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STABILITY OF GENERAL PLASMA EQUILIBRIA

PART I. FORMAL THEORY

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STABILITY OF GENERAL PLASMA EQUILIBRIA, PART I. FORMAL THEORY

by

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ABSTRACT

A method is described for the detailed investigation of electrostatic instabilities in real experimental geometries. These have frequently been discussed in the plane slab model, and modifications of it, but the present work includes all geometrical effects from the outset. The starting point is the collisionless Boltzmann equation with the approximation that the scale length of the equilibrium is long compared to the ion gyro-radius. The main interest is in perturbations of low frequency but of arbitrary wavelength, which may be comparable to the ion Larmor radius. Thus several instabilities such as Drift wave, Flute or Trapped Particle, come within the scope of the theory.

Expressions are first obtained for the contribution to the charge density produced by an arbitrary electrostatic perturbation affecting particles whose unerpturbed orbits are (i) trapped between magnetic mirrors; (ii) circulating around closed field lines; (iii) tracing out a magnetic surface. Together with Poisson's equation these expressions lead, via the appropriate Nyquist contours, to stability criteria valid for arbitary equilibria. Finally it is shown how this method leads to a differential equation whose solution will determine the stability of an experimental configuration such as the multipole.

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CONTENTS

I. INTRODUCTION

The stability of a magnetically confined plasma has been thoroughly discussed in terms of the one-dimensional, plane slab configuration. The flute instability, with finite Larmor radius effects included, was investigated by ROSENBLUTH et al. (1962), and in the long wavelength limit drift wave instabilities due to temperature gradients and current along field lines were discovered (RUDAKOV and SAGDEEV (1961), KADOMTSEV (1963a)). The investigation of these drift waves in the short wavelength limit, (GALEEV et al. (1963), MIKHAILOVSKII and RUDAKOV (1963), KADOMTSEV and TIMOFEEV (1963)) revealed that such waves were unstable even in the absence of temperature gradients and parallel currents, provided only that there was a density gradient. Consequently the term 'Universal' was introduced to describe this mode. A review of the literature up to this time is given by KADOMTSEV (1963b).

The next generation of papers on the subject studied possible stabilising mechanisms, such as finite machine length, magnetic field shear, and gravity-simulated curvature, (GALEEV (1963), MIKHAILOVSKAYA and MIKHAILOVSKII (1964), KRALL and ROSENBLUTH (1965a)), and more recently this trend has continued with investigations of the effect of sinusoidally varying gravity (to simulate $\oint \frac{d\ell}{B}$ properties) (COPPI et al. (1967), ROHLENA and JUKES (to be published)). All these investigations of plasma stability are based on the plane slab model.

However real confinement systems are geometrically complex and introduce additional effects due to field gradients and curvatures, and to the fact that particles may be reflected at mirrors or circulate with varying speed round closed lines of force or over magnetic surfaces, rather than moving at uniform speed as in the plane slab. These effects as we have seen above, can be partially simulated in the plane slab, e.g. by a fictitious gravity, but in this paper we abandon the plane slab model and investigate the stability of arbitrary plasma equilibria in a way which permits all the effects due to geometrical complexity to play their full rôle. The only restriction is that the Larmor radius must be small compared to the scale lengths over which the equilibrium (not the perturbation) varies.

Thus, the present theory is applicable to any plasma equilibrium and at the same time provides a basis for the detailed investigation of micro-instabilities in real experimental situations. The instabilities with which we are most concerned are electrostatic modes of low frequency (compared to the ion gyro-frequency) with a scale length longer than, or comparable with, the ion Larmor radius. These include the flute instability, and the drift wave instability - to which we devote most attention.

The present method of studying stability is an extension of that used earlier to study equilibria in arbitrary magnetic field configurations, (HASTIE, TAYLOR and HAAS (1967a,b)), henceforth referred to as H.T.H. That is we expand in powers of λ, the ratio of the Larmor radius to a scale length of the equilibrium, (but not necessarily to the scale length of the perturbation). When studying equilibria it was shown that each order of the calculation reduced the arbitrariness in F_1, the first order correction to the zero order distribution F_o, until it was completely determined in terms of F_o. It will be shown here that in a similar way each order of calculation also decreases the arbitrariness of any perturbation f, until it too is completely specified in terms of the perturbed electrostatic field and the equilibrium quantities. The form which this specification takes depends on whether the corresponding particles are trapped, circulating round closed field lines, or tracing out magnetic surfaces. Together with Poisson's equation this expression for f determines the stability of general equilibria in an arbitrary magnetic field.

Some immediate comparisons can be made with the plane slab model. For example, in the plane slab an important rôle is played by the resonance between an electrostatic wave and particles moving at the wave's phase velocity. In three-dimensional equilibria the velocity of a particle varies from point to point and there are no independent plane wave oscillations. However, our analysis shows that a corresponding resonance phenomenon still arises. This is because in any confined three-dimensional equilibrium the orbits of the particles must be quasi-periodic and the important resonance is between the period of this quasi-cyclic motion and the frequency of a normal mode of electrostatic oscillation of the system.

A variety of sufficient criteria for stability of an arbitrary equilibrium can also be derived from our equation. For example, if one considers only perturbations with a long transverse wavelength, then a set of sufficient stability criteria are

$$\left(\frac{\partial F_o}{\partial K}\right)_{\mu J} < 0 \qquad \text{and} \qquad \left(\frac{\partial F_o}{\partial K}\right)_{\mu\alpha\beta} < 0$$

If perturbations of short perpendicular wavelength are also allowed, so that the drift wave instability is included, then the stability criteria are naturally more restrictive. In this case an equilibrium of the form $F_o = F_o(\mu, K)$ is stable at low density if

$$\frac{\partial F_o}{\partial K} < 0$$

and is also stable at high densities if in addition

$$\sum \frac{e^2}{m} \int \frac{B}{q} \, d\mu dK \, (1 - J_o^2(z)) \left(\frac{1}{B} \frac{\partial F_o}{\partial \mu} + \frac{\partial F_o}{\partial K} \right) < 0$$

where $z = \frac{v_\perp}{\omega_c} |\nabla S_o|$ and $\underset{\sim}{\nabla} S_o$ is a measure of the scale length of the perturbation.

