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SINGLE PARTICLE MOTION IN TOROIDAL STELLARATOR FIELDS

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by

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A B S T R A C T

The discussion of the motion of single particles in stellarators is simplified by identifying three distinct groups of particle. These groups are treated by approximate analysis and by numerical methods. It is shown that, unlike an axisymmetric system, the region of a stellarator with surfaces closed about a magnetic axis need not form a perfect trap. In fact, unless the toroidal curvature is very small, particles in one of the groups, those mirrored between maxima of the helical field, will drift out beyond the separatrix no matter how large the magnetic field strength.

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CONTENTS

1. INTRODUCTION

Single particles moving in toroidally helical stellarator fields can be divided into three categories. (a) Passing particles which pass completely around the minor cross section without being reflected. (b) Blocked particles which, although they pass through several helical field periods, do not pass completely round the minor cross section: instead they are reflected, symmetrically about the median plane, because of the gradient of the toroidal magnetic field (B_{θ}). (c) Localised particles which are trapped within a single helical field period and have orbits which are not symmetric about the median plane.

The drift motion of passing and blocked particles is averaged over many field periods, consequently their motion can be treated approximately by replacing the real field with an axisymmetric field having the same rotational transform. This approximation is discussed in Section 3. The localised particles, being restricted to one helical field period, move in a system which is not axisymmetric, and an approximate treatment of their motion is given in Section 5.

As is well known, the drift surfaces of the passing particles are similar to, but displaced from, the magnetic surfaces; the displacement tending to zero with the Larmor radius. The blocked particles make larger excursions from the magnetic surface but this excursion still tends to zero with Larmor radius. On the other hand, because they are confined to a single field period, localised particles have drift surfaces which are not strongly related to the magnetic surfaces. In many cases they may 'escape' no matter how small the Larmor radius.

In Section 6 these results of the approximate theory are supported by comparison with numerical guiding centre computations for specific $\ell = 3$ toroidal stellarators.

2. DEFINITION OF CONTAINMENT

Stellarator fields consist of two regions separated by a separatrix (which may be broadened into a separatrix region[1]). Magnetic field lines inside the separatrix encircle the magnetic axis, whereas lines outside the separatrix pass between the conductors generating the helical field. Particles which drift onto these outer lines may undergo mirror reflection and sooner or later return to the interior region. However they will then have sampled lines which intersect, or pass near to, material conductors and the behaviour on these lines will be typical of mirror confinement rather than of stellarators. Consequently we shall consider a particle to be confined only if its guiding centre never passes outside the separatrix. We shall show that in this sense a stellarator need not form a perfect single particle trap.

This concept of confinement is different from that used by Morozov and Solov'ev[2] and Popryadukhin[3] where a particle is assumed to be confined if it has a finite displacement from an initial magnetic surface.

3. APPROXIMATE TREATMENT OF THE MOTION OF PASSING AND BLOCKED PARTICLES

The coordinate system is shown in Fig.1. All lengths will be expressed in units of r_m, the maximum separatrix radius, so that r must always be less than one for confinement. The Larmor radius r_L is defined in terms of the total velocity V and the field B_o on the ℓ-winding axis.

$$r_L = \frac{mcV}{qB_o r_m} \qquad \ldots (1)$$

where m and q are particle mass and charge. The rotational transform in passing once around the machine will be taken as:

$$\iota(r) = \iota_o \, r^{2n} \qquad \ldots (2)$$

where, $n = \ell - 2$ and ℓ = number of pairs of conductors in the ℓ winding. We will set ι_o to the maximum value for small aspect ratio stellarators[1] i.e.

$$\iota_o = p\pi/6 \qquad \ldots (3)$$

where p is the number of field periods on the torus.

In discussing passing and blocked particles the real stellarator field will be replaced by an axisymmetric field which has the same rotational transform:

$$B_\varphi/B_o = (p/12R_o)r^{2n+1} \qquad \ldots (4)$$

$$B_\theta/B_o = R_o/R = \left[1 + (r/R_o)\cos\varphi\right]^{-1} \qquad \ldots (5)$$

If $\underset{\sim}{A} = \text{curl } \underset{\sim}{B}$, is the vector potential of this field, then the motion of the particles in the axisymmetric field is given by,

$$RA_\theta + (mc/e)RV_\theta = \text{const.} \qquad \ldots (6)$$

and the flux surfaces by,

$$RA_\theta = \text{const.} \qquad \ldots (7)$$

which may be expressed in terms of the field as,

$$RA_\theta = \int_0^r R_o \, B_\varphi \, dz$$

or
$$RA_\theta = r_m \, pB_o \, r^{2n+2}/12(2n+2) \qquad \ldots (8)$$

The velocity V_θ can be expressed in terms of the usual invariants

$$\left.\begin{aligned} 2\varepsilon/m &= (V_\parallel^2 + V_\perp^2) = V^2 \\ 2\mu/m &= V_\perp^2/B \end{aligned}\right\} \qquad \ldots (9)$$

where $V_\parallel$ and $V_\perp$ are velocity components parallel and perpendicular to $\underset{\sim}{B}$. Then

$$(mc/e)V_\theta = r_L \;\; r_m \;\; B_o \, B_\theta \left[1 - \mu B/\varepsilon\right]^{\frac{1}{2}}/B \qquad \ldots (10)$$

and substituting (8) and (10) into (6) we obtain for the trajectory of the guiding centre

$$\frac{p \quad r^{2n+2}}{12 \, R_o \, r_L (2n+2)} + \frac{B_\theta\left[1 - \mu B/\varepsilon\right]^{\frac{1}{2}}}{B} = \text{const} \qquad \ldots (11)$$

where the quantity $\left[1 - \mu B/\varepsilon\right]^{\frac{1}{2}}$ changes sign on reflection of the particle.

