CLM - P 141

CLM - P 141

United Kingdom Atomic Energy Authority

RESEARCH GROUP

Preprint

THE PROPAGATION OF A GUIDED SLOW ELECTROMAGNETIC WAVE IN A COLD DENSE MAGNETIZED PLASMA SLAB

E. HOTSTON

Culham Laboratory
Abingdon Berkshire

1967

UNCLASSIFIED
(Approved for Publication)

CLM-P 141

THE PROPAGATION OF A GUIDED SLOW ELECTROMAGNETIC WAVE IN A COLD DENSE MAGNETIZED PLASMA SLAB

by

E. HOTSTON

(Submitted for publication in Proc. Phys. Soc.)

ABSTRACT

The propagation of a slow electromagnetic wave in a cold magnetized plasma slab located between two planar slow wave structures is analysed. The slow wave structures are assumed to be equivalent to a surface with an anisotropic impedance. For a collisionless plasma of uniform density it is possible to obtain an exact solution in terms of elementary functions. Under certain circumstances there is a simple relationship between the surface impedance of the slow wave structure and the phase velocity of the wave. The presence of electron-ion collisions which cause attenuation of the wave is treated by an approximate method.

In the presence of density gradients in the plasma numerical methods are used to solve the problem. It is shown that there are no singularities in the solution in the region where the plasma density is the critical density for the frequency of the wave (collisionless plasma). An example of a numerical solution of the problem is given. It is shown that the presence of density gradients introduces an extra constraint into the system.

U.K.A.E.A. Research Group,
Culham Laboratory,
Nr. Abingdon,
Berks.

September, 1967 (MEJ)

CONTENTS

<u>List of Most Commonly Used Symbols</u>

A_b A_s	Amplitude of wave
a	Semi thickness of plasma slab
a_o	Scale factor of density distribution
B_o	Static magnetic field
b	Semi separatron of slow wave structure
b (as subscript)	Bulk wave
c_1	Coordinate of plasma of critical density
$\underline{D}$ (D_x, D_y, D_z)	Electric displacement
$\underline{E}$ (E_x, E_y, E_z)	Electric field
$E = \varepsilon_o E_y$	
F_p	Constant of integration
f	Frequency of wave
f_c	Electron gyro frequency
H (H_x, H_y, H_z)	Magnetic field
K	Dielectric constant
k	Wave number in z direction
k_o	Free space wave number
k_1	Wave number for x direction
L	Power e folding length
M_p	Constant of integration
$n = \pm 1$	
n_e	Electron density of plasma
n_c	Critical plasma density for frequency f
p	Any integer
$R = k/k_o$	Retardation of wave
r	Subscript denoting b or s
s	Subscript denoting surface wave

T_e	Electron temperature
$U = 1 - j\,\nu/\omega$	
$X = n_e/n_c$	Normalized electron density
x	Cartesian coordinate
$Y = f_c/f$	Normalised electron gyro frequency
y	Cartesian coordinate
$Z = \nu/\omega$	
Z_o	Impedance of free space
Z_y, Z_z, Z'_z	Normalized surface impedances
z	Cartesian coordinate
α	Density gradient
$\gamma = (k^2 - k_o^2 K)^{1/2}$	Propagation constant
$\gamma_1 = (k^2 - k^2)^{1/2}$	Propagation constant
$\gamma_2 = [k^2(1 - Y^{-2} - Y^{-2}(R^2-1)^{-1}) - k_o^2]^{1/2}$	
$\eta = (\gamma_2^2 a)^{1/3}\, x_1$	Dimensionless distance coordinate
ε_o	Permittivity of free space
μ_b, μ_s	Refractive indices of plasma
ν	Electron ion collision frequency
ρ	Wave polarisation
φ	Angle of propagation of elementary wave
Ω	Skew angle of slow wave structure
$\omega = 2\pi f$	

1. INTRODUCTION

This paper considers the propagation of an electromagnetic wave in a cold dense magnetized plasma located between two planar slow wave structures which serve to define the phase velocity of the wave. The attenuation of the wave arises from electron-ion collisions in the plasma. Under favourable circumstances the characteristic attenuation length can be small, a few tens of free space wavelengths. It has been suggested (Wort, Weaver, Hotston 1967) that the high attenuation of such a wave could form the basis for heating the plasma electrons.

It has been found that when the result of the analysis can be expressed in terms of elementary functions, the expressions are often so cumbersome that the properties of the solution are only realised when specific numerical examples are considered. The values of the plasma parameters chosen for numerical illustration was considered typical of a small scale experiment designed to test the method of plasma heating. The following symbols are defined:

f sec^{-1},	is the frequency of the wave (~ 3000 Mc/s)
$n_c \sim 1{\cdot}2\ 10^{-7} f^2 \text{cm}^{-3}$,	is the critical electron density for frequency f
f_c sec^{-1},	is the electron gyro frequency in the static magnetic field impressed on the plasma.
n_e cm^{-3},	is the electron density of the plasma.
ν sec^{-1},	is the electron ion collision frequency (fully ionized plasma).
$X = n_e/n_c$,	is the normalized plasma density.
$Y = f_c/f$,	is the normalized electron gyro frequency.
$Z = \nu/\omega$	

It will be assumed for the purpose of numerical illustration that $X \sim 10$, $Y \sim 3$, $Z \sim 10^{-5}$. These numerical values correspond to a fully ionized plasma of density $\sim 10^{12}$ cm^{-3}, with an electron temperature of 40 eV, (Heald and Wharton 1965), immersed in a static magnetic field of approximately 3000 G. The transverse dimensions of the plasma are assumed to be of the order 10 cm. The phase velocity of the wave is assumed to be R^{-1} times the velocity of light where $R \sim 10$. Although a slab model has been used here any experimental apparatus would use cylindrical geometry with approximately circular symmetry. The solutions of the slab model have been restricted to those which can be carried over into cylindrical geometry with circular symmetry.

If the problem is analysed in cylindrical coordinates, the arguments of the Bessel functions which appear in the solution are so large over most of their range that they can be replaced by their approximations in terms of trignometric functions. Because of this the solutions obtained for the slab model have quantitative as well as qualitative significance for the cylindrical case.

Problems of excitation of the wave are not considered.