The first condition was derived earlier on the assumption that the magnetic moment μ was a constant (TAYLOR (1967b)). The fact that a second inequality is involved was pointed out recently by WIMMEL and SAISON (1966), and its relation to the earlier results, and to the variation of μ has been discussed by TAYLOR (1967). To demonstrate that this theory also leads to normal modes with a well-defined growth rate the WKB method is extended to deal with the present situation in which there is a small expansion parameter valid in two directions, (perpendicular to the magnetic field) but not valid for the third direction. This leads to a formal dispersion equation for the normal modes of any equilibrium in an arbitrary magnetic field, and is in a form which can be compared in detail with the dispersion equations obtained for the plane slab model or for modifications of it in which a fictitious periodic gravity has been added to simulate geometrical effects.

Finally, we show how, for a class of microinstabilities, the basic equations can be reduced to a simple form convenient for the detailed investigation of real experimental configurations. These applications will be discussed fully in a second paper.

II. METHOD

The method by which we analyse instabilities in an arbitrary geometrical situation is similar to that used previously for the study of equilibrium in general geometry, particularly in the mathematical manipulations. Accordingly, we shall omit the detailed algebraic steps and refer the interested reader to the earlier paper and report (H.T.H. (a), (b)).

The starting point is the Vlasov equation

$$\frac{\partial f}{\partial t} + \underset{\sim}{v} \cdot \underset{\sim}{\nabla} f + \frac{e}{m} [\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}] \frac{\partial f}{\partial \underset{\sim}{v}} = 0 \qquad \ldots (1)$$

linearised about an arbitrary equilibrium F. This equilibrium may include an electric field $E = -\nabla\Phi$ which gives rise to $\underset{\sim}{E} \times \underset{\sim}{B}$ drifts of the same order as the gradient B and curvature drifts. Details of the equilibrium are given in H.T.H.(a). In the study of equilibrium we expanded the solution of the Vlasov equation in the small parameter λ, related to the ratio of the Larmor radius to the scale length over which the solution varies. If we were to do this in a straightforward way for the perturbations as well as the equilibrium we would exclude important classes of instability which have wavelengths of the same order as the ion Larmor radius. These instabilities can be included if we expand only the equilibrium quantities in a simple power series in λ and employ a different expansion for the perturbation. The perturbed potential is expressed as

$$\psi = \hat{\psi} \exp\,(iS(\underset{\sim}{x})/\lambda) \qquad \ldots (2)$$

and the perturbed distribution as

$$f = \hat{f} \exp\,(iS(\underset{\sim}{x})/\lambda) \qquad \ldots (3)$$

where $\hat{\psi}$, $\hat{f}$ and S are all expressed as simple power series in λ. This description of the perturbation allows us to include oscillations with wavelengths comparable to the ion Larmor radius while still having to deal throughout only with functions which vary slowly on the scale of the Larmor radius and which have simple power series expansions in λ.

Accordingly we write the solution of (1) as

$$f = F + \hat{f}\, e^{iS(\underset{\sim}{x})/\lambda}\, e^{i\omega t}$$

where F is the equilibrium distribution function and λ is the expansion parameter proportional to r_L/L (where r_L is a typical Larmor radius and L the typical scale length of variation of the equilibrium F, and of $\hat{f}$, $\hat{\psi}$ and S).

The linearised form of the Vlasov equation is then

$$i\omega\hat{f} + \frac{i\hat{f}}{\lambda}\,\underset{\sim}{v}\cdot\underset{\sim}{\nabla}S + \underset{\sim}{v}\cdot\underset{\sim}{\nabla}\hat{f} - \frac{e}{m}\,\underset{\sim}{\nabla}\Phi\cdot\frac{\partial\hat{f}}{\partial\underset{\sim}{v}} - \frac{e}{m}\left[\underset{\sim}{\nabla}\hat{\psi} + \frac{i\hat{\psi}}{\lambda}\,\underset{\sim}{\nabla}S\right]\cdot\frac{\partial F}{\partial\underset{\sim}{v}} = -\frac{e}{m}\,\underset{\sim}{v}\times\underset{\sim}{B}\cdot\frac{\partial\hat{f}}{\partial\underset{\sim}{v}}\ . \qquad \ldots (4)$$

Now, bearing in mind that all functions in (4) vary over a length L, large compared to the Larmor radius it is clear that, if equation (4) is divided by the cyclotron frequency ω_c, then the operator $\dfrac{\underset{\sim}{v}\cdot\underset{\sim}{\nabla}}{\omega_c}$ always produces a term of order $r_L/L \sim \lambda$ and it is convenient to indicate this explicitly by introducing a factor λ into the operator. The equation is then

$$i\,\frac{\omega}{\omega_c}\,\hat{f} + i\hat{f}\,\frac{\underset{\sim}{v}\cdot\underset{\sim}{\nabla}S}{\omega_c} + \lambda\,\frac{\underset{\sim}{v}\cdot\underset{\sim}{\nabla}\hat{f}}{\omega_c} - \lambda\,\frac{e}{m}\,\frac{\underset{\sim}{\nabla}\Phi}{\omega_c}\cdot\frac{\partial\hat{f}}{\partial\underset{\sim}{v}} - \frac{e}{m\omega_c}\left[\lambda\underset{\sim}{\nabla}\hat{\psi} + i\hat{\psi}\underset{\sim}{\nabla}S\right]\cdot\frac{\partial F}{\partial\underset{\sim}{v}} = -\frac{\underset{\sim}{v}\times\underset{\sim}{B}}{B}\cdot\frac{\partial\hat{f}}{\partial\underset{\sim}{v}} \qquad \ldots (5)$$

in which the order of each operator is now explicitly indicated by the power of λ which accompanies it and the solution can proceed formally, writing $F = F_o + \lambda F_1 + \ldots$, $\Phi = \Phi_o + \lambda\Phi_1 + \ldots$, $\hat{f} = f_o + \lambda f_1 + \ldots$, $\hat{\psi} = \psi_o + \lambda\psi_1 \ldots$, $S = S_o + \lambda S_1 + \ldots$.

The requirement that the equilibrium electric field $-\nabla\Phi$ be of order λ (so that the second, or longitudinal, invariant J is preserved and long term equilibrium ensured) means that $\Phi_o = 0$. Further, although $S(\underset{\sim}{x})$ appears as an arbitrary complex function it will be found later that only real S_o need be considered.