Since all the passing and blocked particles intersect the plane $\varphi = 0$ we will consider the containment condition for particles starting in this plane. Two such orbits are sketched in the (r,φ) plane in Fig.2. Consider a particle starting with a guiding centre at the point A; this will be most difficult to contain if it is just passing or just reflected at a point such as B on $\varphi = \pi$ where the B_θ field is a maximum. AEBFA and AEBGC are limiting orbits for passing and blocked particles.

If we make the additional approximations $B_\theta/B \sim 1$ and $r/R \ll 1$ then we can obtain from (11) simple conditions for these limiting orbits, and hence all the guiding centres of the passing and blocked particles, to remain within $r = 1$ (i.e. within a radius equal to the maximum radius of the separatrix in the original fields). For the case $\ell = 3$ the condition for confinement of all passing particles originating at $r = r_0$ is,

$$\frac{1}{r_L} \geqslant \frac{48R_0^{1/2}}{p} \cdot \frac{(1+r_0)^{1/2}}{(1-r_0^4)} \qquad \ldots (12)$$

and for containment of the blocked particles is;

$$\frac{1}{r_L} \geqslant \frac{48R_0^{1/2}}{p} \cdot \frac{(r_0+r_1)^{1/2} + (r_1+1)^{1/2}}{(1-r_0^4)} \qquad \ldots (13)$$

Since $r_0 < r_1 < 1$ it will be sufficient to put $r_1 = \frac{1}{2}(1+r_0)$ when evaluating (13).

4. CONDITION FOR A PARTICLE TO BE LOCALISED

The localised particles are trapped within one helical field period and we will represent the field for these particles, for $\ell = 3$, by the approximate expressions:

$$\begin{aligned} B_r &= B_t\, r^2 \operatorname{Sin}(3\varphi - p\theta) \\ B_\varphi &= B_t\, r^2 \cos(3\varphi - p\theta) \\ B_\theta &= B_0\,[1 - r\cos\varphi/R_0] \end{aligned} \qquad \ldots (14)$$

where $\qquad (B_t/B_0)^2 = (p/R_0)^2/18$

and the rotational transform is given by (2). For a given ratio of separatrix to ℓ-winding radius the quantity (p/R_0), which is proportional to the tangent of the mean winding pitch angle, is typically 1.5 in a high shear stellarator with a 45° degree winding[4].

The condition for a particle to be trapped within a field period is

$$(V_{\parallel}/V_{\perp})_0^2 \leqslant (B_{max} - B_{min})/B_{min} \qquad \ldots (15)$$

where $(V_{\parallel}/V_{\perp})_0$ is the value in the centre of the helical mirror and the variation of B is taken over one field period. For r/R small,

$$(B/B_0)^2 = 1 + (B_t/B_0)^2\, r^4$$

and with $B_{max}/B_{min} \sim 1$ and $(B_t/B_0)^2 \ll 1$, (15) becomes:

$$(V_{\parallel}/V_{\perp})_0^2 \leqslant 0.5\ (B_t/B_0)^2\, [r_{max}{}^4 - r_{min}{}^4]$$

The bracketed term can be evaluated by integrating

$$\frac{Rd\theta}{B_\theta} = \frac{dr}{B_r}$$

over a complete field period and the condition for localisation becomes

$$(V_{\parallel}/V_{\perp})_0^2 \leqslant (\sqrt{2}/27) \cdot (p/R_0)^2\, r^5 \qquad \ldots (16)$$

where r is the normalised radius at the centre of the helical mirror.

The fraction f of an isotropic velocity distribution within the region of velocity space defined by (16) is obtained by integrating over r, and for $(V_{\parallel}/V_{\perp})$ small is

$$f \approx p/12R_0 \qquad \ldots (17)$$

In an $\ell = 3$ system this is numerically equal to the value of the shear parameter [4], important for considerations of plasma stability. Thus we see that an appreciable number of localised particles necessarily occur in a high shear stellarator. In an $\ell = 2$ stellarator, the fraction of localised particles will be greater for the same shear. In higher ℓ number stellarators, where the shear and rotational transform occur mainly in an annular shell near the separatrix, the fraction of localised particles will be smaller.

5. APPROXIMATE TREATMENT OF THE MOTION OF THE LOCALISED PARTICLES

A particle which is just localised on a field line with the maximum transform per field period $(\pi/6)$ has its projection in the (r,φ) plane move through $2\pi/3$ between reflections; all other localised particles move through smaller angles. Thus there can be no complete cancellation of the toroidal drift for these particles and they can be expected to drift through the separatrix in the z direction. In fact other drifts, particularly that due to the gradient of the B_φ field, modify this expectation, and a condition for the localised particles to be confined can be obtained by examining their total drift motion.

Since a localised particle necessarily has small $(V_{\parallel}/V_{\perp})$ the velocity of the guiding centre is given approximately by:

$$\underset{\sim}{V}_g \approx \text{const. } \underset{\sim}{B} \wedge \underset{\sim}{\nabla} (B^2)/B^4 + \underset{\sim}{V}_{\parallel}$$

and the average drift velocity of the guiding centre is thus

$$\langle \underset{\sim}{V}_g \rangle \approx \text{const. } \underset{\sim}{B}_o \wedge \underset{\sim}{\nabla} \langle B^2 \rangle /B^4$$

where B_0 is the toroidal field at $r = 0$ and $\langle\rangle$ indicates the average over the orbit between reflections.