2. FORMULATION OF THE DIFFERENTIAL EQUATIONS

Fig.1 shows the geometry of the system, the static magnetic field B_0 is parallel to the z axis and the slow wave structures are located at $x = \pm b$. The wave is assumed to depend on time t and the z coordinate as the real part of $\exp j(\omega t - kz)$, and to have no dependence on the y coordinate. The plasma density is independent of z. Using the dielectric tensor of a plasma and Maxwell's equations (Budden 1961 A) the following pair of differential equations can be deduced.

$$k_o^2 \left[\frac{-nYX}{U(U - X) - Y^2} \right] H = \left[k^2 - k_o^2 \frac{(U - X)^2 - Y^2}{U(U - X) - Y^2} - \frac{\partial^2}{\partial x^2} \right] E \quad \ldots (2.1)$$

$$k^2 \left[\frac{nYX}{U(U - X) - Y^2} \right] E = \left[k_o^2 - k^2 \frac{U^2 - Y^2}{U(U - X) - Y^2} + \frac{\partial}{\partial x}\left(\left(\frac{U}{U - X} \right) \frac{\partial}{\partial x} \right) \right] H \quad \ldots (2.2)$$

The variables E,H are related to the fields $\underline{E},\underline{H},\underline{D}$ of the wave by

$$E = \varepsilon_o E_y \quad , \quad H = j D_x = j \frac{k}{\omega} H_y$$

The other symbols are defined as follows: ε_o is the permittivity of free space, $U = 1 - jZ$, and $n = +1$ if B_o is anti parallel to the z axis and $n = -1$ if it is parallel: $k_o = k R^{-1}$ is the free space wavenumber.

In sections 3-7 wave propagation in a uniform plasma is considered, the attenuation being obtained by an approximate method. In section 8 approximate solutions of equations 2.1, 2.2 are given for the collisionless plasma $(U = 1)$. These solutions are valid in the region $X = 1$ and have no singularities. Also in section 8 an example is given of the numerical integration of equations 2.1, 2.2 for a plasma with a density gradient.

3. THE PLASMA OF UNIFORM DENSITY

The plasma is assumed to fill the region $a \geqslant x \geqslant - a$. There is no power flow parallel to the x axis. The required solution of equations 2.1, 2.2 is one in which E_y, H_y are both odd functions of x, so that they correspond to the solutions in cylindrical coordinates which have circular symmetry.

where $$E_{yr} = A_r \,\mathrm{Sin}\, k_{1r}x \quad \exp j(\omega t-kz)$$

$$\rho_r = \frac{n}{Y}\,\frac{\mu_r}{R}\left[U + \frac{X}{\mu_r^2 - 1}\right]$$

Z_o is the wave impedance of free space and r can take the meanings $r = b$, $r = s$.

TABLE I

Values of μ calculated from equation 3.3 for a collisionless plasma

Y = 1·5								
R	2·0		5·0		10·0		26·0	
X	μ_b	μ_s	μ_b	μ_s	μ_b	μ_s	μ_b	μ_s
0·033	1·11	0·998	1·57	1·00	02·61	1·00	06·36	1·00
0·533	1·82	0·984	4·18	0·998	08·24	1·00	21·3	1·00
1·2	2·02	1·02	5·22	1·00	10·5	1·00	27·3	1·00
2·1	1·76	1·31	5·79	1·02	11·8	1·01	30·9	1·00
5·6			6·08	1·26	13·3	1·06	35·2	1·01
20.8					13·1	1·81	37·4	1·13

Y = 4·5								
R	2·0		5·0		10·0		26·0	
X	μ_b	μ_s	μ_b	μ_s	μ_b	μ_s	μ_b	μ_s
0·033	1·05	1·00	1·36	1·00	02·11	1·00	04·96	1·00
0·533	1·63	0·998	3·76	1·00	07·42	1·00	19·2	1·00
1·2	2·13	1·00	5·43	1·00	10·9	1·00	28·3	1·00
2·1	2·61	1·02	7·02	1·03	12·6	1·00	36·9	1·00
5·6	3·48	1·24	10·4	1·11	21·3	1·01	55·6	1·00
20·8			15·2	1·39	32·0	1·10	84·2	1·02

Y = 9·5								
R	2·0		5·0		10·0		26·0	
μ_X	μ_b	μ_s	μ_b	μ_s	μ_b	μ_s	μ_b	μ_s
0·033	1·05	1·00	1·35	1·00	2·08	1·00	4·87	1·00
0·533	1·62	1·00	1·72	1·00	7·35	1·00	23·8	1·00
1·2	2·14	1·00	5·45	1·00	10·9	1·00	28·4	1·00
2·1	2·69	1·00	7·18	1·00	14·5	1·00	37·7	1·00
5·6	4·05	1·05	11·4	1·01	23·0	1·00	60·1	1·00
20·8	6·55	1·69	20·1	1·09	41·0	1·02	107·0	1·00

TABLE II

Limiting values of various parameters
for a collisionless plasma for $X \sim 1$

Parameter	b wave	s wave
μ^2	$X R^2 + (1 - X)$	1
$\dfrac{X}{\mu^2 - 1}$	$\dfrac{1}{R^2 - 1}$	$\dfrac{Y^2(R^2 - 1)}{X - 1} - 1$
$n\rho$	$\dfrac{1}{Y}\;\dfrac{R^2 X^{1/2}}{R^2 - 1}$	$\dfrac{Y}{R}\;\dfrac{R^2 - 1}{X - 1}$
$\dfrac{n\mu}{\rho(X - 1)}$	$\dfrac{Y(R^2 - 1)}{(X - 1)R}$	$\dfrac{R}{Y(R^2 - 1)}$
$\dfrac{\mu^2 \mathrm{Sin}^2\varphi}{1 - X}$	$1 - R^2$	$\dfrac{1 - R^2}{1 - X}$
$-\dfrac{n\mu^2 \mathrm{Sin}^2\varphi}{\rho(1 - X)}$	$\dfrac{Y(R^2 - 1)^{3/2}}{R(X - 1)^{1/2}}$	$-\dfrac{R}{Y}\;\dfrac{1}{(1 - R^2)^{1/2}}$

It is necessary to consider the amplitudes of the field components given by equations 3.4 in the region where $X \to 1$. These values calculated from equation 3.3 are given in the Table II for a collisionless plasma. From this table it is seen that the amplitude of the transverse fields is never infinite, so that a wave of finite energy density can propagate in the plasma as $X \to 1$. Of the axial fields only E_z for the b wave ($\propto (1-X)^{-1/2}$) tends to infinity as $X \to 1$; this arises from the neglect of collisions. However D_z for the b wave is proportional to $(1-X)^{1/2}$ so the energy associated with this singularity is finite.