To solve equation (5), order by order, it is first necessary to express it in the 'natural' variables used in H.T.H. (a), (b). That is we express the velocity in terms of μ, ε, φ and σ where

$$\mu = v_\perp^2/2B \qquad \varphi = \tan^{-1}\frac{\underset{\sim}{e}_3\cdot\underset{\sim}{v}}{\underset{\sim}{e}_2\cdot\underset{\sim}{v}}$$

$$\varepsilon = \frac{1}{2}\left(v_\parallel^2 + v_\perp^2\right) \qquad \sigma = v_\parallel/|v_\parallel| \qquad \ldots (6)$$

and $\underset{\sim}{e}_2$, $\underset{\sim}{e}_3$ are unit vectors orthogonal to $\underset{\sim}{e}_1 = \underset{\sim}{B}/B$. Position $\underset{\sim}{x}$ is expressed in terms of α, β, ℓ, where α, β label a line of force through $\underset{\sim}{B} = \underset{\sim}{\nabla}\alpha \times \underset{\sim}{\nabla}\beta$ and ℓ is arc length along the line. It proves convenient to make a further change and replace ε by $K = \varepsilon + \frac{e}{m}\Phi$, then equation (5) becomes

$$i\frac{\omega}{\omega_c}\hat{f} + i\frac{\hat{\hat{f}}}{\omega_c}BDS + \lambda\frac{B}{\omega_c}D\hat{f} - \frac{\lambda}{\omega_c}\frac{e}{m}\underset{\sim}{\nabla}\Phi\cdot\left[\frac{\underset{\sim}{c}_\perp}{B}\frac{\partial}{\partial\mu} + \frac{\underset{\sim}{e}_1\times\underset{\sim}{c}_\perp}{c_\perp^2}\frac{\partial}{\partial\varphi}\right]\hat{f}$$

$$- \frac{e}{m\omega_c}\left[\lambda\underset{\sim}{\nabla}\hat{\psi} + i\hat{\psi}\underset{\sim}{\nabla}S\right]\cdot\left[\sigma q\,\underset{\sim}{e}_1\frac{\partial}{\partial K} + \underset{\sim}{c}_\perp\left(\frac{1}{B}\frac{\partial}{\partial\mu} + \frac{\partial}{\partial K}\right) + \frac{\underset{\sim}{e}_1\times\underset{\sim}{c}_\perp}{c_\perp^2}\frac{\partial}{\partial\varphi}\right]F = \frac{\partial\hat{f}}{\partial\varphi} \qquad \ldots (7)$$

where

$$q = \left[2\left(K - \mu B - \frac{e}{m}\Phi\right)\right]^{\frac{1}{2}}$$

$$c_\perp = \sqrt{2\mu B} \qquad \ldots (8)$$

$$\underset{\sim}{c}_\perp = c_\perp\left(\underset{\sim}{e}_2\cos\varphi + \underset{\sim}{e}_3\sin\varphi\right)$$

and D is an operator introduced and discussed in H.T.H. (a);

$$D = \frac{1}{B}\left\{\sigma q\frac{\partial}{\partial\ell} + c_\perp\cos\varphi\,\underset{\sim}{e}_2\cdot\left(\underset{\sim}{\nabla} - \underset{\sim}{V}\frac{\partial}{\partial\mu}\right) + c_\perp\sin\varphi\,\underset{\sim}{e}_3\cdot\left(\underset{\sim}{\nabla} - \underset{\sim}{V}\frac{\partial}{\partial\mu}\right)\right.$$

$$\left. + \mu\sigma q(\rho_3 - \rho_2)\cos2\varphi\frac{\partial}{\partial\mu} + \mu\sigma q\sin2\varphi(\tau_2 - \tau_3)\frac{\partial}{\partial\mu}\right\}$$

$$+ \frac{1}{B}\left\{\sigma q\left[\tau_1 + \frac{1}{2}(\tau_2 + \tau_3)\right] + \cos\varphi\left[\frac{q^2}{c_\perp}\sigma_1 - c_\perp\sigma_2\right] + \sin\varphi\left[\frac{q^2}{c_\perp}\rho_1 - c_\perp\sigma_3\right]\right.$$

$$\left. + \frac{1}{2}\sigma q(\tau_2 - \tau_3)\cos2\varphi + \frac{1}{2}\sigma q\sin2\varphi(\rho_2 - \rho_3)\right\}\frac{\partial}{\partial\varphi}$$

where $\underset{\sim}{V} = \mu\frac{\underset{\sim}{\nabla}B}{B} + \frac{q^2}{B}(\underset{\sim}{e}_1\cdot\underset{\sim}{\nabla})\underset{\sim}{e}_1$ and the σ_i, ρ_i, and τ_i are related to the curvature, shear and torsion of the field lines and are defined in H.T.H.(a), (b).

As yet no assumption has been made about the frequency of the perturbation and the method deals with both high and low frequency instabilities. However we shall concentrate on the low frequency modes for which

$$\frac{\omega}{\omega_c} \sim \lambda \qquad \text{and} \qquad \frac{\partial S_0}{\partial\ell} = 0\ .$$

The latter condition implies long, but not infinite, wavelength in the direction of the magnetic field and both conditions are appropriate to the drift wave instability - our main concern - and to flute and flute-like instabilities.

III. SOLUTION

In lowest order equation (7) gives

$$\frac{\partial f_o}{\partial\varphi} = i f_o \frac{BD}{\omega_c} S_o - i \frac{e}{m}\frac{\psi_o}{\omega_c} \underset{\sim}{\nabla} S_o \cdot \underset{\sim}{c}_\perp \left[\frac{1}{B}\frac{\partial F_o}{\partial\mu} + \frac{\partial F_o}{\partial K} \right] \quad \ldots (9)$$

with solution

$$f_o = \frac{e}{m}\psi_o \left(\frac{1}{B}\frac{\partial F_o}{\partial\mu} + \frac{\partial F_o}{\partial K} \right) + g(\mu, K, \sigma, \alpha, \beta, \ell) e^{\frac{i}{\omega_c}(\underset{\sim}{c}_\perp \times \underset{\sim}{e}_1 \cdot \underset{\sim}{\nabla} S_o)} \quad \ldots (10)$$

where g is an arbitrary function.