The average drift motion is therefore along the surfaces,

$$\langle B^2 \rangle = \text{const.} \qquad \ldots (18)$$

or from (14)

$$p(p/R_0)r^4 A_1 - 2r\cos\varphi = \text{const.} \qquad \ldots (19)$$

where

$$A_1 = 1/18 - \sqrt{2}\langle u \rangle/9 \qquad \ldots (20)$$

and

$$u = \cos(3\varphi - p\theta)$$

The condition that a particle with guiding centre starting at (r_0, φ_0) always remains within $r = 1$ is

$$p \geqslant \frac{2(r_0 \cos\varphi_0 - \cos\varphi) \quad (R_0/p)}{A_1(1 - r_0^4)}$$

or if we replace $(r_0 \cos\varphi_0 - \cos\varphi)$ by its maximum value $(1 + r_0)$ confinement of the localised particles is ensured if,

$$p \geqslant 2(R_0/p) / A_1(1 - r_0)(1 + r_0^2) \qquad \ldots (21)$$

The quantity u has the value -1 at the minima of the helical field and for particles which make only small excursions about this point $\langle u \rangle = -1$, on the other hand for particles which are only just localised $\langle u \rangle = 0$. For these extremes we see from (21) (with $R_0/p = 1.5$) that the value of p necessary to contain all the particles with $r_0 \leqslant 0.9$ (i.e. approximately 80%), is

$$35 < p < 130$$

Such a large number of field periods is not usually attainable in practice so that a substantial fraction of the localised particles will not be confined within the separatrix. Whether or not localised particles are confined depends only on the gradients of $\underset{\sim}{B}$ and <u>is independent of the Larmor radius.</u>

If positive values of $\langle u \rangle$ are possible, as will be the case if the orbit is not symmetric about u = − 1, then a resonance $(A_1 = 0)$ can occur so that r becomes unlimited.

6. NUMERICAL COMPUTATION OF PARTICLE ORBITS

The approximate treatment of the previous sections describes the qualitative features of particle motion in stellarator fields and provides quantitative estimates of the orbits. However, it is not clear to what extent the approximations will be valid in a real case. Consequently, we have computed guiding centre orbits, in the drift approximation, for a toroidal, high shear, $\ell = 3$ stellarator which is of special interest to us. The currents and the filamentary conductor configuration are as described in reference 1; at the separatrix the aspect ratio is 6.5 and the ratio of toroidal to helical field is 3.7.

The motion of the guiding centre is obtained by integrating the equation

$$\underset{\sim}{V}_g = \frac{m(V_\perp^2 + 2V_\parallel^2)\, \underset{\sim}{B} \wedge \underset{\sim}{\nabla}\, (B^2)}{4qB^4} + \underset{\sim}{V}_\parallel$$

where ε and μ defined in (9) are assumed to be invariant. A predictor-corrector routine[5] is used for the integration and the accuracy is verified by varying the step length.

Examples of computed orbits are shown in Figs.3 to 8. The complete orbit of the guiding centre in the (r,φ) plane is shown, the position of the guiding centre at each field period is numbered and the numbered points joined to give a cross section of the drift surface. The helical nature of the trajectories arises from the helical form of the field lines. Figs. 3 to 7 were computed for the same magnetic configuration as Fig.2 of reference 1. Fig.8 is for a winding with the same mean pitch angle $(\simeq 45^{\circ})$ but with 64 field periods and a winding aspect ratio of 21. This shows the orbit of a localised particle which is confined. The figures show examples of confined and escaping particles of each of the three categories of passing, blocked and localised. Fig.9 shows the intersection of the $|B| =$ const. surfaces with the plane $\theta = 0$ and is to be compared with eq.(19). The intersection of the orbit of a localised particle with this same plane is also shown, it is parallel to the contours of $|B|$ showing that in this case the averaging described in section 5 does not greatly affect the motion.

Fig.10 shows a comparison between the approximate containment conditions (12) and (13) and the containment found in the computations. Tamm [6] has shown that, in axisymmetric systems, considerations of the conservation of energy and angular momentum alone ensure that a particle does not depart from its initial magnetic surface by more than a Larmor radius in the B_{φ} field. This limit is also indicated on the figures.

Fig.11 shows a comparison between eq.(16) and the value of $(V_{\parallel}/V_{\perp})_0$ found necessary for localisation in the computations.

7. DISCUSSION

Fig. 10 shows that it is much easier to contain the passing and blocked particles than indicated by Tamm's upper limit [6]; this is to be expected because of the extra constraint introduced by assuming μ to be invariant. On the other hand localised particles, which have no analogue in axisymmetric systems, escape even when Tamm's condition is satisfied. The agreement between the approximate treatment and the computed results in Fig. 10 is to within a factor 2, the computations showing that it is more difficult to contain the particles than indicated by approximate theory. Fig. 11 shows eq. (16) to be suprisingly accurate and together with Figs. 8 and 9 offers support for the validity of (21) as a criterion for confinement of localised particles.

As an illustration the approximate expressions may be used to calculate the parameters of a stellarator trap with a 45^o winding which is such that 5% of an isotropic distribution of 20 keV H^+ ions has drift orbits which intersect the separatrix. According to (17) about 12% of the ions will be localised and from (21) a value of $p \simeq 100$ will retain some 80% of these so that 2% of the total ions are localised and lost. The condition that less than 3% of the ions are lost as passing and blocked ions, is, from (13), $1/r_L \geqslant 140$. If we assume the ℓ - winding radius to be twice the separatrix radius and the B_θ field to be 10^5 gauss these values of p and r_L lead to a ℓ - winding minor and major radii of 0.8 and 21 metres.

It does not at all follow that such large dimensions will be necessary to confine plasma for a specific time in such a trap. The condition (16) in a stellarator defines the loss region in velocity space as a disc centred on $V_{\parallel} = 0$, so that the loss time of a sufficiently small density of particles in the trap will be of the order t_{90} the time for small angle collisions to rotate the velocity vector by 90^o. At higher densities however a particle entering the loss region may well be scattered out again before it has time to drift to the separatrix, consequently it will be contained for longer than t_{90}. The collisional diffusion loss of plasmas in stellarators due to this mechanism will be considered in a later paper.