4. THE SOLUTION OUTSIDE THE PLASMA SLAB

The region between the plasma and the slow wave structure is assumed to be filled with a lossless dielectric of dielectric constant K. For reasons of symmetry only the region $x > a$ need be considered, and the solutions of Maxwell's equations in this region are,

$$E_y = F_1 e^{-\gamma x} + F_2 e^{\gamma x} \;, \quad H_y = F_3 e^{-\gamma x} + F_4 e^{\gamma x} \;, \quad (b \geqslant x \geqslant a) \quad \ldots (4.1)$$

where the F's are constants of integration and $\gamma^2 = k^2 - k_o^2 K > 0$. The other field components can be found from equation 4.1 by use of Maxwell's equations. If the solutions given by equation 4.1 are matched at the plasma boundary $x = a$ to those found in section 3 it is found that outside the plasma

$$E_y = \sum_r A_r \left[\sin k_{1r}a \;\; \cosh \gamma(x-a) + \frac{k_{1r}}{\gamma} \cos k_{1r}a \;\; \sinh \gamma(x-a) \right]$$

$$H_y = \sum_r \frac{-j}{Z_o} \frac{\mu_r A_r}{\rho_r} \left[\frac{k_{1r}}{\gamma} K \frac{\cos k_{1r}a}{1 - X} \sinh \gamma(x-a) + \sin k_{1r}a \;\; \cosh \gamma(x-a) \right]$$

$$\ldots (4.2)$$

5. THE SLOW WAVE STRUCTURES

Two slow wave structures which match the wave are considered, differing in the boundary conditions they impose at the plane $x = b$. The first structure is similar to that of a particle accelerator, consisting of a series of grooves parallel to the axis cut in the surface of a perfect conductor at $x = b$, the spacing of the grooves being small compared with a wavelength. The boundary conditions at $x = b$ are then

$$E_y = 0 \;, \quad \frac{E_z}{H_y} = -j \frac{Z_o}{k_o K} \frac{1}{H_y} \frac{\partial H_y}{\partial x} = -jZ_o Z_z \quad \ldots (5.1)$$

where Z_z, the normalized surface impedance of the slow wave structure (Slater 1950), is a function of the geometry of the surface. It will be shown later that to match the wave Z_z has to be positive, so this type of surface is called inductive.

The second type of structure is the complement of the first and imposes at $x = b$ the conditions:

$$E_z = 0 \quad , \quad \frac{E_y}{H_z} = - j\, k_o Z_o\, E_y \left(\frac{\partial E_y}{\partial x}\right)^{-1} = - j\, Z_o Z_y \qquad \text{... (5.2)}$$

$- Z_y$ is the normalized surface impedance and for this structure has to be negative so that the structure is capacitative. A structure of this type could be made by cutting the grooves parallel to the z axis and filling them with a material of dielectric constant greater than $R^{\frac{1}{2}}$.

The condition that $E_y = 0$ or $E_z = 0$ at $x = b$ gives the ratio A_s/A_b, from equations 4.2 if $E_y = 0$ at $x = b$, for a collisionless plasma

$$\frac{A_s}{A_b} = \frac{-\left(\text{Sin}\, k_{1b}a \;\; \text{Cosh}\, \Upsilon(x-a) + \frac{k_{1b}}{\Upsilon} \;\; \text{Cos}\, k_{1b}a \;\; \text{Sinh}\, \Upsilon(b-a)\right)}{\text{Sin}\, k_{1s}a \;\; \text{Cosh}\, \Upsilon(b-a) + \frac{k_{1s}}{\Upsilon} \;\; \text{Cos}\, k_{1s}a \;\; \text{Sinh}\, \Upsilon(b-a)} \qquad \text{... (5.3)}$$

and if $E_z = 0$ at $x = b$

$$\frac{A_s}{A_b} = - \frac{\frac{\mu_b}{\rho_b}\left[\text{Sin}\, k_{1b}a - \frac{K}{X-1}\,\frac{k_{1b}}{\Upsilon}\,\text{Cos}\, k_{1b}a \;\; \tanh \Upsilon(b-a)\right]}{\frac{\mu_s}{\rho_s}\left[\text{Sin}\, k_{1s}a - \frac{K}{X-1}\,\frac{k_{1s}}{\Upsilon}\,\text{Cos}\, k_{1s}a \;\; \tanh \Upsilon(b-a)\right]} \qquad \text{... (5.4)}$$

The surface impedance of the slow wave structure required to obtain any value of R can be obtained by substituting from equations 4.2 into equation 5.1 or 5.2, but the resulting expressions are cumbersome and are not given. In an experimental arrangement it is reasonable to assume that there will be a finite separation (b-a) between the plasma and slow wave structure, for the parameters considered typical of an experiment it will only require (b-a) to be greater than two or three millimetres for $\tanh \Upsilon(b-a) \sim 1$. With this condition the expressions of Z_z, Z_y take a simple form,

$$Z_z = \frac{\Upsilon}{Kk_o} \quad , \quad Z_y = \frac{k_o}{\Upsilon}$$

so that the phase velocity of the wave is governed directly by the impedance of the slow wave structure.

6. THE ATTENUATION OF THE WAVE

The attenuation due to the electron ion collisions is now calculated. The spatial dependence of the waves is assumed to be the same as in the collisionless case except that the z dependence becomes proportional to

$$e^{-z/2L} \exp j(\omega t-kz) \qquad \ldots (6.1)$$

where L is the attenuation length for the power flow. If the plasma is in contact with the slow wave structure and only the bulk wave propagates then

$$L^{-1} = 2\, k_o\, I_m(\mu_b)/\text{Cos}\ \varphi_b \qquad \ldots (6.2)$$

where $I_m(\mu_b)$ is the imaginary part of μ_b and is calculated from equation (3.3) or the Appleton-Hartree equation (Radcliffe 1959 A).

