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CRITICAL ALFVÉN-MACH NUMBERS FOR TRANSVERSE FIELD MHD SHOCKS

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CRITICAL ALFVÉN-MACH NUMBERS FOR TRANSVERSE FIELD MHD SHOCKS

by

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ABSTRACT

Just as ordinary gas dynamic shocks in a gas of zero viscosity, but finite thermal conductivity do not possess a continuous structure above a certain critical Mach number, MHD shocks in a two-fluid plasma also exhibit discontinuities when viscosity is neglected. These discontinuities occur when the Alfvén-Mach number of the shock, M_A, exceeds a certain critical value M_A^* ; the precise value of M_A^* depends on which of the three dissipative mechanisms of thermal conductivity for ions and electrons and electrical resistivity are retained and is independent of their actual values. In the practically important case of strong narrow shocks in which ion collisions can be neglected, but the electron gas still exhibits fluid behaviour, one finds that $M_A^* = 3{\cdot}46$. This is the case $\tau_i/\tau_s \sim \infty$ and $\tau_e/\tau_s < 1$, where τ_i, τ_e are collision times in the ion and electron gases respectively and τ_s is the characteristic shock time (shock thickness divided by shock speed). The ion viscosity tends to zero in both of the limits $\tau_i/\tau_s \to 0, \infty$ and while the resulting discontinuity at $M_A > M_A^*$ is known to be stable for the first of these limits, in the second limit ion-ion streaming instabilities are likely to occur. The addition of electron inertia and of collisionless electron viscosity due to finite Larmor radius results in a structural instability for all shocks, regardless of the value of M_A.

CONTENTS

I. INTRODUCTION

It is well known in the theory of ordinary gas dynamic shocks that when the Prandtl number is zero, continuous solutions for the shock structure can be found only for upstream Mach numbers, M, lying in a finite range $1 < M < M^*$. The upper limit can be shown to be

$$M^* = \sqrt{\left\{\frac{3\gamma-1}{\gamma(3-\gamma)}\right\}}, \qquad [\simeq 1{\cdot}2 \quad \text{for} \quad \gamma = 1{\cdot}4] ,$$

and when M exceeds this critical value, an isothermal discontinuity must appear. The difficulty is eliminated by the introduction of some viscosity, and the adoption of a singular-perturbation technique in the rapidly changing region.

A similar phenomenon can be expected with the more complicated shocks occurring in magnetic plasmas, and as these have two fluids and hence four dissipative mechanisms, several distinct Alfvén-Mach[2] numbers M_A^* can be anticipated. While in general each of viscosity (mainly due to the ions), ion thermal conductivity, electron thermal conductivity and electrical resistivity (mainly due to electrons) will normally play dissipative roles in plasma shocks, there are some circumstances in which the shock occurs over a length λ_s so small that the ions behave as if collisionless, and only the electron transport coefficients can be invoked to produce the dissipation. It is in these 'near-collisionless' shocks that critical Alfvén numbers have some practical importance, for the instabilities that must be provoked by exceeding these limits in shock speed probably act to restore a continuous shock structure by producing turbulent viscosity. Thus one would expect to see some change in shock structure as M_A is increased beyond M_A^*, and it appears[3,4] that this situation has in fact been met in some strong shocks in laboratory plasmas. The instabilities mentioned here are not, of course, the structural instabilities of continuum theory[5], but the kinetic theory instabilities due to the counterstreaming of ions[6] - a counterstreaming made possible by the absence of collisions and the relative mass motion of the upstream and downstream ions. On continuum theory the velocity discontinuities can be shown to be structurally stable in the limit of vanishing viscosity.

As the shocks for which critical Alfvén numbers are significant are nearly collisionless, it is reasonable to neglect the transfer of energy between ions and electrons in these shocks.

II. GENERAL THEORY

Let the shock be situated in the $x = 0$ plane, with the plasma moving along the positive Ox-axis. Let $\hat{\underset{\sim}{x}}, \hat{\underset{\sim}{y}}, \hat{\underset{\sim}{z}}$ be unit vectors along Ox, Oy, Oz and let this coordinate frame be chosen so that

$$\underset{\sim}{B} = B\hat{\underset{\sim}{z}}, \quad \underset{\sim}{E} = E_0\hat{\underset{\sim}{y}} + E_x\hat{\underset{\sim}{x}} \qquad \ldots (1)$$

where by the $\hat{\underset{\sim}{z}}$-component of $\nabla \wedge \underset{\sim}{E} = 0$ E_0 is a constant. The Debye length will be supposed to be negligible compared with the shock thickness λ_s, permitting charge separation and displacement current to be neglected. Thus no current flows in the Ox-direction and by Maxwell's equation the electric current is

$$\underset{\sim}{j} = -\frac{1}{\mu} B' \hat{\underset{\sim}{y}} \qquad \ldots (2)$$

where the dash denotes $\partial/\partial x$.