8. CONCLUSIONS

Numerical computations for real stellarator fields confirm the existence of passing and blocked particles which have similar orbits to their counterparts in axisymmetric systems. In agreement with previous approximate analysis [2,3,7] the orbits are found to lie on surfaces similar to but displaced from the magnetic surfaces[1]. The displacement tends to zero with Larmor radius and the surfaces appear closed within the numerical accuracy for at least 40 field periods.

However, just because a real stellarator is not axisymmetric, a further group of particles exists which is much more difficult to contain. These localised particles, reflected in the gradients of the helical field, have incomplete cancellation of the toroidal drift and escape from particular stellarators no matter how strong the field. Furthermore ions and electrons of the same energy will leave with the same drift velocity. These particles can be retained only by making the number of field periods, and hence the aspect ratio, so large that the drifts due to the helical field dominate the toroidal drift.

Thus the stellarator does not form a perfect single particle trap in the sense of our definition. However, the particles which can drift through the separatrix are just those (with large $V_{\perp}/V_{\parallel}$) which will most easily be reflected in the increasing field towards the conductors. Thus the hybrid stellarator-mirror field, which includes the regions inside and outside the separatrix may well form a perfect single particle trap.

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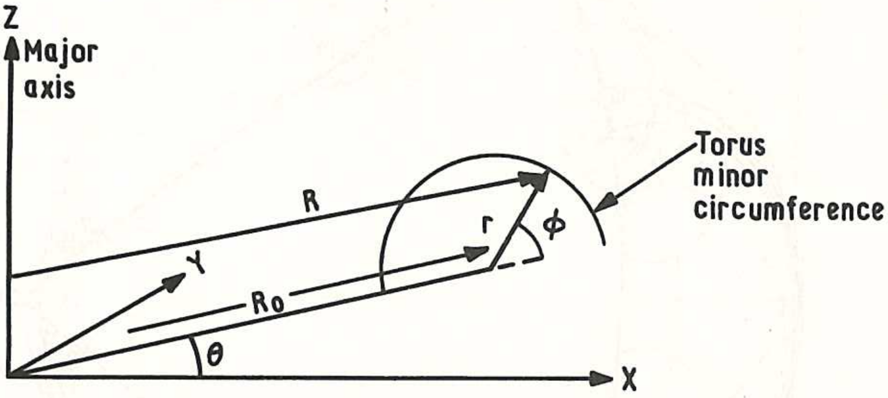

Fig. 1 Co-ordinate systems (CLM-P 137)

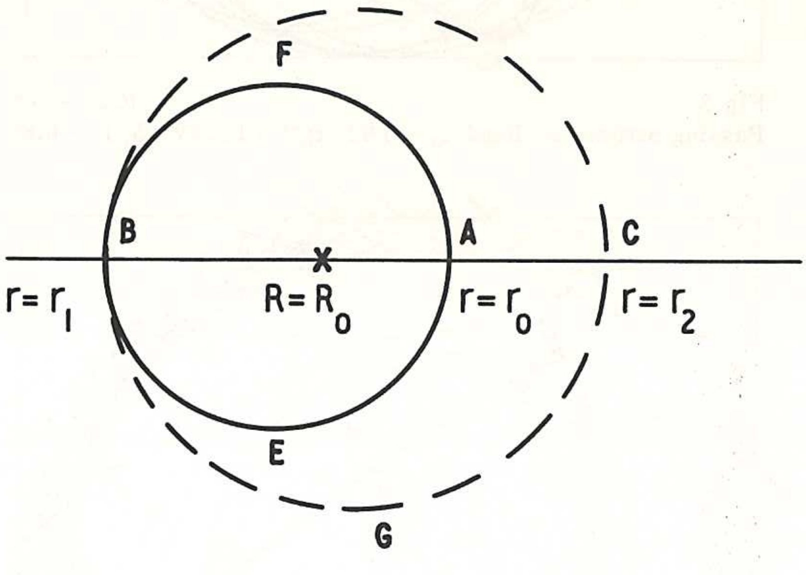

Fig. 2 Idealised limiting passing and blocked orbits (CLM-P 137)

Figs. 3 to 7 show electron orbits in the (R, Z) plane for a filamentary $\ell = 3$ winding having 8 field periods on a toroid of minor and major radii 11.4 and 30 cms and carrying 25 kA; the toroidal magnetic field at the winding axis is 1 kG. The maximum separatrix radius is 4.6 cms about a centre at R = 29.3 cms and is indicated by the circle r = 1. The position of the guiding centre at each field period is indicated by X and these points are joined to give a cross section of the drift surface. The numbers indicate the order of the points and thus show the magnitude of the particle transform