In general L is found by calculating the power flow in the z direction across the planes $z, z + \Delta z$ and equating the difference to the power loss by collisions between the planes. Following Ratcliffe (1959 C) the power loss in the plasma is written in the form

$$\int_o^a \nu \left[B_1^2(E_x^2 + E_y^2) + B_2\, E_z^2 \right] dx \qquad \ldots (6.3)$$

where B_1, B_2 are independent of x. For the first slow wave structure ($E_y = 0$, $x = b$) numerical calculation shows the power loss is almost all due to the bulk wave (> 99·8%), over the following range of parameters $2{\cdot}5 \leqslant Y \leqslant 3{\cdot}5$; $7 \leqslant X \leqslant 13$; $7 \leqslant R \leqslant 15$. For the capacitative slow wave structure the bulk wave is again the main cause of the power loss. Assuming the attenuation is due solely to the bulk wave the attenuation length becomes

$$L^{-1} = \frac{2k_o I_m(\mu_b)}{\cos \varphi_b} \; \frac{\int_o^a (\underline{E}_b \times \underline{H}_b^*)\, dx}{\int_o^b (\underline{E} \times \underline{H}^*)\, dx} \qquad \ldots (6.4)$$

This form is more convenient for numerical computation than the direct use of equation 6.3.

7. NUMERICAL VALUES OF THE ATTENUATION LENGTH

The attenuation length L has been calculated for a plasma of electron temperature of 40 eV. Fig.2 shows values of L for the first (inductive) type of slow wave structure for values of R of 7,10,15. The attenuation length L is seen to be strongly dependent on R and on the separation of the plasma from the slow wave structure (b-a). Figs.3,4 show the dependence of L upon X,Y with R = 10.

For the curves of Fig.2 the presence of the surface wave accounts for less than 1% of the power flow in the plasma and the fluctuation of L with variation of b-a is due to changes in the power flow outside the plasma relative to the power flow inside the plasma. The amplitude of the surface wave at the edge of the plasma $A_s \sin k_{1s} a$ is of order A_b or less.

Repeating the calculations for the second capacitative type of slow wave structure it is found that $A_s \sin k_{1s}a$ can be of order $200A_b$, so that the variation of L with b-a is much greater.

In all cases L scales with electron temperature T_e as $T_e^{3/2}$.

8. THE PLASMA OF VARYING DENSITY

The general wave equations 2.1, 2.2 are now solved for a collisionless plasma of varying density. The density distribution of the slab is assumed symmetrical about the plane $x = 0$ and to decrease monotonically from its maximum at $x = 0$ to zero at $x = \pm b$ (Fig.5).

To solve the equations the origin is shifted to a surface where $X = 1$ so that $x = x_1 + c$ (Fig.5), and we assume that $X = 1 = - \alpha X$ in the region where $X_1 \sim 0$, where α is the density gradient. Introduce new parameters $\Upsilon_1, \Upsilon_2, \eta$ where

$$\Upsilon_1^2 = k^2 - k_o^2$$

$$\Upsilon_2^2 = k^2 \left[1 - Y^{-2} - Y^{-2}(R^2-1)^{-1} \right] - k_o^2 \qquad \ldots (8.1)$$

$$\eta = (\Upsilon_2^2 \alpha)^{1/3} x_1$$

On substituting these quantities into equations 2.1, 2.2 and putting $X \sim 1$ the following pair of equations is obtained,

$$\frac{nk_o^2}{Y} H = \left[\Upsilon_1^2 - (\Upsilon_2^2 \alpha)^{2/3} \frac{\partial^2}{\partial \eta^2} \right] E \qquad \ldots (8.2)$$

$$\frac{nk^2}{Y} E = \left[\Upsilon_2^2 + \frac{k^2}{Y^2(R^2-1)} - \Upsilon_2^2 \frac{\partial}{\partial \eta} \left(\frac{1}{\eta} \frac{\partial}{\partial \eta} \right) \right] H \qquad \ldots (8.3)$$

No approximations have been made in obtaining the last terms on the right hand sides of equations 8.2, 8.3 so they should be valid for a plasma with a small density gradient $(\alpha \sim 0)$.

A solution of equations 8.2, 8.3 can be obtained in the form of a power series in η which contains four constants of integration M_1, M_2, M_3, M_4 (see Appendix). The first terms of the series are

$$H = M_1 + M_2\eta^2 + M_3\eta^3 + M_4\eta^4 + O(\eta^5) \qquad \ldots (8.4)$$

$$E = \frac{1}{b_2}\left[M_1\left(a_2^2 + O(\eta^2)\right) - M_2\left(\frac{b_1 b_2}{12}\eta^4 + -\right) - M_3\left(3 + O(\eta^2)\right) - M_4\left(4\eta + O(\eta^3)\right)\right] \qquad \ldots (8.5)$$

where

$$a_1^2 = \frac{\Upsilon_1^2}{(\Upsilon_2^2\alpha)^{2/3}}, \qquad b_1 = \frac{nk_o^2}{Y}\frac{1}{(\Upsilon_2^2\alpha)^{2/3}}$$

$$a_2^2 = 1 + \frac{k^2}{Y^2\Upsilon_2^2(R^2-1)}, \qquad b_2 = \frac{nk^2}{Y\Upsilon_2^2}$$

The four constants of integration correspond to the four waves found to exist in the plasma of uniform density. In that example the restriction that there was no energy transfer parallel to the x axis caused the four waves to reduce to two resultant waves referred to as the s,b waves. The same pairing of the solutions happens when density gradients are present so that the four constants of integration are not independent. To effect this pairing it is assumed that E/H for the resultant waves is independent of the density gradient. Equations 3.4 show that for the uniform plasma $E/H = (n/YR^2)(1 + X/(\mu^2-1))$, and using the results of Table II it is found that near $\eta = 0$ for the s wave $E/H = -nY(R^2-1)(\Upsilon_2^2\alpha)^{1/3}(R^2\alpha\,\eta)^{-1}$ and for the b wave $E/H = n/(R^2-1)Y$. Examination of equations 8.4, 8.5 shows that the s wave must be formed from a combination of the second and fourth terms and the b wave corresponds to the first and third terms, so that

$$H = M_2\,\eta^2 + O(\eta^4)\ ;\ E = -nY\frac{(R^2-1)}{R^2}\left(\frac{\Upsilon_2}{\alpha}\right)^{2/3} M_2\eta + O(\eta^3)$$

for the s wave and

$$H = M_1 \, [1 + \eta^3/3 + O(\eta^5)] \quad ; \quad E = \frac{n\,M_1}{(R^2-1)Y} \, [1 + O(\eta^2)] \quad \ldots \; (8.6)$$

for the b wave.