The assumption that there is no transverse momentum generated in the shock is reasonable in the circumstances of the laboratory experiments mentioned above, and means that the plasma velocity can be written as $\underset{\sim}{v} = v\underset{\sim}{\hat{x}}$. Thus the equations for the conservation of mass and momentum read

$$\rho v = G \ , \quad Gv + p - a_\nu v' + \frac{1}{2\mu}B^2 = A \qquad \dots (3)$$

where G and A are constant and $a_\nu = a_{\nu e} + a_{\nu i}$ is a viscosity coefficient, the precise nature of which is not important in the present study.

The equation of motion of the electrons provides a suitable form for Ohm's law. The transverse component of this equation plus (2) give

$$a_\eta B' + a_{\eta T} T_e' = vB - E_o \qquad \dots (4)$$

where a_η is a resistivity coefficient and $a_{\eta T}$ is a coefficient coupling the electron temperature gradient T_e' with the transport of electrons. In (4) it has been assumed that the collision-free skin depth $d \equiv (m_e/e^2 n\mu)^{\frac{1}{2}}$ for electrons is much less than λ_s, i.e. that electron inertia can be ignored. The effect of retaining electron inertia and non-collisional electron viscosity in (4) will be considered later.

Two energy equations are required to complete the system of equations. We choose the energy equation for the whole plasma and that for the ions alone. The former can be written

$$a_{\eta\kappa} B' + a_{\kappa e} T_e' + a_{\kappa i} T_i' = B + E_o B/\mu - vB^2/2\mu + vA + \alpha_i p_i v + \alpha_e p_e v - \tfrac{1}{2} v^2 G, \qquad \dots (5)$$

where B is a constant, $a_{\kappa e}$, $a_{\kappa i}$ are heat conductivity coefficients for ions and electrons, $a_{\eta\kappa}$ is a coupling coefficient and

$$\alpha_i = \frac{1}{\gamma_i - 1} \ , \quad \alpha_e = \frac{1}{\gamma_e - 1} \ , \qquad \dots (6)$$

γ_i, γ_e being the ratio of specific heats for ions and electrons. The value of γ_i varies from 5/3 for collision dominated ions to either 2 for $(\omega_i\tau_i) \gg 1$ (strong magnetic field) or 3 for $(\omega_i\tau_i) \ll 1$ (weak magnetic field) when the ions are collisionless. A similar remark applies to γ_e. Thus both α_i and α_e will be slowly varying functions through the kind of shocks in which we are interested.

The rate at which entropy is produced in the electron gas is positive provided

$$\frac{4a_{\kappa e}\, a_\eta}{T\mu} > \left\{ \frac{a_{\eta T}}{\mu} + \frac{a_{\eta\kappa}}{\mu} \right\}^2 . \qquad \ldots (7)$$

The energy equation for the ions alone gives

$$\alpha_i T_i' \, v + v' T_i = a_{\nu i}(v')^2 + (a_{\kappa i} T_i')' \qquad \ldots (8)$$

where in accordance with the previous remarks the transfer of energy between ions and electrons has been ignored.

Subscripts "1" and "2" will be used to denote upstream and downstream values respectively, in a frame in which the shock is stationary. The shock speed v_s is thus the same as v_1. Some useful non-dimensional numbers are as follows:

$$M_A = \frac{v_s \sqrt{(\mu\rho_1)}}{B_1}, \quad w = \frac{v}{v_s}, \quad b = \frac{B}{B_1}, \quad \left(n = \frac{n_1}{w}\right),$$

$$\theta = \frac{\mu n_1 k}{B_1^2} T_e = \frac{\mu p_e v}{B_1^2 v_1}, \quad \varphi = \frac{\mu n_1 k}{B_1^2} T_i = \frac{\mu p_i v}{B_1^2 v_1}$$

and

$$X = x/\lambda_s$$

where λ_s is the (unknown) shock thickness. On evaluating the constants E_0, A, G and B by the condition that the derivatives vanish at the upstream point 1 and introducing non-dimensional coefficients

c_η, c_ν, ... etc. to replace a_η, a_ν, ... we can write (4), (3), (5) and (8) in the forms

$$c_\eta b' + \delta\theta' = bw - 1$$

$$c_\nu w' = M_A^2 w^2 - w(M_A^2 + \theta_1 + \varphi_1 + \tfrac{1}{2} - \tfrac{1}{2}b^2) + \theta + \varphi$$