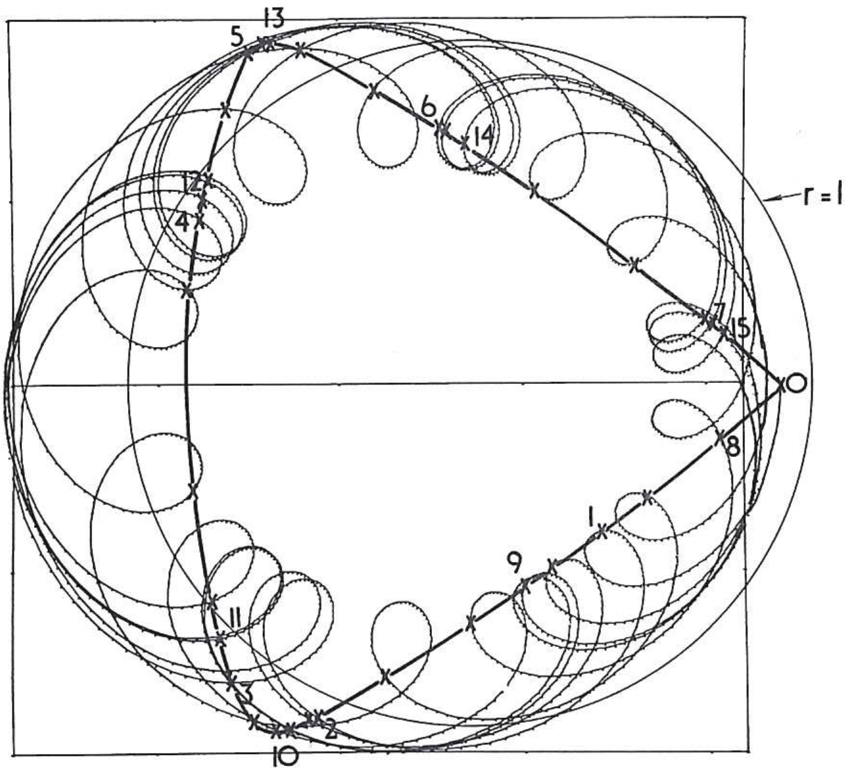

Fig. 3 (CLM-P 137)
Passing particle confined $r_0 = 0.98$, $r_L^{-1} = 16$, $(V_{\parallel}/V_{\perp}) = -0.85$

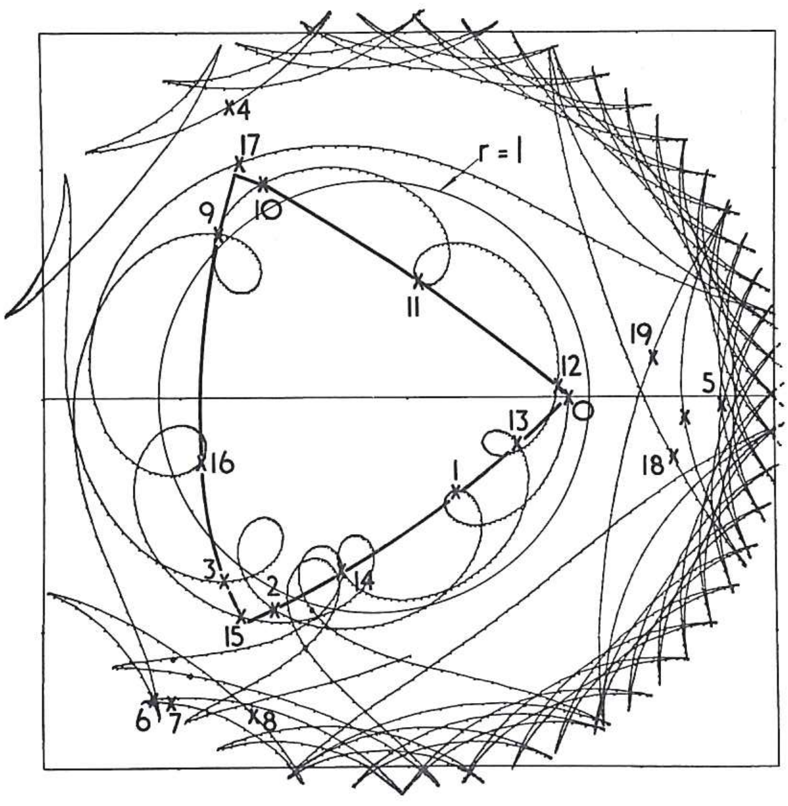

Fig. 4 (CLM-P 137)
Passing particle escapes $r_0 = 0.98$, $r_L^{-1} = 16$, $(V_{\parallel}/V_{\perp})_0 = -0.8$

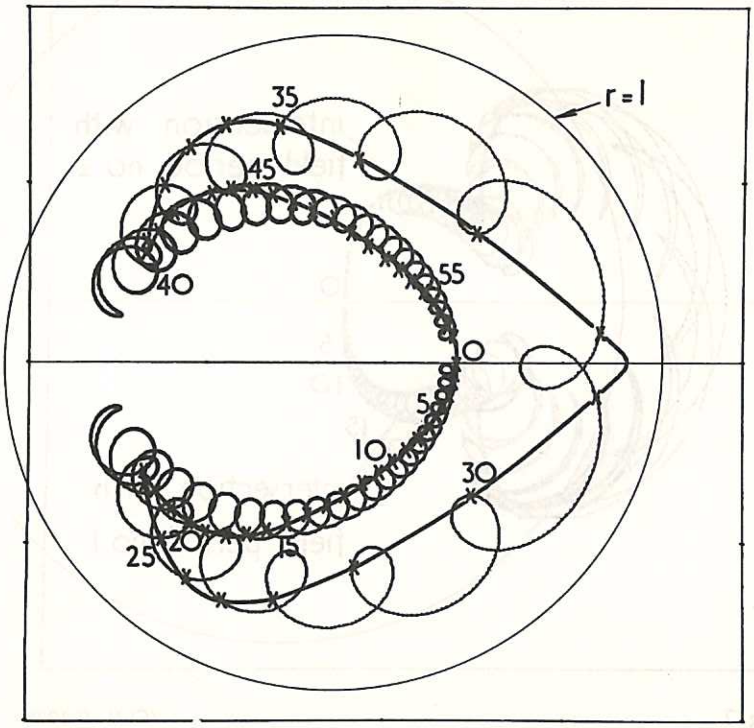

Fig. 5 (CLM-P 137)
Blocked particle confined $r_o = 0.37$, $r_L^{-1} = 80$, $(V_{\parallel}/V_{\perp})_o = -0.44$