The solution for the b wave in a plasma of arbitrary density profile is obtained by numerically integrating equations 2.1, 2.2 using equations 8.6 as starting values. This has been done assuming a density profile of

$$X = X_o \, (1 - x^2 \, a_o^{-2}) \qquad x < a_o$$

$$X = 0 \qquad x > a_o$$

where X, a_o are constants. The problem is an eigenvalue problem because for a given X, a_o the solution only exists for those values of R which give $\partial H/\partial x = 0$ at surfaces where $X = 1$. This introduces a constraint into the system absent in the case of the uniform slab.

To obtain an illustrative example it is easiest to assume values of X_o, R and to integrate equations 2.1, 2.2 for a series of values of a_o until a solution is obtained which gives E, H as odd functions of x. This has been done assuming $X = 10$, $Y = 3$, $R = 10$, $k_o = \pi/5 \ cm^{-1}$, and Fig.6 shows the result obtained for $a_o = 4 \cdot 9$ cm, which is an approximation for one of the eigenvalues. The solution has nodes of field in the region where $X > 1$ which is characteristic of the b wave. The assumption made in obtaining the starting values appear to be justified by the agreement between the local refractive index and that calculated from equation 3.3. As a random example successive zeros of H occur at $x = 4 \cdot 26$ cm and $3 \cdot 89$ cm giving a local wavelength of $0 \cdot 84$ cm, the mean density X being $3 \cdot 15$.

Equation 3.3 predicts $\mu \sin\varphi = 12{\cdot}33$ at $X = 3{\cdot}15$ corresponding to a wavelength of 0·811 cm which is good agreement.

The slow wave structures required to match the wave are assumed to be situated at $x = \pm a_o$, and to consist of grooves cut in the surface of a perfect conductor at angle Ω to the y axis (Fig.7). Axes O_1y^1, O_1z^1 are then taken in the plane $x = a_o$, such that O_1y^1 is parallel to the grooves in the plane $x = a_o$ and O_1z^1 is perpendicular to them. Equations 8.6 show that E,H are in phase, so that the resultant electric field is inclined to the z axis. This direction is chosen as the direction of the O_1z^1 axis, so that

$$\tan\Omega = Ek/(\partial H/dx) \quad \text{at} \quad x = 0$$

For the example given $\tan\Omega = 4{\cdot}875\ 10^{-3}$ $(\Omega \sim 0{\cdot}29^o)$. The normalized surface impedance of the slow wave structure required to match the wave is Z_z^1 where

$$Z_z^1 = \frac{j}{Z_o} \frac{E_z \cos\Omega - E_y \sin\Omega}{H_y \cos\Omega + H_z \sin\Omega}$$

For the example given $Z_z^1 = 6{\cdot}44$ so the structure is inductive. For the treatment to be valid it is necessary that a standing wave can be formed along the depth of the slot: this requires $R\,(\sin\Omega) \ll 1$ which is true in this case.

It is concluded that the b wave can exist by itself in a plasma of variable density if the slow wave structure is chosen correctly as regards surface impedance and orientation of the grooves. To treat propagation in a plasma of arbitrary density profile surrounded by an arbitrary slow wave structure it would appear necessary to use b,s waves with a range of phase velocities (R values). In the case of a uniform plasma only one value of R is required.

9. CONCLUSIONS

The equations governing the propagation of a slow electromagnetic wave in a plasma slab have been formulated. The case of the uniform plasma has been treated extensively and it has been shown that the phase velocity of the wave is determined by the geometry of the system and the surface impedance of the slow wave structure. For separations of plasma from the structure which are likely to be found under experimental conditions the phase velocity is directly proportional to $(Z_z^2 + 1)^{1/2}$ (inductive structure). The damping of the waves due to electron ion collisions has been considered and its variation with the plasma parameters determined. When density gradients are present in the plasma, the solution of the problem becomes more complicated but the wave responsible for heating the plasma can still propagate. The wave equations have been solved near the region of critical plasma density.

In view of the need for a skewed slow wave structure if a pure b wave is to be propagated in the plasma of variable density it might be a fruitful subject of investigation to solve the problem for a sheared or helical magnetic field.

ACKNOWLEDGEMENTS

The author would like to thank Dr. R.S. Pease, Dr. P. Davenport, Dr. G. Francis, Mr J. Weaver and Mr D.J.H. Wort for helpful discussions and criticism.

REFERENCES

BUDDEN, K. 1961 . Radio waves in the ionosphere. London, Cambridge University Press. Section 3·1 to 3·4. pp.24-27

BUDDEN, K. 1961 . Ibid. Section 6·10. p.65

BUDDEN, K. 1961 . Ibid. Section 8·17. p.120

HEALD, M. and WHARTON, C. 1965. Plasma diagnostics with microwaves. New York, Wiley. Section 2·5·3. p.82

RATCLIFFE, J. 1959 . The magneto-ionic theory and its application to the ionosphere. London, C.U.P. Chapter 6, section 1. p.53

RATCLIFFE, J. 1959 . Ibid. Chapter 7, section 3. p.69

RATCLIFFE, J. 1959 . Ibid. Chapter 5. p.40

SLATER, J. 1950. Microwave electronics. Princeton, Van Nostrand. Section 13·2.

WORT, D., WEAVER, J. and HOTSTON, E. 1967. UKAEA Culham Report CLM - R 78.

APPENDIX

The solution of equations 8.2, 8.3

Equations 8.2, 8.3 can be written in the form

$$\left(a_1^2 - \frac{\partial^2}{\partial\eta^2}\right) E = b_1 H \quad , \quad \left(a_2^2 - \frac{\partial}{\partial\eta}\left(\frac{1}{\eta}\frac{\partial}{\partial\eta}\right)\right) H = b_2 E$$

where the notation is defined in section 8.

