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# RADIAL OSCILLATIONS OF A PLASMA CYLINDER WITH ARBITRARY DENSITY DISTRIBUTION

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RADIAL OSCILLATIONS OF A PLASMA CYLINDER
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A B S T R A C T

This paper discusses the radial hydromagnetic oscillations of a plasma cylinder confined by an axial magnetic field for the general case of an arbitrary distribution of mass.

The oscillation frequency is expressed in terms of the value for an annular distribution multiplied by a correction factor, g. Values of g are calculated analytically for several simple distributions and the results compared with those obtained by a numerical method and using a variational principle. The numerical and variational methods are used to calculate g values for other distributions, including examples from the Hain-Roberts theta-pinch code and from a theta-pinch experiment. The value of g varies from 1 for an annulus to about 1.4 for a distribution peaked on the axis. These simple calculations show that an accurate value of the plasma mass can be obtained when the oscillation frequency and density distribution are known.

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C O N T E N T S

## 1. INTRODUCTION

Radial hydromagnetic oscillations of a plasma cylinder confined by an axial magnetic field in the theta pinch have been studied by several authors. NIBLETT and GREEN (1959) calculated the frequency, $\omega$ , analytically for the simplest case where the plasma lies in an annulus; the result is $\omega^2 = \sqrt{\frac{B^2}{M}}$ where M is the mass/cm length and B the external magnetic field. Good agreement was found between theory and experiment for a reversed field theta pinch in which the plasma was observed to lie in an annulus. TAYLOR (1959) treated the more general case which includes different radial density distributions and calculated analytically the period for uniform distribution and also for some other distributions which could be conveniently expressed in simple analytic form.

The present work extends Taylor's theory to include any arbitrary distribution of mass. The results are expressed in terms of a correction factor g by which the expression for oscillation period for an annulus must be multiplied; g is defined by the expression

$$\omega = g\sqrt{\frac{B^2}{M}}$$

and for an annulus $g = 1$

Correction factors are computed for the distribution used by Taylor and also for a number of other distributions including examples from the HAIN-ROBERTS (1960) hydromagnetic theta pinch code and from the megajoule theta pinch experiment (BODIN et al, 1965).

## 2. THEORY

The model and equations used in calculation of the natural periods of oscillation of a plasma cylinder are the simplest possible cases of those considered by TAYLOR (1959). Consider an infinitely long cylinder of plasma of pressure $p_o(r)$ containing a magnetic field, $B_o(r)$ and surrounded by an infinite vacuum containing another uniform magnetic field, $B_V$, such that

$$p_o + \frac{B_o^2}{8\pi} = \frac{B_v^2}{8\pi} \qquad \ldots (1)$$

Azimuthally symmetric radial oscillations of this plasma cylinder are assumed to be governed by the linearised magnetohydrodynamic equations:

$$\rho_o \frac{dV_1}{dt} = -\nabla\left(p_1 + \frac{B_o B_1}{4\pi}\right) \qquad \ldots (2)$$

$$\frac{\partial \rho_1}{\partial t} = -\nabla(\rho_o V_1) \qquad \ldots (3)$$

$$\nabla \times (V_1 \times B_o) = \frac{\partial B_1}{\partial t} \qquad \ldots (4)$$

where $\rho$ = plasma density, and the subscript, $_1$, refers to perturbation quantities. As the motion of the plasma is only two dimensional the equation of state will be approximated by the adiabatic law $p = \rho^2$ so that

$$\frac{1}{p_o} \frac{dp_1}{dt} = \frac{2}{\rho_o} \frac{d\rho_1}{dt} \qquad \ldots (5)$$

Assuming that perturbation quantities are of the form $q_1 = q_1(r)^{i\omega t}$ equations (1) - (5) reduce to

$$\frac{B_v^2}{4\pi} \frac{d}{dr}\left(\frac{1}{r} \frac{d}{dr}(rV_1)\right) + \omega^2 \rho_o V_1 = 0 \qquad \ldots (6)$$

Integrating this equation through the boundary at $r = r_o$ gives the boundary condition

$$\left[\frac{d}{dr}(rV_1)\right]_{r = r_o} = 0 \qquad \ldots (7)$$

and at the origin, $V_1(0) = 0$.

For purposes of numerical computation the equations are put in dimensionless form as follows:

The mass of the plasma per unit length is

$$M = 2\pi \int_0^{r_o} \rho_o (r) r \, dr$$

$$= 2\pi \, n_o \, m_i \, r_o^2 \int_0^1 \eta (\xi) \, \xi \, d\xi \ , \quad r = \xi r_o$$

where $n_o$ is the maximum number density and $m_i$ the ion mass, so that equation (6) may be rewritten as

$$\frac{d}{d\varepsilon} \left( \frac{1}{\xi} \frac{d}{d\xi} (\xi V_1) \right) + \frac{2g^2}{f} \eta (\xi) \, V_1 = 0 \qquad \ldots (8)$$

and

$$f = \int_0^1 \eta (\xi) \, \xi \, d\xi$$

A variational principle for $g^2$ may be found by multiplying equation (8) by $V_1$ and integrating the result from 0 to 1:

$$\frac{2g^2}{f} = \left\{ V_1^{\,2} (1) + \int_0^1 \left[ \left( \frac{dV_1}{d\xi} \right)^2 + \frac{V_1^2}{\xi^2} \right] \xi d\xi \right\} \left( \int_0^1 \eta \, V_1^2 \xi d\xi \right)^{-1} \qquad \ldots (9)$$