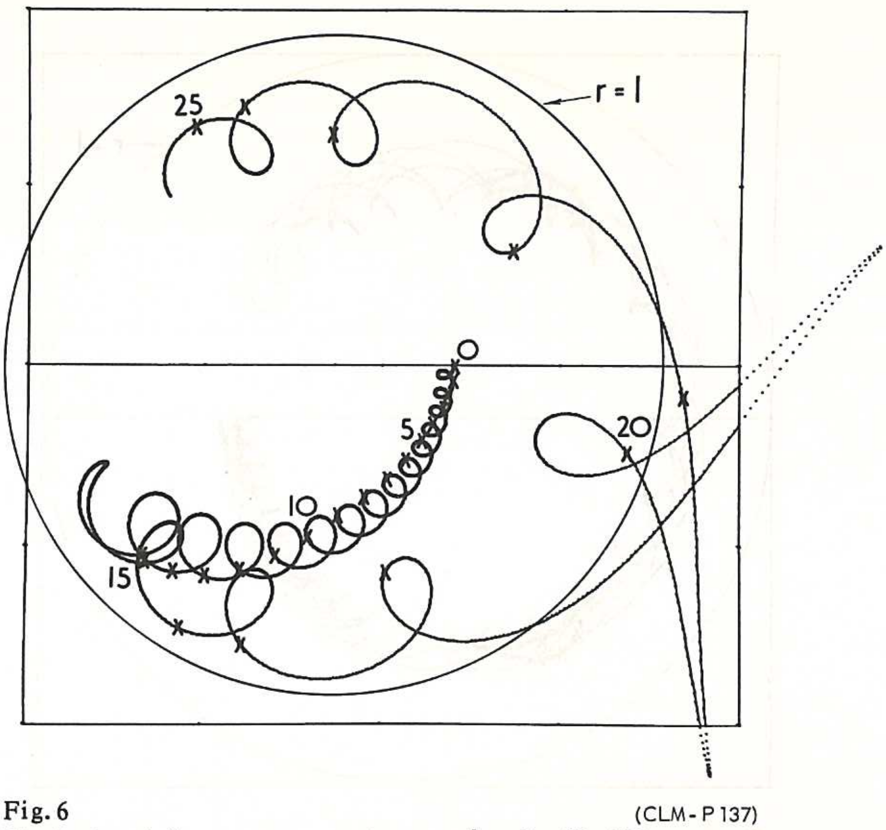

Fig. 6 (CLM-P 137)
Blocked particle escapes $r_o = 0.37$, $r_L^{-1} = 40$, $(V_{\parallel}/V_{\perp})_o = -0.45$

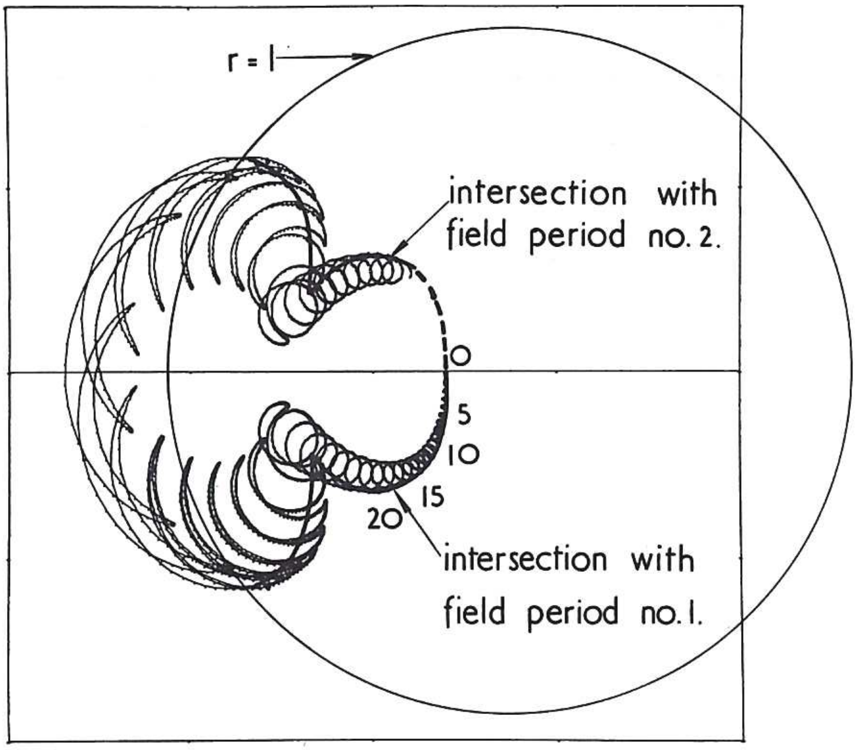

Fig. 7 (CLM - P 137)

Localised particle escapes $r_0 = -0.17$, $r L^{-1} = 60$, $(V_{\|}/V_{\perp})_0 = -0.3$

This particle is initially able to pass freely around the torus, but becomes localised as it moves to small major radius. Other examples show particles which are localised for the complete orbit.