In order to determine g we consider the next order in the expansion of (7) which provides an equation for f_1, the first order correction to f_o. This is

$$\frac{\partial}{\partial\varphi} f_1 e^{-\frac{i}{\omega_c}(\underset{\sim}{c}_\perp \times \underset{\sim}{e}_1 \cdot \underset{\sim}{\nabla} S_o)} = e^{-\frac{i}{\omega_c}(\underset{\sim}{c}_\perp \times \underset{\sim}{e}_1 \cdot \underset{\sim}{\nabla} S_o)} \Bigg\{ i \frac{\omega}{\omega_c} f_o + \frac{B}{\omega_c} D f_o$$

$$+ \frac{iB}{\omega_c} f_o DS_1 - \frac{\underset{\sim}{\nabla}\Phi_1}{B} \cdot \left[\frac{\underset{\sim}{c}_\perp}{B}\frac{\partial}{\partial\mu} + \frac{\underset{\sim}{e}_1 \times \underset{\sim}{c}_\perp}{c_\perp^2}\frac{\partial}{\partial\varphi} \right] f_o - \frac{\underset{\sim}{\nabla}\psi_o}{B} \cdot \left[\sigma q \underset{\sim}{e}_1 \frac{\partial F_o}{\partial K} + \underset{\sim}{c}_\perp \left(\frac{1}{B}\frac{\partial F_o}{\partial\mu} + \frac{\partial F_o}{\partial K} \right) \right]$$

$$- \frac{i\psi_o}{B} \underset{\sim}{\nabla} S_1 \cdot \left[\sigma q \, \underset{\sim}{e}_1 \frac{\partial F_o}{\partial K} + \underset{\sim}{c}_\perp \left(\frac{1}{B}\frac{\partial F_o}{\partial\mu} + \frac{\partial F_o}{\partial K} \right) \right] - \frac{i\psi_1}{B} (\underset{\sim}{c}_\perp \cdot \underset{\sim}{\nabla} S_o) \left(\frac{1}{B}\frac{\partial F_o}{\partial\mu} + \frac{\partial F_o}{\partial K} \right)$$

$$- \frac{i\psi_o}{B} \underset{\sim}{\nabla} S_o \cdot \left[\underset{\sim}{c}_\perp \left(\frac{1}{B}\frac{\partial F_1}{\partial\mu} + \frac{\partial F_1}{\partial K} \right) + \frac{\underset{\sim}{e}_1 \times \underset{\sim}{c}_\perp}{c_\perp^2}\frac{\partial F_1}{\partial\varphi} \right] \Bigg\}. \quad \ldots (11)$$

In order that f_1 be periodic in φ the right hand side of equation (11) must vanish when averaged over φ and when f_o is expressed by (10) this condition leads to an equation for g. After a great deal of algebra, this can be reduced to the relatively simple form

$$\sigma q \frac{\partial}{\partial\ell}\left(g + \frac{e}{m}\psi_o J_o(z) \frac{1}{B}\frac{\partial F_o}{\partial\mu} \right) + i \left(g + \frac{e}{m}\psi_o J_o(z)\frac{1}{B}\frac{\partial F_o}{\partial\mu} \right)\left(\omega + \sigma q \frac{\partial S_1}{\partial\ell} + \underset{\sim}{V}d \cdot \underset{\sim}{\nabla} S_o + \underset{\sim}{V}_E \cdot \underset{\sim}{\nabla} S_o \right)$$

$$- i \frac{e}{m}\psi_o J_o(z) \left(\frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla} F_o \cdot \underset{\sim}{\nabla} S_o}{\omega_c} - \omega \frac{\partial F_o}{\partial K} \right) = 0 \quad \ldots (12)$$

where J_o is the zero order Bessel function, and

$$V_d = \frac{\underset{\sim}{e}_1 \times (q^2 \frac{\partial \underset{\sim}{e}_1}{\partial\ell} + \mu\underset{\sim}{\nabla} B)}{\omega_c}$$

$$V_E = - \frac{\underset{\sim}{\nabla}\Phi_1 \times \underset{\sim}{B}}{B^2}$$

$$z = \frac{c_\perp}{\omega_c} |\underset{\sim}{\nabla} S_o|$$

The general solution of (12) is

$$g\, e^{\frac{i}{\lambda}(S_0 + \lambda S_1)} = e^{-i\sigma M(\ell_0,\ell)} \, h(\mu, K, \sigma, \alpha, \beta) - \frac{e}{m} \psi J_0 \frac{1}{B} \frac{\partial F_0}{\partial \mu}$$

$$- i \frac{e}{m} e^{-i\sigma M(\ell_0,\ell)} \left(\omega \frac{\partial F_0}{\partial K} - \frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla} F_0 \cdot \underset{\sim}{\nabla} S_0}{\omega_c} \right) \int_{\ell_0}^{\ell} \sigma \psi J_0 \, e^{i\sigma M(\ell_0,\ell')} \frac{d\ell'}{q}$$

... (13)

where we have written

$$\psi \equiv \psi_0 \, e^{\frac{i}{\lambda}(S_0 + \lambda S_1)} ,$$

and

$$M(x,y) = \int_x^y (\omega + \underset{\sim}{V}_d \cdot \underset{\sim}{\nabla} S_0 + \underset{\sim}{V}_E \cdot \underset{\sim}{\nabla} S_0) \frac{d\ell}{q} \equiv \int_x^y \frac{\omega^*}{q} \, d\ell . \qquad \text{... (14)}$$

Thus, although the ℓ dependence of g has been determined, it still contains an arbitrary function h. However this can be determined without going to higher order in the λ expansion. Three separate cases must be considered which correspond physically to particles which

(i) Circulate round closed field lines

(ii) Are reflected between magnetic mirrors

(iii) Trace out a magnetic surface with small rotational transform.

(i) Closed Field Lines

If in a toroidal confinement system, the field lines form closed curves, ℓ is a periodic variable and the function g must be periodic in ℓ, i.e. $g(\ell) - g(\ell + L) = 0$ where L is the length of the field line. Using this condition to fix the unknown function h one finds

$$g\, e^{\frac{i}{\lambda}(S_0 + \lambda S_1)} = -\frac{e}{m} \left\{ \psi J_0 \frac{1}{B} \frac{\partial F_0}{\partial \mu} + \frac{i\sigma \left(\omega \frac{\partial F_0}{\partial K} - \frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla} F_0 \cdot \underset{\sim}{\nabla} S_0}{\omega_c}\right)}{(e^{i\sigma M_0} - 1)} \oint_{\ell}^{\ell + L} \psi J_0 \, e^{i\sigma M(\ell,\ell')} \frac{d\ell'}{q} \right\}$$

... (15)

where $M_0 = \oint \frac{\omega^*}{q} d\ell$. We have taken the limit of integration ℓ_0 in (13) at the same position as the independent variable ℓ (since f_0 and g are, in fact, independent of the choice of ℓ_0), in order to write (15) in its most concise form.

