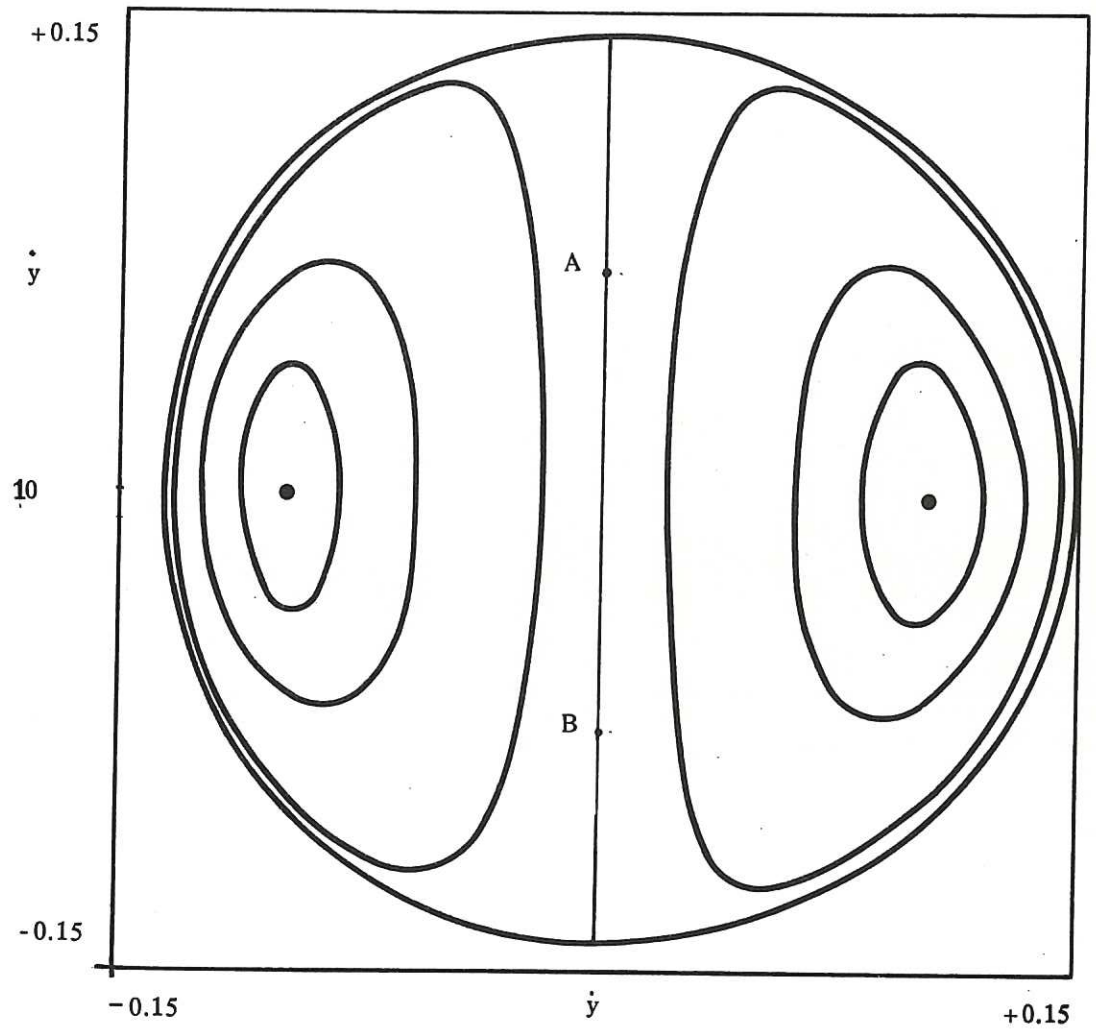

Fig. 3 Surfaces of constant J_1 (CLM-P 111)