$$\varepsilon b' + c_e\theta' + c_i\varphi' = -\tfrac{1}{2} M_A^2(w-1)^2 + (w-1)(\theta_1 + \varphi_1) + a_e(\theta - \theta_1) + a_i(\varphi - \varphi_1) + b-1 + \tfrac{1}{2} w(1-b^2),$$

$$a_i w\varphi' + \varphi w' = c_{\nu i}(w')^2 + w(c_i\varphi')', \qquad \ldots (9)$$

where $c_\nu = c_{\nu i} + c_{\nu e}$, the symbol κ has been omitted from $c_{\kappa e}$ and $c_{\kappa i}$ and the dashes now denote derivatives with respect to X. Equation (7) takes the form

$$4\, c_e c_\eta > (\delta + \varepsilon)^2. \qquad \ldots (10)$$

From (9) it follows that downstream of the shock, where the derivatives again vanish,

$$w_2 = \frac{1}{b_2}\ , \quad M_A^2 = b_2\, \frac{1 + \alpha + b_2(\alpha - 1)}{2\alpha + 1 - b_2} + b_2\, \frac{2(\alpha+1)}{2\alpha + 1 - b_2}\,(\theta_1 + \varphi_1)\ , \qquad \ldots (11)$$

and

$$\theta_2 + \varphi_2 = \frac{(b_2 - 1)^3}{2b_2(2\alpha + 1 - b_2)} + \frac{(2\alpha + 1)b_2 - 1}{b_2(2\alpha + 1 - b_2)}\,(\theta_1 + \varphi_1)\ , \qquad \ldots (12)$$

where to simplify the algebra here and below we have set $a_i = a_e = \alpha$, where α is a constant through the shock. The method is easily extended to the general case, but a large number of combinations of the upstream and downstream values of Υ_i and Υ_e must be considered, and until there is firm experimental evidence for the existence of critical Alfvén numbers in real plasmas, there seems little point in undertaking this additional algebra. As Table I shows, M_A^* is not unduly sensitive to the choice of $\alpha = 1/(\Upsilon - 1)$.

III. NATURE OF THE SINGULAR POINTS

To investigate the nature of the singular points, 1 and 2, it is necessary to linearize (9) about these points. Let the subscript r denote values at points 1 and 2, and denote by $\hat{b}$, $\hat{\theta}$, $\hat{\varphi}$ and $\hat{w}$ the perturbations $(b-b_r)$, $(\theta-\theta_r)$, $(\varphi-\varphi_r)$ and $(w-w_r)$, then on linearizing (9) we arrive at

$$\begin{bmatrix} c_\eta & \delta & 0 & 0 \\ \varepsilon & c_e & 0 & 0 \\ 0 & 0 & c_i & 0 \\ 0 & 0 & 0 & c_\nu \end{bmatrix} \begin{bmatrix} \hat{b} \\ \hat{\theta} \\ \hat{\varphi} \\ \hat{w} \end{bmatrix}' \qquad \begin{bmatrix} w_r & 0 & 0 & b_r \\ 0 & \alpha & 0 & k_r \\ 0 & 0 & \alpha & \varphi_r b_r \\ 1 & 1 & 1 & g_r \end{bmatrix} \begin{bmatrix} \hat{b} \\ \hat{\theta} \\ \hat{\varphi} \\ \hat{w} \end{bmatrix}, \quad \ldots (13)$$

$$\left.\begin{aligned} \text{where} \quad g_r &= (2w_r - 1)M_A^2 - \tfrac{1}{2} + \tfrac{1}{2}b_r^2 - (\theta_1 + \varphi_1) \\ \text{and} \quad k_r &= (1 - w_r)M_A^2 + \tfrac{1}{2} - \tfrac{1}{2}b_r^2 + (\theta_1 + \varphi_1) - b_r\varphi_r \end{aligned}\right\}. \quad \ldots (14)$$

By (11) one finds the values

$$g_1 = M_A^2 - (\theta_1 + \varphi_1) \quad , \quad k_1 = \theta_1 \; , \quad \ldots (15)$$

and

$$g_2 = \frac{3 + 2\alpha + (2\alpha-5)b_2 + 3b_2^2 - b_2^3}{2(2\alpha + 1 - b_2)} + \frac{\{2\alpha + 3 - (2\alpha+1)b_2\}(\theta_1 + \varphi_1)}{2\alpha + 1 - b_2} \; , \quad \ldots (16)$$

$$k_2 = b_2\theta_2 \; . \quad \ldots (17)$$

For the case $(\omega_e\tau_e) \gg 1$, ω_e being the electron cyclotron frequency and τ_e the collision time for the electrons, the transport coefficients on the left-hand side of (13) can be shown[7] to have the form