If E is eliminated from these equations the following equation is obtained

$$\frac{\partial^3}{\partial\eta^3}\left(\frac{1}{\eta}\frac{\partial H}{\partial\eta}\right) - a_1^2 \frac{\partial}{\partial\eta}\left(\frac{1}{\eta}\frac{\partial H}{\partial\eta}\right) - a_2^2 \frac{\partial^2 H}{\partial\eta^2} + \left(a_1^2 a_2^2 - b_1 b_2\right) H = 0$$

The solution of this equation can be found in the usual way by assuming a power series solution of the form $H = \eta^\beta \sum_p C_p \eta^p$. The solution is found to be:

$$H = M_1 \left[1 - \frac{a_1^2 a_2^2 - b_1 b_2}{30}\eta^5 + \frac{a_1^2 a_2^2 - b_1 b_2}{840}\eta^7 + \ldots\right]$$

$$+ M_2 \eta^2 \left[1 + \frac{a_2^2}{15}\eta^3 + \frac{b_1 b_2}{420}\eta^5 + \ldots\right]$$

$$+ M_3 \eta^3 \left[1 + \frac{a_1^2 \eta^2}{10} + \frac{a_2^2}{240}\eta^3 + \frac{a_1^4}{280}\eta^4 + \frac{2a_1^2 a_2^2 + b_1 b_2}{960}\eta^5 + \frac{1}{1890}\left(\frac{a_1^6}{8} - \frac{5}{4}a_2^4\right)\eta^6 + \ldots\right]$$

$$+ M_4 \eta^4 \left[1 + \frac{a_1^2}{18}\eta^2 + \frac{a_2^2}{35}\eta^3 + \frac{a_1^4}{720}\eta^4 + \frac{5/3\, a_1^2 a_2^2 - b_1 b_2}{1890}\eta^5 + \frac{a_1^6}{42000}\eta^6 + \ldots\right]$$

Substitution into the original equation

$$E = \frac{1}{b_2}\left(a_2^2 - \frac{\partial}{\partial\eta}\left(\frac{1}{\eta}\frac{\partial}{\partial\eta}\right)\right) H$$

shows that

$$E = \frac{1}{b_2}\Bigg[M_1\left(a_2^2 + \frac{a_1^2a_2^2 - b_1b_2}{2}\eta^2 - \frac{a_1^2a_2^2 - b_1b_2}{24}a_1^2\eta^4 + \dots\right)$$

$$- M_2\left(\frac{b_1b_2}{12}\eta^4 + \dots\right)$$

$$- M_3\left(3 + 3/2\ a_1^2\eta^2 - \frac{9}{10}a_2^2\eta^3 + \dots\right)$$

$$- M_4\left(4\eta + \frac{4}{3}a_1^2\eta^3 + \dots\right)\Bigg]$$

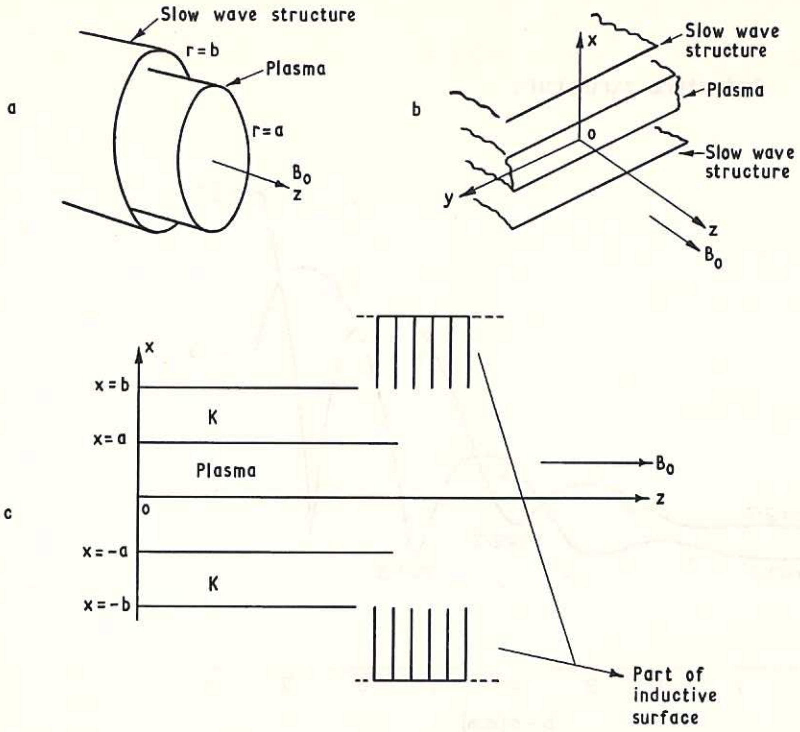


Fig. 1 (CLM-P 141)
Illustrating the geometry of the system (a) Cylindrical plasma with slow wave structure; (b) Planar plasma between two slow wave structures; (c) Showing the position of the slots of the slow wave structure.

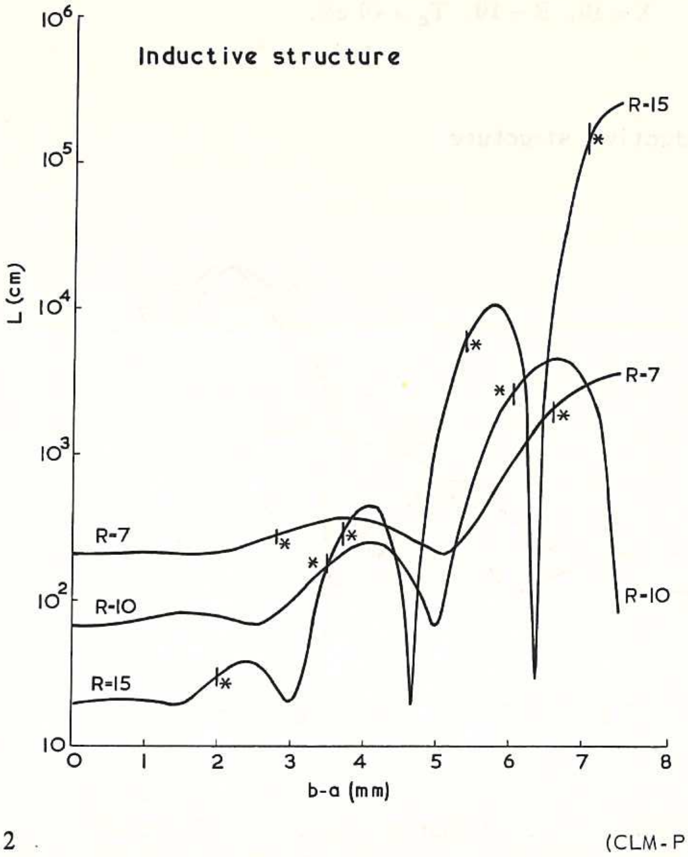


Fig. 2 (CLM-P 141)
Showing the variation of the attenuation length L with the separation of the plasma from the slow wave structure for different values of R the retardation of the wave. The stars indicate points where the surface wave is absent. ($A_s = 0$). The parameters are, b = 5.35 cm, X = 10, Y = 3, T_e = 40 eV.