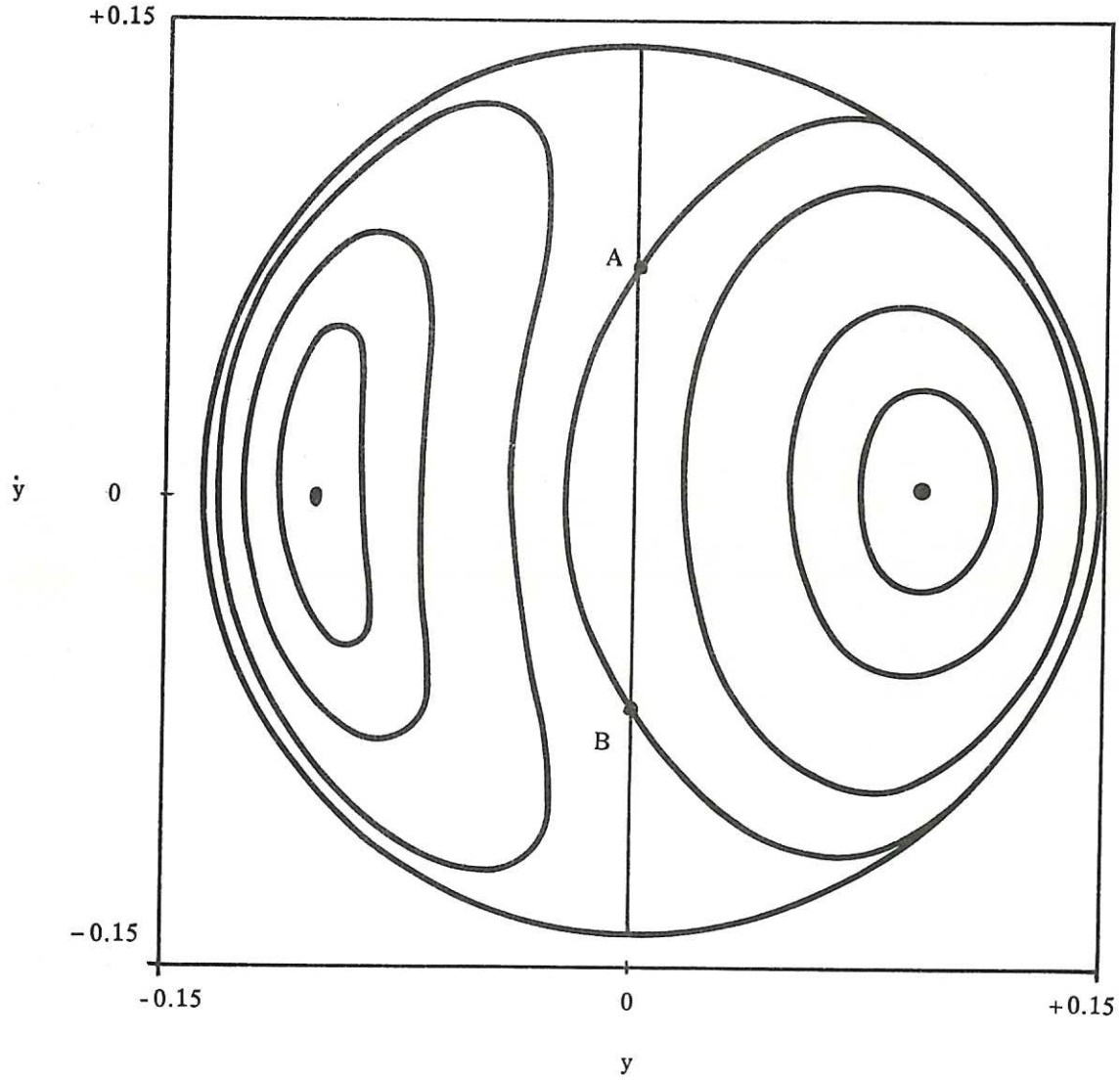

Fig. 4 Surfaces of constant $J_1 = \epsilon J_2$ (CLM-P 111)