$$\left.\begin{aligned} c_\eta &= m_e/(e^2 n\tau_e\mu v_s\lambda_s) \; , & \delta &= 3\, c_\eta/(2wb) \; , \\ c_\nu &= v_s(f\varphi\tau_i + 0{\cdot}24\, \theta\tau_e) \; , & c_i &= 3{\cdot}91\, F\varphi\tau_i v_s/(\lambda_s w M_A^2) \; , \\ c_e &= 4{\cdot}66\, \theta\, c_\eta/(wb)^2 \; , & \varepsilon &= 0{\cdot}322\, c_e\, wb \; , \end{aligned}\right\} \quad \ldots (18)$$

where F and f are functions of $(\omega_i \tau_i)$. Let σ denote the subscript i or e, then (13) are valid only if

$$\tau_s \gg \tau_\sigma , \quad \lambda_s \gg \min(r_\sigma , \lambda_\sigma) ,$$

for both ions and electrons, where τ_s is a characteristic shock time and r_i, r_e are the gyro-radii. The case of interest in the present study is when the above inequalities are satisfied for the electrons but not for the ions, i.e. the formulae for c_η, δ, c_e and ε are correct but those for c_ν and c_i are not.

To solve the equations in (13) assume that each dependent variable can be written in the form $\hat{A} = \tilde{A} \exp(X/\lambda)$, where $\tilde{A}$ is a constant, and then find in the usual way that λ must be a solution of the equation:

$$E_r\lambda^4 + D_r\lambda^3 + C_r\lambda^2 + B_r\lambda + A_r = 0, \quad \ldots (19)$$

where

$$A_r = c_{i\nu}U, \; B_r = -(GU + c_{i\nu}S), \; E_r = \alpha(w_rH - R),$$

$$C_r = HU + \alpha c_{i\nu}w_r + SG - c_iT, \; D_r = -(HS + \alpha w_rG - c_iR - \alpha T) \quad \ldots (20)$$

with

$$c_{i\nu} = c_ic_\nu, \; U = c_\eta c_e - \delta\varepsilon, \quad S = c_ew_r + \alpha c_\eta ,$$

$$T = k_r(c_\eta - \delta) + b_r(c_e - \varepsilon), \; R = k_rw_r + \alpha b_r ,$$

non-negative numbers (see (10)) and

$$G = \alpha c_\nu + c_ig_r, \; H = \alpha g_r - \varphi_rb_r , \quad \ldots (21)$$

numbers able to assume both positive and negative values. Notice that it is always the case that

$$A_r \geqq 0 . \quad \ldots (22)$$

By (15) to (17) it follows that the values of E are

$$E_1 = \alpha^2\{ M_A^2 - \Upsilon(\theta_1 + \varphi_1) - 1 \} ,$$

$$E_2 = \frac{\alpha(b_2 - 1)}{2b_2(2\alpha+1-b_2)} \left\{(\alpha-1)b_2^2 - 2(\alpha-1)(2\alpha+1)b_2 - (\alpha+1)(2\alpha+1) - 2(2\alpha+1)(\alpha+1)(\theta_1+\varphi_1)\right\}.$$

Thus, as $1 < b_2 < 2\alpha + 1$, we find that $E_2 < 0$. Also $E_1 > 0$ provided

$$M_A^2 > 1 + \Upsilon\,(\theta_1 + \varphi_1) \; ; \qquad \ldots (23)$$

in terms of the Alfvén speed v_{A1} and the sound speed c_{s1} this result takes the familiar form:

$$v_s^2 > v_{A1}^2 + c_{s1}^2 \; . \qquad \ldots (24)$$

A solution curve can <u>enter</u> the singular point at $X = \pm\,\infty$ only if all the roots λ_j of (19) are such that $\underline{\mathcal{R}e}\,\lambda_j \lessgtr 0$. Of course one can always suppress a root leading to a divergent exponential by setting the corresponding amplitude factors $\tilde{A}$ equal to zero, however in practice this is insufficient to allow a solution curve to approach a singular point and finally enter it, for small errors will revive the suppressed root and lead to divergence. On the other hand solution curves can <u>leave</u> the singular point at $X = \pm\,\infty$ provided <u>at least one</u> of the roots λ_j satisfies $\underline{\mathcal{R}e}\,\lambda_j \lessgtr 0$. Any roots not satisfying this condition can be suppressed without difficulty. Let n_1, n_2 be the number of roots at $x = -\,\infty$, $x = \infty$, i.e. at $r = 1$, $r = 2$, for which $\underline{\mathcal{R}e}\,\lambda < 0$, then asymptotic stability[8] of the singular point $r = 1$, plus the requirement that solution curves can leave $r = 2$, can be expressed $n_2 \geqq 1$, $n_1 = 0$. At the point $r = 2$ (initial point) the solution curves will form an $(n_2 - 1)$ - parameter family, this many amplitude factors $\tilde{A}$ being available after one degree of freedom is absorbed in locating the origin. If $n_2 > 1$ small errors in the solution near $r = 2$ will alter the ratios of the $\tilde{A}$'s and as the solution curve moves away from $r = 2$, these errors will be amplified. Thus structural stability near $r = 2$ requires that $n_2 \leqq 1$. Near $r = 1$ the non-uniqueness of the solution curves is unimportant because errors will be suppressed as solution curves converge to $r = 1$, which convergence is ensured by the condition $n_1 = 0$. The condition for overall structural stability now reads