(ii) Mirror Containment

If over part of the line of force $\mu B > (K - \frac{e}{m} \Phi)$ then the corresponding particles undergo mirror reflection. This may occur for some particles in a toroidal system or for all particles in a mirror system. As discussed in H.T.H.(a), the distribution functions corresponding to $\sigma = +1$ and $\sigma = -1$ are then no longer independent and if particles of a certain μ, K have turning points ℓ_1 and ℓ_2 then

$$g(\mu, K, \ell_i, \sigma = +1) = g(\mu, K, \ell_i, \sigma = -1).$$

This condition again determines h but it is now more convenient to identify the arbitrary lower limit of integration ℓ_o with one of the turning points ℓ_i; then

$$g\, e^{\frac{i}{\lambda}(S_o + \lambda S_1)} = -\frac{e}{m}\, e^{-i\sigma M(\ell_1,\ell)} \left(\omega \frac{\partial F_o}{\partial K} - \frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla} F_o \cdot \underset{\sim}{\nabla} S_o}{\omega_c}\right) \left\{ \frac{\int_{\ell_1}^{\ell_2} \psi J_o \cos M(\ell',\ell_2) \frac{d\ell'}{q}}{\sin M(\ell_1,\ell_2)} \right.$$

$$\left. + i\sigma \int_{\ell_1}^{\ell} \psi J_o\, e^{i\sigma M(\ell_1,\ell')} \frac{d\ell'}{q} \right\} - \frac{e}{m} \psi J_o \frac{1}{B} \frac{\partial F_o}{\partial \mu} \qquad \text{... (16)}$$

(iii) Magnetic Surfaces with Small Rotational Transform

In a toroidal confinement system the lines of force need not close on themselves: instead, if followed indefinitely, a line of force may trace out a magnetic surface. In this case the periodicity condition on the distribution function of particles which do not undergo reflection, must be reconsidered.

First it is necessary to introduce a 'cut' in the torus to render the α, β coordinate system single valued. Then in following a line of force across the cut there is a discontinuity in the (α, β) value. In fact it is convenient to make one of the coordinates (say β) a magnetic surface coordinate which does not change in crossing the cut; then the other coordinate α changes by an amount ι - a generalised rotational transform, and we shall assume that this transform is "small" in the sense discussed in H.T.H. (a), i.e., is to be treated as $O(\lambda)$. Instead of the condition $g(\ell) - g(\ell + L) = 0$ we now have the condition

$$\left[g\, e^{iS_o/\lambda} \right]_{\ell} \left[g\, e^{iS_o/\lambda} \right]_{\ell + L} = O(\lambda) \qquad \text{... (17)}$$

where

$$S_o(\ell) - S_o(\ell + L) = \iota \frac{\partial S_o}{\partial \alpha} \qquad \text{... (18)}$$

These conditions determine the unknown function h and if we again identify the arbitrary lower limit ℓ_o with the running point ℓ we find

$$g\, e^{iS_1}\, e^{iS_o/\lambda} = -\frac{e}{m} \left\{ \psi J_o \frac{1}{B} \frac{\partial F_o}{\partial \mu} + \frac{i\sigma \left(\omega \frac{\partial F_o}{\partial K} - \frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla} F_o \cdot \underset{\sim}{\nabla} S_o}{\omega_c}\right)}{\left(e^{i(\sigma M_o + \iota \frac{\partial S_o}{\partial \alpha})} - 1\right)} \oint_{\ell}^{\ell + L} \psi J_o\, e^{i\sigma M(\ell,\ell')} \frac{d\ell'}{q} \right\} \qquad \text{... (19)}$$

This equation gives the generalisation of (15) to the case of magnetic surfaces with small rotational transform. It is clear that when $\iota \to 0$ (19) reduces to (15).

Before proceeding further it is convenient to summarise the results of this section. These show that if an arbitrary equilibrium is perturbed by an arbitrary electrostatic potential

$$\psi = \hat{\psi}\,(\underset{\sim}{x})\; e^{iS(\underset{\sim}{x})/\lambda}\; e^{i\omega t}$$

then the perturbed distribution function is given by $\hat{f}\; e^{iS(\underset{\sim}{x})/\lambda}\; e^{i\omega t}$ where

$$\hat{f} = + \frac{e}{m}\,\psi_0 \left(\frac{1}{B}\frac{\partial F_0}{\partial \mu} + \frac{\partial F_0}{\partial K} \right) + g\; e^{\frac{i}{\omega_c}(\underset{\sim}{c}_\perp \times \underset{\sim}{e}_1 \cdot \underset{\sim}{\nabla} S_0)} + O(\lambda) \qquad \ldots (20)$$

and g is defined by (15) for particles circulating round closed field lines, by (16) for mirrored particles and by (19) for particles tracing out a magnetic surface. Later we shall use these expressions to derive stability criteria and a dispersion equation which determines the normal modes of perturbation (i.e. the functions ψ_0, S_0, S_1 and the corresponding complex normal frequencies ω).

The Perturbed Charge Density

To obtain stability criteria the perturbed charge density corresponding to (15), (16) or (19) is needed. This is given in general by

$$\rho = \sum_{j,\sigma} e_j \int \frac{B}{q}\, d\mu\, dK\, d\varphi\, f_0\; e^{iS/\lambda}\, e^{i\omega t} + O(\lambda) \qquad \ldots (21)$$

i.e. using (20)

$$\rho = 2\pi\, e^{i\omega t} \sum_{j,\sigma} \frac{e_j^2}{m_j} \int \frac{B}{q}\, d\mu\, dK \left[\psi \left(\frac{1}{B}\frac{\partial F_0}{\partial \mu} + \frac{\partial F_0}{\partial K} \right) + \frac{m}{e} J_0(z) g\, e^{iS/\lambda} \right] + O(\lambda) \qquad \ldots (22)$$

Because the expression valid for closed lines can always be obtained from the 'magnetic surface' expression by putting $\iota = 0$ we need write down only the expression for the charge density valid in cases (ii) and (iii).