An approximate value, $G^2$, of $g^2$ for any density profile $\eta (\xi)$ may be calculated as follows: insert into equation (9) any trial function, V, which satisfies the boundary conditions and evaluate $G^2$. By allowing V to depend on one or more parameters, $\alpha_i$, an an upper bound to $g^2$ may be found by minimising $G^2$ with respect to the $\alpha_i$. Thus, the best values of the $\alpha_i$ are given by the solution of the equation:

$$\frac{\partial G^2}{\partial \alpha_i} = 0$$

A simple trial function which satisfies the boundary conditions is

$$V = \xi - \tfrac{2}{3} \xi^2$$

The correction factor is then given by

$$g^2 \leqslant \frac{f_o}{6} \Bigg/ \left( f_2 - \frac{4}{3} f_3 + \frac{4}{9} f_4 \right) \qquad \ldots (10)$$

where the $f_i$ are the moments of the density distribution,

$$f_i = \int_0^1 \eta\ (\xi)\ \xi^{i+1}\ d\xi$$

Evidently, $g$ cannot be simply related to any one feature of the density profile but a knowledge of the first four moments will enable an upper bound to the correction factor to be calculated with ease. For the case of uniform density, $\eta\ (\xi) = 1.0$ , the variational principle gives $G = 1.21$. In this case the solution of equation (8) is $V_1 = J_1 (2g\xi)$ and the boundary condition gives

$$J_o(2g) = 0$$

So, $g = 1.20$, and the simple trial function has given a very good approximation.

## 3. NUMERICAL STUDIES

Values of $g$ appropriate to several density profiles computed by the Hain-Roberts program and profiles measured experimentally in the Thetatron were calculated using a general complex eigenvalue program (McNAMARA, 1966). As a check on the program $g$ was calculated for density profiles $\rho(r) \propto r^o, r^2, r^4, r^\infty$ for which analytic solutions were calculated by Taylor. Using a ten-point difference scheme to compute the eigenvalues of the differential equation (8) gave errors in $g$ less than 3%. The computer program increased the number of points in the difference scheme until the computed value of $g$ was constant to 1%. The values corresponding to the above profiles are $g = 1.20,\ 1.11,\ 1.09,\ 1.00$ respectively. It is interesting to compare these results with the values of $g$ derived from the variational principle (10), $g_v$ = 1.21, 1.15, 1.14, 1.23. These values provide an upper limit on g and it is not surprising that the simple trial function used should give a poor result for the last case, where all the plasma is concentrated in an annulus.

Two experimental density profiles measured at different times (2 and 4 μsec respectively) in the megajoule theta pinch experiment BODIN et al, 1965) are shown in dimensionless form in Fig.1. Because the plasma tends to be concentrated near the axis in these profiles the g values (1.20, 1.26) are large compared with the value, g = 1.0, for an annular distribution.

A direct comparison with experiment cannot be made as the oscillations are not observed in the highly compressed plasma when good measurements of density profile are possible. However, on the basis of other measurements (BODIN et al, 1965) it is known that all of the plasma is swept up in the initial implosion and simultaneous measurements of frequency and magnetic field give a g value of 1.1 to 1.15. This agrees with the theory in this paper and the density profiles expected early in the discharge.

Three further examples are shown in Fig.2 for theoretical profiles taken from the Hain-Roberts computations. The bias field is 0, ± 2 kg and the values of g are (1.14, 1.13, 1.16). A precise comparison with the Hain-Roberts code is difficult because the external field is changing rapidly, but the average values of g are g = (1.09, 1.11, .99). Agreement is fair for parallel and zero bias field but not very good for reversed bias field. These results are summarised in Table 1.

Table 1. Comparison of Various Calculations of g.

| Profile | Analytic and/or computed g. | g from simple variational principle, equ. (10) |
|---|---|---|
| $r^0$ | 1.20 | 1.21 |
| $r^2$ | 1.11 | 1.15 |
| $r^4$ | 1.09 | 1.14 |
| $r^\infty$ | 1.00 | 1.23 |
| Experimental 1 | 1.21 | 1.29 |
| " 2 | 1.28 | 1.45 |
| | | Numerical Experiment |
| H-R 1 | 1.14 | 1.09 |
| H-R 2 | 1.13 | 1.11 |
| H-R 3 | 1.16 | 0.99 |

## 4. CONCLUSIONS

Three methods of calculating the frequency of radial hydromagnetic oscillations of a plasma cylinder using a very simple model have been studied. Numerical computations were shown to agree with analytic results and then applied to cases where an analytic solution is not possible. Comparison with computer calculations with the Hain-Roberts code show this simple model to be quite good. A variational principle for the frequency gives good results, for the cases where an analytic solution is possible, even with a very simple trial function. Using this variational principle and a knowledge of the density profile the g-factor can be calculated and the plasma mass determined from the oscillation frequency. In the early stages of the megajoule theta pinch experiment the g value was 1.1 - 1.15.