$$r=2 \rightarrow r=1 : \quad n_2 = 1, \quad n_1 = 0 . \qquad \ldots (25a)$$

Similarly for a solution curve starting at $r = 1$ and terminating at $r = 2$,

$$r=1 \rightarrow r=2 : \quad n_1 = n-1, \quad n_2 = n , \qquad \ldots (25b)$$

where n is the degree of the characteristic equation (19). If one of (25) is satisfied for a solution curve joining the two singular points, it can be termed a "structurally stable" solution curve[9].

Now the number of roots of (19) with $\mathcal{R}e\,\lambda > 0$ equals the number of sign changes in the sequence

$$\left\{A_r, B_r, C_r - A_r D_r/B_r, D_r - B_r E_r/(C_r - A_r D_r/B_r), E_r\right\}. \qquad \ldots (26)$$

Provided (23) is satisfied, one finds by (20) and (21) that at $r=1$ (26) has the sign sequence $\{+, -, +, -, +,\}$ and it follows that all four roots of (19) have positive real part at $r = 1$, so solution curves can enter this point. This property of $r = 1$ is independent of the values of c_e, c_η, ε, δ, c_i and c_ν (subject to $c_e c_\eta > \delta\varepsilon$) some of which may take the value zero in special cases. If either c_η or c_e vanishes then (10) requires that both δ and ε vanish.

By (22) and $E_2 < 0$, it follows that (26) is $\{+,\pm,\pm,\pm, -\}$, at $r = 2$, showing that $n_2 \geqq 1$. Thus a solution curve can leave $r = 2$ and enter $r = 1$. By a detailed analysis of the possible signs in this sequence, or by the method adopted by Anderson[10] one can show that $n_2 = 1$. The conclusion follows that in the general case structurally stable solutions exist for all Alfvén numbers satisfying (23).

IV. CRITICAL ALFVÉN NUMBERS

While a solution curve can always enter $r = 1$ even when some of the transport coefficients vanish, it cannot always leave $r = 2$ if c_ν is zero. In these cases there is a finite critical Alfvén number, M_A^*, such that for Alfvén numbers M_A satisfying

$$1 + \gamma(\theta_1 + \varphi_1) \leqq M_A^2 < M_A^{*2} , \qquad \ldots (27)$$

solution curves can leave $r = 2$ and enter $r = 1$, whereas if $M_A \geqq M_A^*$ they cannot leave $r = 2$. In other words continuous solution curves exist only for a finite range of Alfvén numbers. Below our interest will be confined to finding the value of M_A at which the condition $n_2 \geqq 1$ is changed to $n_2 = 0$. A fuller study shows that when n_2 is not zero it is always unity (exceptions are given in Section V), and hence the condition for structural stability, equation (25), is always satisfied.

First let us confirm that if $c_\nu > 0$, $M_A^* = \infty$ regardless of the values of c_e, c_i and c_η. For example let c_e (and ε) vanish, then $A_2 = 0$, $B = - \alpha c_\nu U < 0$, $E_2 < 0$, and (25), which reduces to

$$\left\{ B_r, C_r, (D_r C_r - B_r E_r)/C_r, E_r \right\} , \qquad \ldots (28)$$

has the sign sequence $\{-,\pm,\pm,-\}$, showing that at least one λ_j has $\mathcal{Re}\, \lambda_j < 0$. Thus solution curves can leave $r = 2$ for all M_A. By similar reasoning we find that this is also the case for zero values of c_i or c_η or any combinations of zero values for c_e, c_i and c_η.