Case (ii) Mirror Containment

For mirrored particles the equilibrium necessarily has the important property that $F_0(\sigma = +1) = F_0(\sigma = -1)$ so that the total charge perturbation is

$$\rho = 4\pi\, e^{i\omega t} \sum_j \frac{e_j^2}{m_j} \int \frac{B}{q}\, d\mu\, dK \Bigg\{ \psi \frac{\partial F_0}{\partial K} + \psi(1 - J_0^2)\, \frac{1}{B}\frac{\partial F_0}{\partial \mu}$$

$$- J_0 \left(\omega \frac{\partial F_0}{\partial K} - \frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla} F_0 \cdot \underset{\sim}{\nabla} S_0}{\omega_c} \right) \left[\frac{\cos M(\ell_1,\ell)}{\sin M(\ell_1,\ell_2)} \int_{\ell_1}^{\ell_2} \psi J_0 \cos M(\ell,\ell_2) \frac{d\ell}{q} - \int_{\ell_1}^{\ell} \psi J_0 \sin M(\ell,\ell') \frac{d\ell'}{q} \right] \Bigg\} \qquad \ldots (23)$$

Cases (i) and (iii), Closed Lines and Magnetic Surfaces

$$\rho = 2\pi e^{i\omega t} \sum_{j,\sigma} \frac{e_j^2}{m_j} \int \frac{B}{q}\, d\mu\, dK \left\{ \psi \frac{\partial F_o}{\partial K} + (1 - J_o^2)\, \psi \frac{1}{B} \frac{\partial F_o}{\partial \mu} \right.$$

$$\left. - i\sigma J_o \frac{\left(\omega \frac{\partial F_o}{\partial K} - \frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla} F_o \cdot \underset{\sim}{\nabla} S_o}{\omega_c}\right)}{\left(e^{i\sigma M_o + i\left(\iota \frac{\partial S_o}{\partial \alpha}\right)} - 1\right)} \int_{\ell}^{\ell+L} \psi J_o\, e^{i\sigma M(\ell,\ell')} \frac{d\ell'}{q} \right\} \quad \ldots (24)$$

IV. STABILITY THEORY

Some sufficient conditions for stability can be obtained directly from the perturbed charge density, expressed in terms of the perturbed potential ψ as described in the preceding sections, and Poisson's equation

$$\frac{1}{4\pi} \nabla^2 \psi + \rho(\omega, \psi) = 0 \ . \quad \ldots (25)$$

If ψ is any solution of this equation then we can define the functional

$$Q(\omega) = \frac{1}{4\pi} \int |\nabla\psi|^2\, d^3x - \int \psi^* \rho(\omega, \psi)\, d^3x \quad \ldots (26)$$

and if the system were unstable there would exist a function ψ for which the equation $Q(\omega) = 0$ has a solution ω, in the lower half of the complex plane, (this being the unstable eigenvalue appropriate to the solution ψ). A Nyquist plot of $Q(\omega)$ shows that $Q(\omega) = 0$ can have no such unstable solutions if

(i) $$P \equiv - \text{Im}\ \omega \int \psi^* \rho\, d^3x < 0 \quad \ldots (27)$$

for all real finite ω (approached from negative imaginary side) and if

(ii) $$\lim_{|\omega| \to \infty} Q(\omega) > 0 \ . \quad \ldots (28)$$

The first of these conditions essentially states that the power transfer from the electrostatic field to the plasma is positive, so that the oscillation is damped. This condition alone was used by KRALL and ROSENBLUTH (1965b) in their analysis of 'Minimum-B' type equilibria in plane slab geometry. The need to supplement this by a second condition was noted (also for the plane slab) by WIMMEL and SAISON (1966).

From the expressions for the charge density given in Section III, one can show by a "stationary phase" argument, that when λ is small the dominant contributions to

$(\omega\int\psi^*\rho)$ arise where ∇S_o is purely real and that the only imaginary contributions to $(\omega\int\psi^*\rho)$ come from the singular points of the integrand where $\rho\to\infty$.

These singularities, or resonances, arise, for mirror trapped particles when

$$M(\ell_1,\ell_2)\equiv\int_{\ell_1}^{\ell_2}(\omega+\underset{\sim}{V}_d\cdot\underset{\sim}{\nabla}S_o+\underset{\sim}{V}_E\cdot\underset{\sim}{\nabla}S_o)\frac{d\ell}{q}=\pi n \qquad \dots (29)$$

and in the closed line or magnetic surface case when

$$\sigma\oint(\omega+\underset{\sim}{V}_d\cdot\underset{\sim}{\nabla}S_o+\underset{\sim}{V}_E\cdot\underset{\sim}{\nabla}S_o)\frac{d\ell}{q}+\iota\frac{\partial S_o}{\partial\alpha}=2\pi n \qquad \dots (30)$$

and are the generalisations of the resonance between a plane wave $\exp i(\omega t-kx)$ and a particle moving with uniform speed ω/k, which occurs in the plane slab calculations. The resonance process in general geometry is that between an electrostatic oscillation and particles whose motion is almost periodic (i.e. periodic when $\lambda\to0$) with quasi-period equal to an integer multiple of the period of electrostatic oscillation.

The contribution to $\mathrm{Im}\,(\omega\int\psi^*\rho)$ arising from a singularity can easily be evaluated by writing

$$M(\ell_1,\ell_2)=\pi n+T(\mu,\ K,\ \alpha,\ \beta) \qquad \dots (31)$$

and $\omega=\omega-i\Upsilon$. Then taking the limit $\Upsilon\to0$ the integral becomes, for particles trapped between mirrors

$$P=\pi^2\sum_j\frac{e_j^2}{m_j}\int d\alpha\,d\beta\int d\mu\,\frac{|R|^2}{\left|\frac{\partial T}{\partial K}\right|}\left(\omega^2\frac{\partial F_o}{\partial K}-\omega\,\frac{\underset{\sim}{e}_1\times\underset{\sim}{\nabla}F_o\cdot\underset{\sim}{\nabla}S_o}{\omega_c}\right) \qquad \dots (32)$$

where

$$R=\int_{\ell_1}^{\ell_2}\psi\,J_o\cos M(\ell,\ell_2)\,\frac{d\ell}{q} \qquad \dots (33)$$

and where the integrand has to be evaluated at those values of K for which $T=0$.