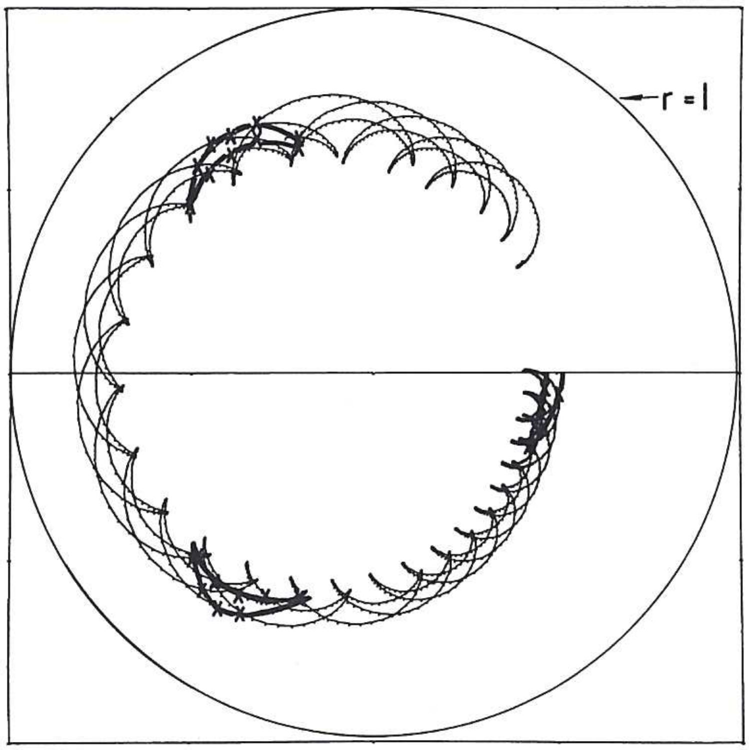

Fig. 8 (CLM - P 137)

Localised particle confined $r_0 = 0.5$, $r L^{-1} = 60$, $(V_{\|}/V_{\perp})_0 = -0.3$

Major radius 240 cms, 64 field periods, other parameters as for Fig. 3.

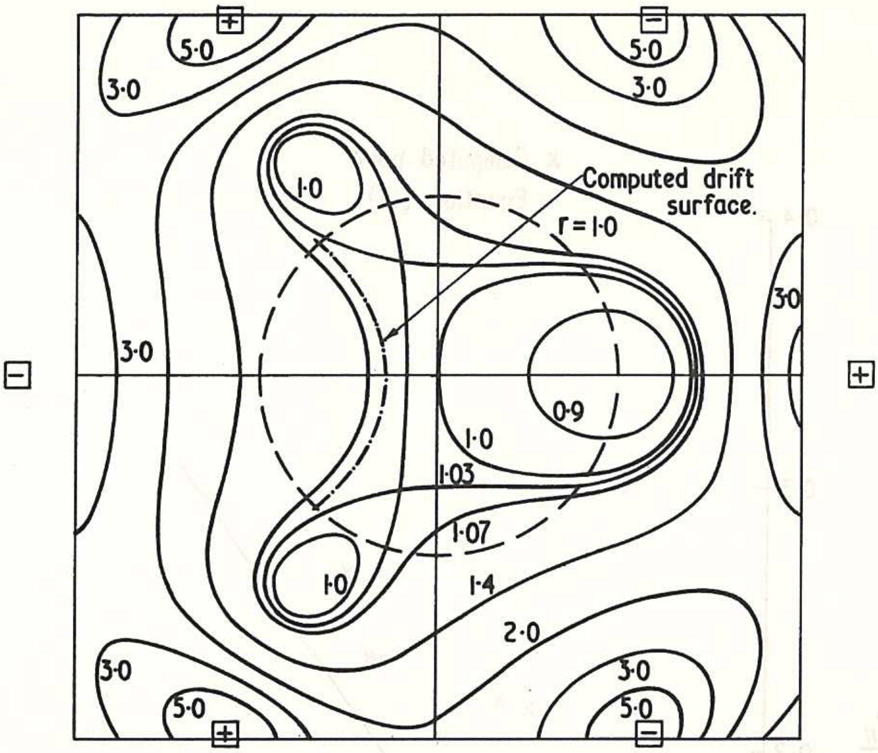

Fig. 9 (CLM-P 137)
Intersection of $|B|$ = constant surfaces with plane $\theta = 0$ parameters as for Fig. 3. The intersection with this plane of a computed drift surface for a localised particle is shown. The numbers indicate the value of $|B|$ in K.G.