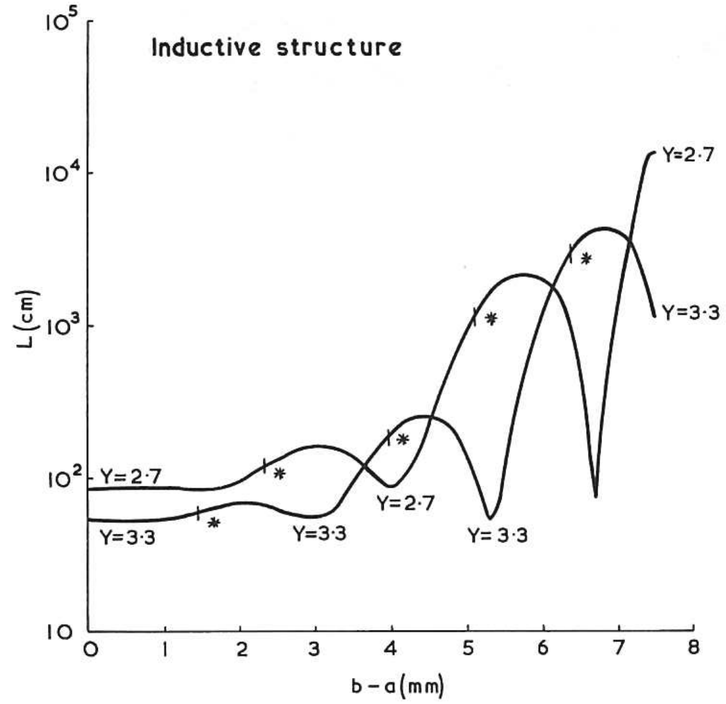


Fig. 3 (CLM-P141)

Showing the variation of the attenuation length L with the separation of the plasma from the slow wave structure for different static magnetic fields. The stars indicate points where the surface wave is absent. ($A_S = 0$). The parameters are, b = 5.35 cm, X = 10, R = 10, $T_e = 40$ eV.

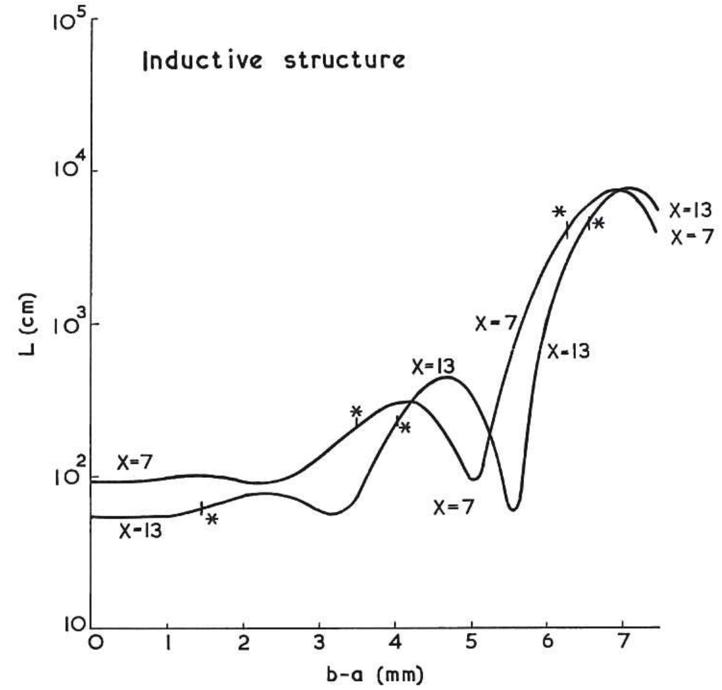


Fig. 4 (CLM-P141)

Showing the variation of the attenuation length L with the separation of the plasma from the slow wave structure for different plasma densities. The stars indicate points where the surface wave is absent. ($A_S = 0$). The parameters are b = 5.35 cm, Y = 3.0, R = 10, $T_e = 40$ eV.

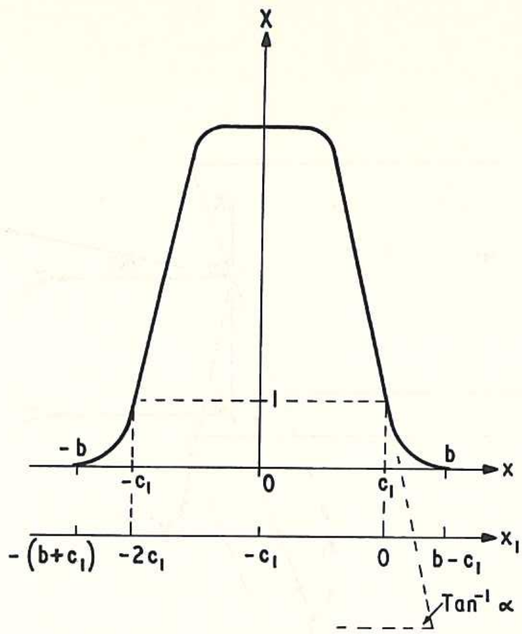


Fig. 5 (CLM-P 141)
A possible density distribution across the plasma column.