In the following it is necessary to have an expression for the number $(D_rC_r - B_rE_r)$ appearing in (28). From (20) we find that

$$C_rD_r - B_rE_r = -U\{H^2S + R(\alpha^2c_\nu + c_i\varphi_rb_r) - \alpha TH\} - \{\alpha w_rSG^2 + \alpha c_iT^2 + c_i^2R + S^2GH\}$$
$$+ c_iRSG + T\{\alpha SG + c_iHS + \alpha w_rc_iG\} + c_{i\nu}\{\alpha^2w_rT - \alpha RS - d^2w_r^2G + \alpha w_rc_iR\}. \quad \ldots (29)$$

Now consider the case $c_\nu = 0$. By (20) $B_r = -GU$. There are two possibilities, viz (i) $G > 0$, when (28) reads $\{-,\pm,\pm,-\}$ and so solution curves can leave $r = 1$, and (ii) $G < 0$, in which case it follows from (21), (20) and (29) that $H < 0$, $C_r < 0$ and $D_r - B_rE_r/C_r > 0$, and (28) reads $\{+,-,+,-\}$, i.e. there are no λ_j with $\mathcal{R}e\,\lambda_j < 0$ and a solution curve cannot leave $r = 2$. The critical Alfvén number is therefore given by $G = 0$, i.e. $g_2 = 0$. By (16) this occurs at a value of b_2, say b_2^*, given by

$$b_2^{*3} - 3b_2^{*2} - (2\alpha-5)b_2^* - (3+2\alpha) + 2[(2\alpha+1)b_2^* - 2\alpha - 3](\theta_1 + \varphi_1) = 0, \quad \ldots (30)$$

and when this has been solved for b_2^*, (11) gives M_A^*.

As a simple example suppose the plasma upstream of the shock is a "low β" plasma, i.e. $(\theta_1 + \varphi_1) \ll 1$, then (30) and (11) give

$$\left.\begin{array}{lll} \gamma = 5/3\,; & b_2^* \simeq 3, & M_A^* \simeq 2\sqrt{3} \simeq 3{\cdot}46 \\ \gamma = 2\,; & b_2^* \simeq 2{\cdot}59, & M_A^* \simeq 3{\cdot}56 \end{array}\right\} \quad \ldots (31)$$

Now suppose that both c_ν and c_i are zero, then (28) reduces to the sequence

$$\{C_r, D_r, E_r\}. \quad \ldots (32)$$

In this case $G = 0$, $c_2 = HU$, $D_2 = -(HS - \alpha T)$, $E_2 < 0$. Thus (32) yields $\{+,\pm,-\}$ for $H > 0$ and $\{-,+,-\}$ for $H < 0$. Thus if $H < 0$ a solution curve cannot leave $r = 2$, and the critical value of b_2 is therefore given by $\alpha g_2^* = b_2^*\varphi_2$. Write $\varphi_2 = R\theta_2$, $f = (\gamma-1)R/(1+R)$, then by (12) and (16) we find

$$b_2^{*3} - 3b_2^{*2} + \frac{3f+5-2\alpha}{1+f} b_2^* - \frac{f+3+2\alpha}{1+f} + 2\left\{(2\alpha+1)b_2^* - \frac{f+3+2\alpha}{1+f}\right\}(\theta_1 + \varphi_1) = 0 \ , \qquad \ldots (33)$$

As c_ν and c_i are zero the ions are heated adiabatically, (9) giving $\varphi_2 = \varphi_1 b_2^{\gamma-1}$. Thus for a given b_2, R can be calculated with the help of (12). If $(\theta_1 + \varphi_2) \ll 1$, (33) and (11) give the values

$$\gamma = 5/3 \ ; \quad b_2^* \simeq 2{\cdot}80 \ , \quad M_A^* \simeq 3{\cdot}01$$

$$\gamma = 2 \ ; \quad b_2^* \simeq 2{\cdot}39 \ , \quad M_A^* \simeq 2{\cdot}80 \ ,$$

for $R = 1$.

Similarly one finds the equations

$$c_\eta \neq 0, \quad c_i \neq 0; \quad \alpha g_2^* = b_2^* \, \theta_2,$$

$$c_e \neq 0, \quad c_i \neq 0; \quad g_2^* = b_2^{*2},$$

$$c_\eta \neq 0; \quad \alpha g_2^* = b_2^*(\theta_2 + \varphi_2) \ ,$$

for the critical Alfvén number. Table 1 is the outcome of calculations for $(\theta_1 + \varphi_1) \ll 1$. The coupling coefficients δ and ε have been retained only when both c_e and c_η are non-zero.