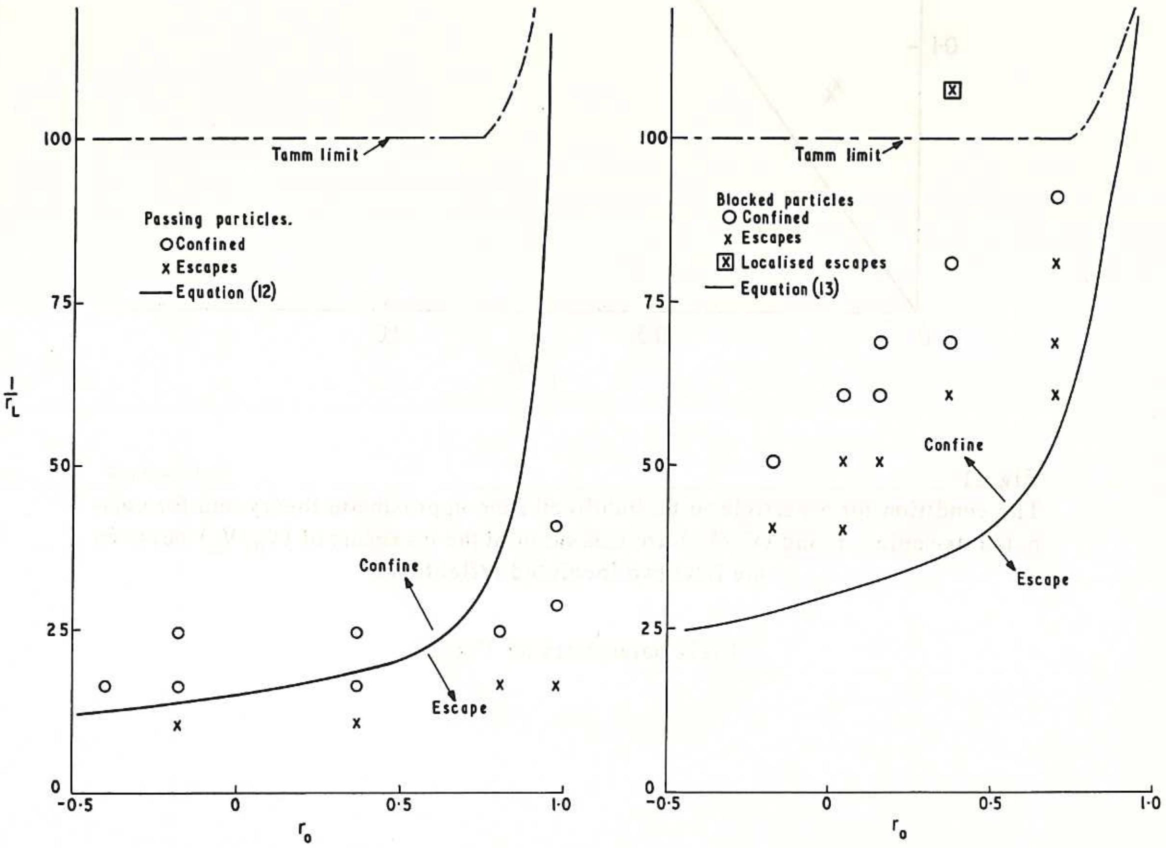

Fig. 10 (CLM-P 137)
The condition for confinement of (a) passing and (b) blocked particles according to approximate theory. Tamm's limiting condition for confinement in axisymmetric systems is shown. Computed results are plotted for just passing, just blocked and localised electrons. Field parameters as Fig. 3.

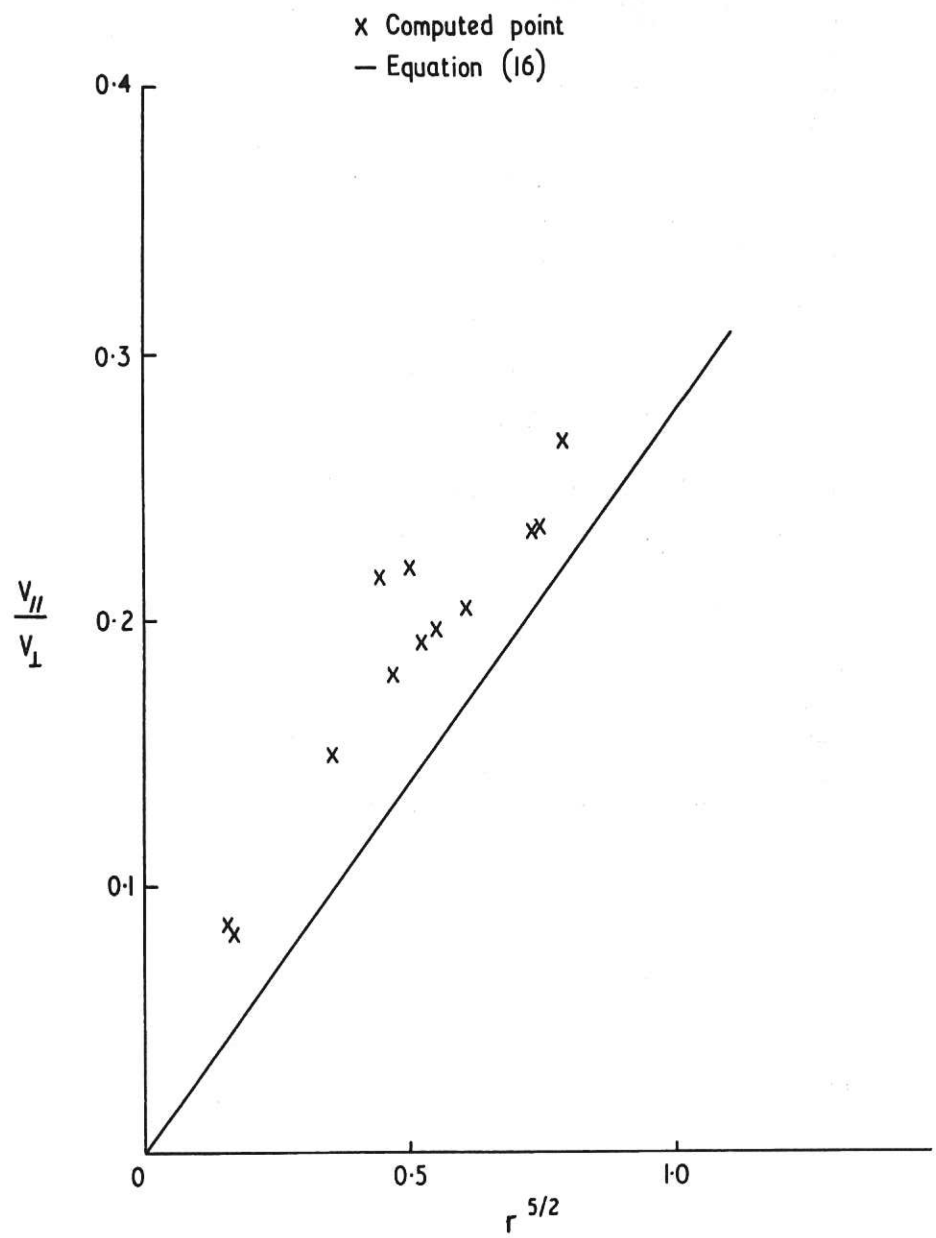

Fig. 11 (CLM - P 137)

The condition for a particle to be localised from approximate theory and for computed examples, r and $(V_{||}/V_{\perp})$ are measured at the maximum of $(V_{||}/V_{\perp})$ between the first two localised reflections

Field parameters as Fig. 3