For particles on closed field lines or on a magnetic surface the contribution to $\mathrm{Im}\,\omega\int\psi^*\rho$ is again given by (32) but with T now defined by

$$T=\sigma M_o+\iota\frac{\partial S_o}{\partial\alpha}-2n\pi \qquad \dots (34)$$

and R by

$$R=\oint_{\ell}^{\ell+L}\psi\,J_o\,e^{i\sigma M(\ell,\ell')}\,\frac{d\ell'}{q} \qquad \dots (35)$$

Some Elementary Criteria

We can now use the basic stability criteria (i) and (ii) to derive simpler criteria expressed in terms of the equilibrium distribution above. The first example concerns the special Minimum-B equilibria (TAYLOR (1963)), $F = F(\mu, K)$. For such equilibria a sufficient set of criteria for stability against all low frequency modes, including those of short perpendicular wavelength becomes

$$P = \pi^2 \sum \frac{e^2}{m} \int d\alpha\, d\beta \int d\mu \frac{|R|^2}{\left|\frac{\partial T}{\partial K}\right|} \left[\omega^2 \frac{\partial F_o}{\partial K} \right] < 0 \qquad \ldots (36)$$

$$Q_\infty = \frac{1}{4\pi\lambda^2} \int |\psi|^2 \; |\nabla S_o|^2 \, d^3x - 2\pi \int d^3x \; |\psi|^2 \sum \frac{e^2}{m} \int \frac{B}{q} \, d\mu dK \left(\frac{1}{B} \frac{\partial F_o}{\partial \mu} + \frac{\partial F_o}{\partial K} \right) (1 - J_o^2(z)) > 0 \qquad \ldots (37)$$

The first is satisfied if $\frac{\partial F_o}{\partial K} < 0$; the second could be satisfied if e.g.

$$\frac{1}{B} \frac{\partial F_o}{\partial \mu} + \frac{\partial F_o}{\partial K} < 0 \qquad \ldots (38)$$

but this precludes the existence of a loss-cone which is essential for the confinement of simple Minimum-B equilibria. Accordingly it seems that (37) is already in its simplest and most useful form. Together with $\frac{\partial F_o}{\partial K} < 0$ it is sufficient for stability of Minimum-B type equilibria at all densities against all low frequency instabilities of both long and short wavelength.

At very low plasma density, where $\omega_p^2 \ll \omega_{ci}^2$, the condition $\frac{\partial F_o}{\partial K} < 0$ alone is sufficient for stability since $Q(\omega)$ is then dominated by the positive term $\int |\nabla S_o|^2 \; |\psi|^2 \, d^3x$.

For more general equilibria $F_o(\mu, K, \alpha, \beta)$ there does not appear to be any particularly succinct expression guaranteeing stability against <u>all</u> low frequency instabilities, but in the limit of long wavelength (equivalent to $\nabla S_o \to 0$ in our expansion scheme) a concise discussion can again be given. In this limit Q_∞ is essentially positive at all densities and if we write F_o in the long term equilibrium form $F_o(\mu, K, J)$ where J is the longitudinal invariant, then P can be written

$$P = \pi^2 \sum \frac{e^2}{m} \int d\alpha d\beta \int d\mu \frac{|R|^2}{\left|\frac{\partial T}{\partial K}\right|} \left[\omega^2 \left(\frac{\partial F_o}{\partial K}\right)_{\mu J} + \omega \left(\frac{\partial F_o}{\partial J}\right)_{\mu K} \left(\omega\tau - \frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla} J \cdot \underset{\sim}{\nabla} S_o}{\omega_c} \right) \right] \qquad \ldots (39)$$

where the integrand in (39) has to be evaluated at resonances, i.e. at those values of K for which

$$M_o = \left[\omega\tau - \frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla} J \cdot \underset{\sim}{\nabla} S_o}{\omega_c} \right] = 2n\pi \qquad \ldots (40)$$

where $\tau \equiv \oint \frac{d\ell}{q}$ is the transit time along the field. In the long wavelength limit we distinguish between two types of instability:

(i) Those whose phase velocity across the field is large compared to typical particle drift velocities, i.e. those for which

$$\omega\tau \gg \left|\frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla}J \cdot \underset{\sim}{\nabla}S_o}{\omega_c}\right| .$$

For such instabilities the $n = 0$ resonance can occur only for a small group of particles in the high energy tail of the distribution and gives a negligible contribution to P. Thus only the $n \neq 0$ resonances are important and for these the ∇S_o term is negligible and P becomes

$$P = \pi^2 \sum \frac{e^2}{m} \int d\alpha d\beta \int d\mu \frac{|R|^2}{\left|\frac{\partial T}{\partial K}\right|} \omega^2 \left(\frac{\partial F_o}{\partial K}\right)_{\mu\alpha\beta} . \qquad \ldots (41)$$

(ii) Those whose phase velocity across the field is comparable to or less than typical particle drifts, i.e. for which

$$\omega\tau \lesssim \left|\frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla}J \cdot \underset{\sim}{\nabla}S_o}{\omega_c}\right| .$$

For these instabilities the $n = 0$ resonance is important and contributes

$$P_o = \pi^2 \sum \frac{e^2}{m} \int d\alpha d\beta \int d\mu \frac{|R|^2}{\left|\frac{\partial T}{\partial K}\right|} \omega^2 \left(\frac{\partial F_o}{\partial K}\right)_{\mu J} \qquad \ldots (42)$$

while for typical particles the long wave limit $\nabla S_o \to 0$ implies also $\omega\tau \to 0$ so that $n \neq 0$ resonances involve only particles with anomalously long transit times.

Thus a set of sufficient stability criteria for long wavelength instabilities, except those driven by a small class of long transit time particles is

$$\left(\frac{\partial F_o}{\partial K}\right)_{\mu J} < 0 \quad \text{and} \quad \left(\frac{\partial F_o}{\partial K}\right)_{\mu\alpha\beta} < 0 . \qquad \ldots (43)$$

The first condition is the usual 'flute' instability criterion. The second condition is usually introduced as a justification for certain stages in the development of small m/e energy principles (KRUSKAL and OBERMAN (1958), TRUBNIKOV (1962), BROSSIER, LAVAL, PELLAT and VUILLEMIN (1964)), but is now seen to be connected with the $n \neq 0$ resonances and with instabilities which while slow compared to cyclotron frequencies do not satisfy $\omega\tau < 1$. For this reason it does not arise in theories which assume $\omega\tau \ll 1$ from the outset (ROSENBLUTH and ROSTOKER (1959)) or invoke the second longitudinal invariant (ANDREOLETTI (1963), TAYLOR (1964)). We note in passing that these instabilities with $\omega\tau \gtrsim 1$ require a small component of electric field parallel to $\underset{\sim}{B}$.