The result for $\gamma = 5/3$ and c_η non-zero was obtained by Golden, Sen and Tréve[11] by a different approach based on De Baggis' theorem[9]. Of course some of the critical Alfvén numbers given in Table 1 are meaningless under the conditions for which (18) hold. The close link between c_η and c_e means that they should both be retained or put equal to zero; and a similar remark applies to c_i and c_ν if electron viscosity can be ignored. Thus the results for $c_\eta, c_e \neq 0$ are the appropriate ones for laboratory plasma experiments on shocks involving collisionless ions.

The physical significance of the critical Alfvén numbers is simply that a normal gas dynamic sub-shock will appear near the downstream singular point only if the velocity v_2 is less than the effective sound speed at this downstream point. Let a_i, a_e and u_i, u_e denote the adiabatic and isothermal sound speeds:

$$a_\sigma = \sqrt{(\gamma k T_\sigma / m_i)} \quad , \quad u_\sigma = \sqrt{(k T_\sigma / m_i)} \quad , \quad (\sigma = i,\ e)$$

then $c_s = \sqrt{(a_i^2 + u_e^2)}$ is the appropriate sound speed for adiabatic ions and isothermal electrons - the case $c_i = 0$ and $c_e \neq 0$ in the first and fifth lines of Table 1. When the electrical resistivity is zero, the condition is modified to $v_2^2 < c_s^2 + v_A^2$, where c_s is the appropriate sound speed and v_A is the Alfvén speed. Anderson[10] gives a similar result in his Table 6·1 for the case of oblique shocks, but restricted to a 'one-fluid' treatment ($T_e = T_i$) of the plasma.

While the above is evident from the theory of ordinary gas dynamic shocks, it can be shown to follow from the above theory as follows. If the subscripts '1' and '2' are reversed in significance, so that '1' now denotes the downstream point, the whole of the above theory would be unchanged in form. Thus the condition $g_2 < 0$ given in the paragraph following (29) for the existence of a discontinuity when only $c_\nu = 0$ (i.e. both the ions and electrons are isothermal) is replaced by $g_1 < 0$ i.e. $M_A^2 < (\theta_1 + \varphi_1)$ by (15). Here $M_A^2 = (v/v_A)^2$ is to be evaluated at the downstream singular point and likewise for $\theta_1 + \varphi_1 = (u_i^2 + u_e^2)/v_A^2$. Thus the condition reads $V^2 < u_i^2 + u_e^2 = c_s^2$, in agreement with the above remarks. Similarly one obtains the critical values given in the last column of Table 1, values not subject to the restriction $(\theta_1 + \varphi_1) \ll 1$ mentioned in the caption.

V. EFFECTS OF ELECTRON INERTIA AND COLLISIONLESS VISCOSITY

Allowance for electron inertia and finite Larmor radius for the electrons results in additional terms on the left-hand side of the first three equations of (19). These are $w(d/\lambda_s)^2\{(wb')' - \frac{1}{2}(\theta w'/bw)'\}$, $-\frac{1}{2}(d/\lambda_s)^2(\theta/b)(wb')'$ and $\frac{1}{2}(d/\lambda_s)^2\{(\theta/b)w'b' - (wb')^2\}$, where d is the collision-free skin depth. Thus to the left-hand side of (13) is added

$$\begin{bmatrix} m_r & 0 & 0 & -p_r b_r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -p_r & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{b} \\ \hat{\theta} \\ \hat{\varphi} \\ \hat{w} \end{bmatrix}'' \qquad \text{where} \quad \begin{cases} m_r = w_r^2(d/\lambda_s)^2 \\ p_r = \frac{1}{2}\theta_r w_r^2(d/\lambda_s)^2 \end{cases} .$$

and equation (19) is replaced by

$$E_r\lambda^6 + D_r\lambda^5 + \tilde{C}_r\lambda^4 + \tilde{B}_r\lambda^3 + \tilde{A}_r\lambda^2 + M_r\lambda + N_r = 0 \quad , \qquad \ldots (34)$$

where

$$\tilde{C}_r = C_r - \alpha(H-k_r)m_r - 2\alpha^2 b_r p_r,$$

$$\tilde{B}_r = B_r + m_r(\alpha G + c_e H - k_r c_i) - \alpha\delta k_r p_r + \alpha b_r p_r(2c_i + 2c_e - \varepsilon),$$

$$\tilde{A}_r = A_r - \alpha^2 b_r p_r^2 - m_r(\alpha c_{i\nu} + c_e G) + c_i p_r(\delta k_r - 2b_r c_e),$$

$$M_r = \alpha p_r^2 b_r(c_i + c_e) + m_r c_e c_{i\nu}$$

$$N_r = -b_r p_r^2 c_i c_e.$$

The first point to notice is that the sign sequence is now of the form $\{N_r, M_r, \ldots, \ldots, \ldots, \ldots, E_r\}$, i.e. at $r = 1$ it is $\{-,+,\pm,\pm,\pm,\pm,+\}$, so $n_1 > 0$, and solution curves can no longer enter the upstream singular point. On the other hand at $r = 2$ the sign sequence is $\{-,+,\pm,\pm,\pm,\pm,-\}$, so $n_2 < n$, and solution curves cannot enter the downstream singular point. The conclusion follows that no continuous solutions can be found possessing structural stability in this general case. The situation is not changed if $p_r = 0$. The critical Alfvén number is thus the lower limit in (27).