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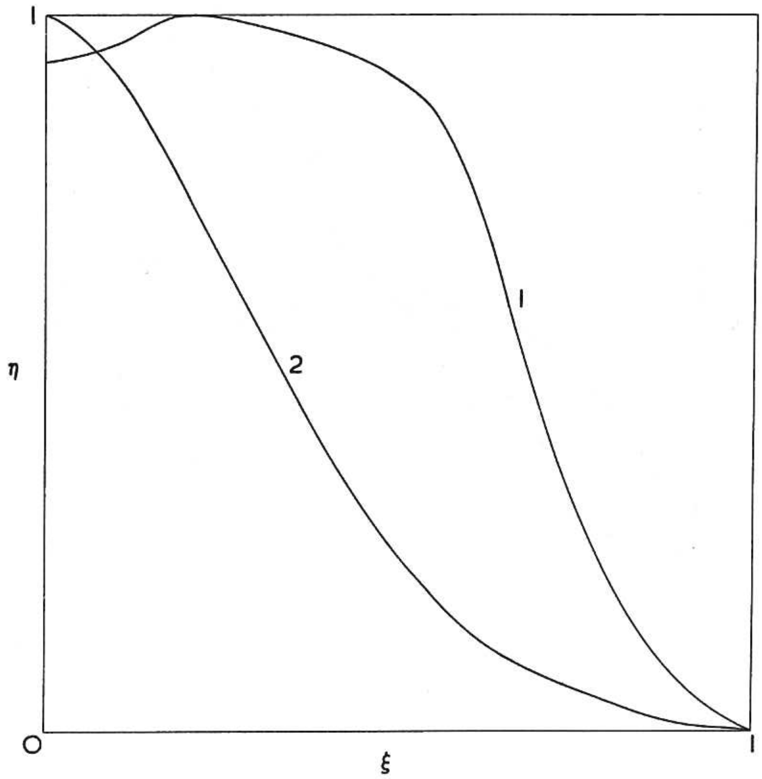


Fig. 1 (CLM-P 117)

Experimental profiles: 1) $n_o = 3.4.10^{16}$ cm$^{-3}$, f = .089, g = 2.52: 2) $n_o = 1.15.10^{16}$ cm$^{-3}$, f = .24, g = 2.39

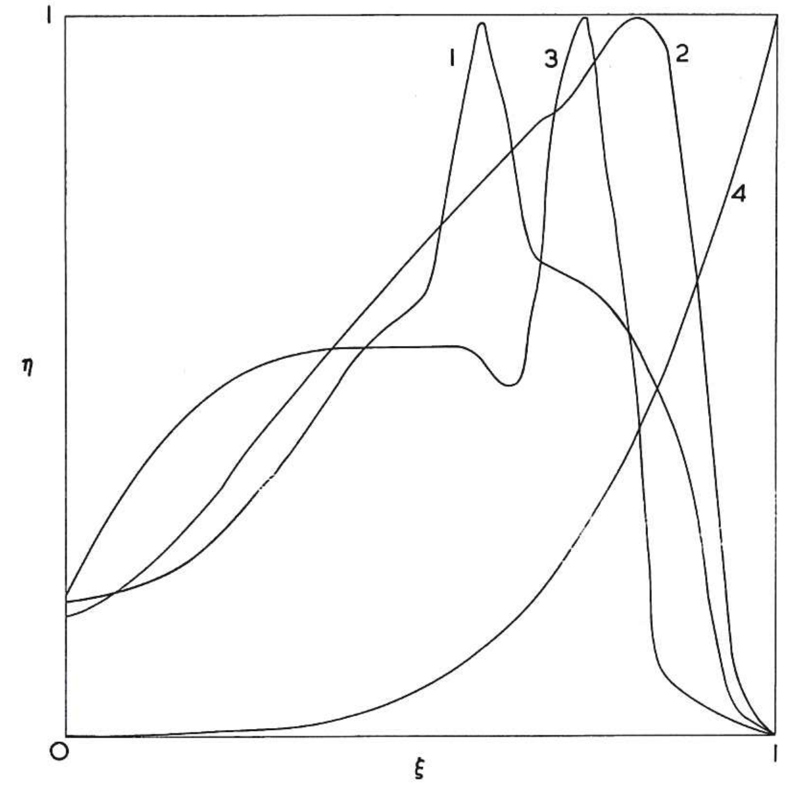


Fig. 2 (CLM-P 117)

Hain Roberts profiles

(1) $n_0 = 1.9 \times 10^{16}$ cm$^{-3}$, no bias, f = .23, g = 2.27;
(2) $n_0 = .95 \times 10^{16}$ cm$^{-3}$, + 2.0 hg bias, f = .32, g = 2.26;
(3) $n_0 = 2.2 \times 10^{16}$ cm$^{-3}$, - 2.0 hg bias, f = .21, g = 2.32;
(4) for comparison, Taylor's $\rho^4$ profile, f = 1/6, g = 2.29