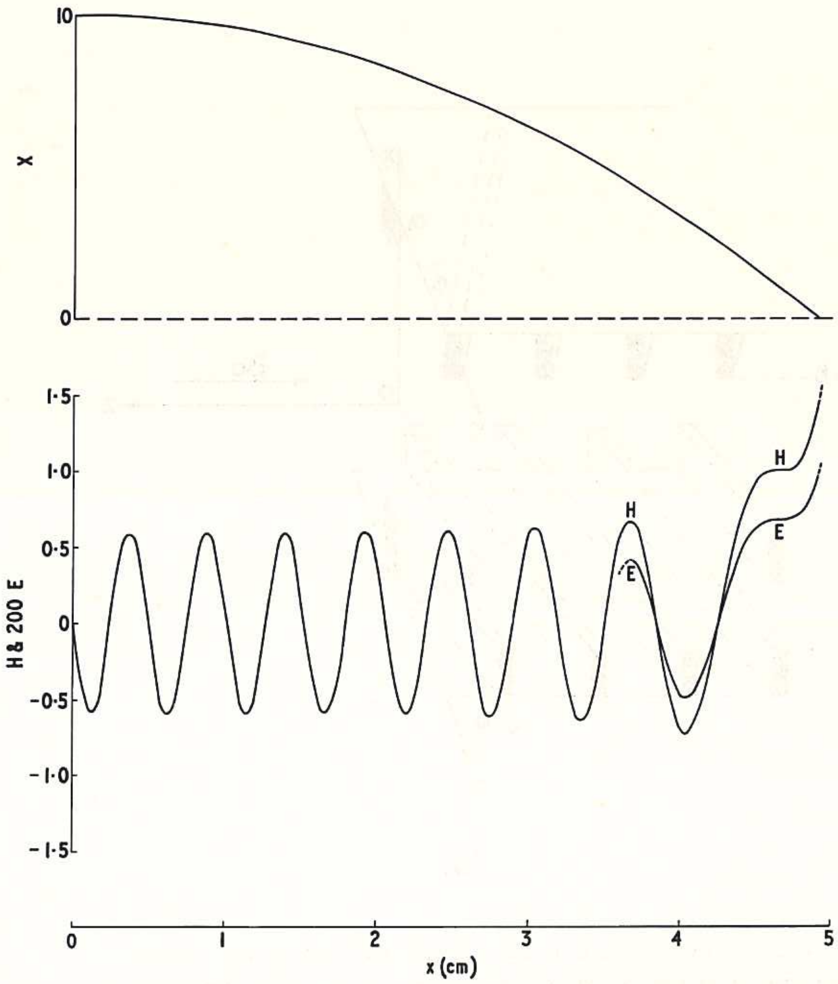
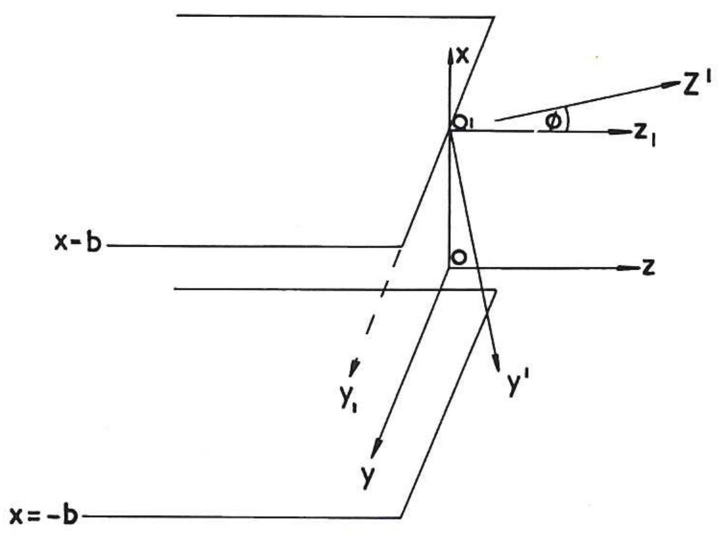
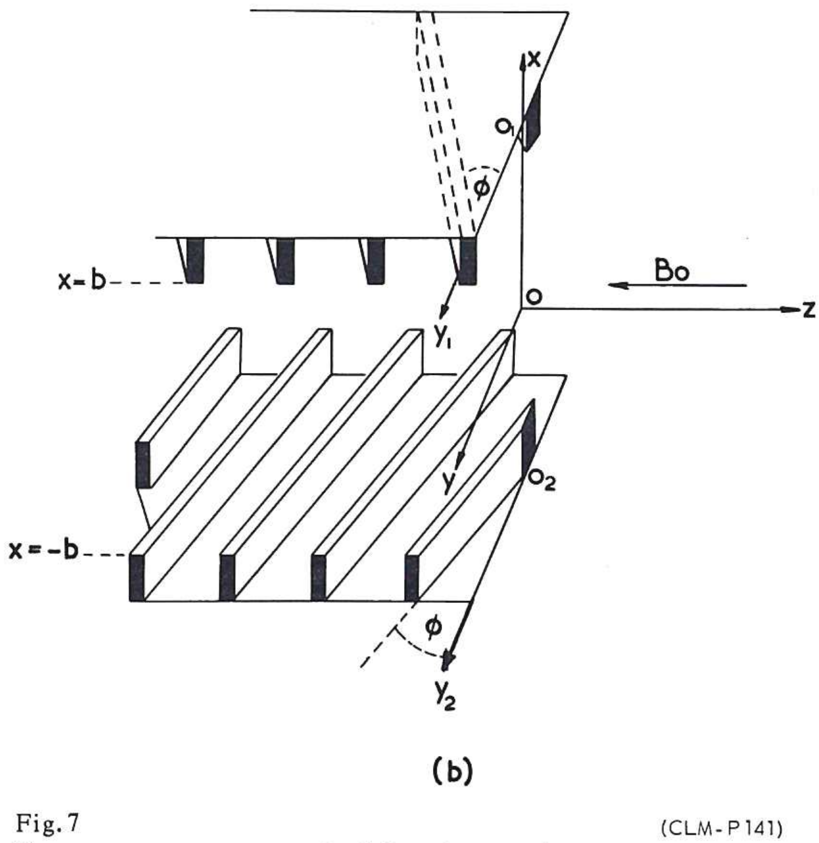


Fig. 6 (CLM-P 141)
(a) Density distribution across the plasma slab for $X = 10(1 - (x/4.9)^2)$
(b) The result of numerically integrating equations 2.1, 2.2 using equations 8.6 as starting values. H is proportional to H_y and E to E_y. ($R = 10$, $Y = 3.0$, $N = 1$, $k_o = \pi/5\ cm^{-1}$).



(a)



(b)

Fig. 7 (CLM-P141)

Slow wave structure required for plasma with density gradient. (a) Showing position of subsidiary axes y', z', the lines O_1Y_1, O_1Z_1 are parallel to the y,z axes. (b) Showing the orientation of the grooves in the slow wave structure relative to y,z axes. The lines O_1Y_1, O_2Y_2 are parallel to the OY axis.