Anderson[10] in extending Germain's work[5] to include the effects of electron inertia, found that this effect increased the values $n_1 = 0$, $n_2 = 1$, to $n_1 = 1$. $n_2 = 2$, an increase in agreement with our results. As Anderson was concerned with mere existence of solution curves, he drew no conclusions about their structural stability.

From the coefficients given above it can be readily shown that $r = 1$ cannot be entered by a solution curve regardless of whether any of the transport coefficients are zero or not. Therefore if a continuous solution is to exist, all the roots of (34) must have negative real parts at $r = 2$. For example consider the important case when the ion thermal conductivity and viscosity are zero and also $p_r = 0$. In this case the first non-zero coefficient is $B_r = c_e m_r H$ and the sign sequence is $\{+,\pm,\pm,(-)^{r+1}\}$, whence $r = 2$ has at least one λ with $\mathcal{R}e\,\lambda > 0$, and no stable solution curve exists. However if one can also assume $c_e = 0$ but $c_\eta \neq 0$ [12], then

$$C_r = - m_r \alpha (H-k_r), \quad D_r = - \alpha c_\eta \{H-k_r[1-\delta/c_\eta]\}.$$

and the sign sequence is favourable to entry at $r = 2$ only if $H > k_2$. This gives a critical Alfvén number exactly the same as the case $c_\eta \neq 0$ of Table 1.

Thus apart from the particular case just given, we conclude that electron inertia has the effect of destabilising the shock wave, in spite of any dissipative mechanisms present.

Finally it is perhaps worth pointing out that the well-known critical Alfvén number of collision-free, large amplitude hydromagnetic waves is rather different in origin from the critical values discussed in this paper. The simplest form of the theory (due to Adlam and Allen[13]) follows from the first two equations of (9) with their left-hand sides replaced by $w(d/\lambda_s)^2\,(wb')'$ - the electron inertia term - and zero respectively, and with $\theta, \varphi, \theta_1$ and φ_1 set equal to zero. When these equations are integrated, one finds that w is everywhere a positive number only if $b \leqslant 3$ and $M_A \leqslant 2$. Sagdeev[14] has discussed the instability that arises from the "overturning" of such waves when $M_A > M_A^* = 2$. Saffman[15] has extended the theory to shocks inclined at an angle to the magnetic field.

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TABLE 1

Critical Values for $(\theta_1 + \varphi_1) \ll 1$

Non zero Transport Coefficients	$\gamma = 5/3$				$\gamma = 2$				$(v_2^2)^*$ [Min. Downstream velocity for continuous flow]
	R = 0		R = 1		R = 0		R = 1		
	b_2^*	M^*	b_2^*	M_A^*	b_2^*	M_A^*	b_2^*	M_A^*	
c_e	1	1	1	1	1	1	1	1	$a_i^2 + u_e^2 + v_A^2$
c_i	1	1	1	1	1	1	1	1	$u_i^2 + a_e^2 + v_A^2$
c_η	2·67	2·76	2·67	2·76	2·44	2·95	2·44	2·95	$a_i^2 + a_e^2$
c_e, c_i	1	1	1	1	1	1	1	1	$u_i^2 + u_e^2 + v_A^2$
c_η, c_i	2·67	2·76	2·80	3·01	2·26	2·48	2·39	2·80	$u_i^2 + a_e^2$
c_η, c_e	3	3·46	2·80	3·01	2·59	3·56	2·39	2·80	$a_i^2 + u_e^2$
c_η, c_i, c_e	3	3·46	3	3·46	2·59	3·56	2·59	3·56	$u_i^2 + u_e^2$
c_ν	4	∞	4	∞	3	∞	3	∞	∞

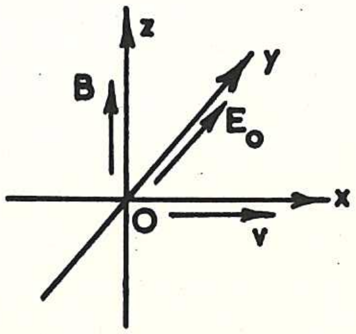

Fig. 1 The Coordinate System (CLM-P 153)