In the case of particles which trace out a magnetic surface the preceding results still apply, but in their derivation the longitudinal invariant J must be replaced by its modified form introduced in H.T.H. namely

$$J^{**} = \oint (q\underset{\sim}{e}_1 + \frac{e}{m}\beta\underset{\sim}{\nabla}\alpha)\cdot \underset{\sim}{d\ell} \;, \qquad \ldots (44)$$

and the equilibrium is in this case $F(\mu, K, J^{**})$. There is thus no discontinuity in the stability criteria as one passes from closed lines to magnetic surfaces. However this is not to say that one situation may not be more stable than the other, for the resonance conditions are different.

V. DISPERSION EQUATION

This far we have used the expression for the charge density ρ to derive certain sufficient criteria for stability. It is not immediately apparent that it also leads to definite values of ω. In the plane slab model ω is determined by a straightforward W.K.B. analysis; in the present case the eigenvalue ω can in principle be determined by an extension of the W.K.B. method to the present situation in which a small expansion parameter is valid in two directions, perpendicular to B but not valid in the third direction. This also provides a dispersion equation for instabilities in arbitrary equilibria in a form which permits direct comparison with that for the plane slab model (MIKHAILOVSKII and RUDAKOV (1963), GALEEV, ORAEVSKII and SAGDEEV (1963), KADOMTSEV and TIMOFEEV (1963)) and its extension (KRALL and ROSENBLUTH (1965a), COPPI, LAVAL, PELLAT and ROSENBLUTH (1967), ROHLENA and JUKES (to be published)).

We start from the Poisson equation, which in its lowest approximation in λ can be written

$$\frac{\psi_o}{4\pi\lambda^2}(\nabla S_o)^2 = \sum_{j,\sigma} e_j \int \frac{B}{q}\, d\mu\, dK\, d\varphi\, f_o \qquad \ldots (45)$$

where f_o has already been expressed in terms of ψ_o, S_o, S_1. If we take the closed line situation as an example then all physical quantities are periodic in ℓ so that we can write

$$\frac{1}{\sqrt{B}}\psi_o\, e^{iS_1} = \sum_{n=1}^{\infty} a_n\, e^{2\pi i n\cdot\frac{\ell}{L}} . \qquad \ldots (46)$$

Not all quantities appearing in the theory are periodic in ℓ however, and $\exp i\, M(\ell_o,\ell)$ increases by $\exp i\, M_o$ in one circuit of the system. Nevertheless $\exp\,(i\, M(\ell_o,\ell) - i\,\frac{\ell}{L} M_o)$ <u>is</u> periodic so we can write

$$\sqrt{B}\,\frac{J_o(z)}{q}\, e^{i\sigma M(\ell_o,\ell)} = \sum_{n=1}^{\infty} b_n\, e^{2\pi i\sigma n\frac{\ell}{L}}\, e^{i\sigma\frac{\ell}{L}M_o} . \qquad \ldots (47)$$

The functions $a_n(\alpha,\beta)$ and $b_n(\alpha,\beta,\mu,K\sigma)$ are given by the usual inverse relationship; a_n depends on S_1 and b_n on S_o.

Using these expressions (46) and (47), Poisson's equation is

$$\sum_n a_n Q_{nm} = 0 \qquad \ldots (48)$$

where

$$Q_{nm} = \oint d\ell \, e^{2\pi i(n-m)\frac{\ell}{L}} \left\{ (\nabla S_o)^2 - \sum_{j,\sigma} \frac{e_j^2}{m_j} \int \frac{B}{q} \, d\mu dK \left[\frac{\partial F_o}{\partial K} + (1 - J_o^2) \frac{1}{B} \frac{\partial F_o}{\partial \mu} \right] \right\}$$

$$+ \frac{L}{2\pi} \sum_p \sum_{j,\sigma} \frac{e_j^2}{m_j} \int d\mu \, dK \frac{b_p^* \, b_{p+\sigma(m-n)} \left(\omega \frac{\partial F_o}{\partial K} - \frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla} F_o \cdot \underset{\sim}{\nabla} S_o}{\omega_c} \right)}{(p + \sigma m + M_o/2\pi)} \qquad \ldots (49)$$

and the dispersion equation we are seeking is

$$\det |Q_{nm}| = 0 \ . \qquad \ldots (50)$$

In the equation only $\nabla S_o(\alpha,\beta)$ and ω appear apart from equilibrium quantities, and it therefore represents an appropriate generalisation of the usual one-dimensional W.K.B. equation. For each value of ω a value of (∇S_o) is determined by (50) and must satisfy a two-dimensional 'phase integral condition' such as

$$\iint (\nabla S_o)^2 \, d\alpha d\beta = 4\pi n \qquad \ldots (51)$$

which in turn determines ω. In fact in symmetric, but entirely realistic configurations, such as the toroidal multipole, perturbations of different azimuthal mode number are independent and S_o can be written

$$S_o = k\theta + S_o(\psi) \qquad \ldots (52)$$

where ψ is the magnetic flux. Then the phase integral condition reduces to the usual one-dimensional form.

Each element of the matrix Q_{nm} has a form similar to the dispersion equation itself in a plane slab. If the present general method were applied to the plane slab then we would have

$$b_s \sim \frac{\sqrt{B}}{q} J_o \cdot \delta_{so}$$

and Q_{nm} is the diagonal matrix with

$$Q_{nm} = \left[(\nabla S_o)^2 - \sum \frac{e^2}{m} \int \frac{B}{q} \, d\mu \, dK \left[\frac{\partial F_o}{\partial K} + (1 - J_o^2) \frac{1}{B} \frac{\partial F_o}{\partial \mu} \right] \right.$$

$$\left. + \sum \frac{e^2}{m} \int \frac{B}{q} \, d\mu dK \, J_o^2 \, \frac{\left(\omega \frac{\partial F_o}{\partial K} - \frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla} F_o \cdot \underset{\sim}{\nabla} S_o}{\omega_c} \right)}{(\omega + \underset{\sim}{V}_d \cdot \underset{\sim}{\nabla} S_o + \underset{\sim}{V}_E \cdot \underset{\sim}{\nabla} S_o + 2\pi nq/L)} \right] \delta_{nm} \ . \qquad \ldots (53)$$

Then the nth root of the general dispersion equation (50) corresponds to the usual plane slab solution with $k_{\parallel} = 2\pi n/L$. In a similar way if the plane slab model is modified by a sinusoidal periodic gravity as in COPPI et al. (1967) the general dispersion equation reduces to their tri-diagonal form.

VI. CONCLUSIONS AND FURTHER APPLICATIONS

We have shown that the method used to discuss arbitrary plasma equilibria can be extended to deal with the stability of these arbitrary equilibria. It leads to an expression for the charge density arising in any perturbation of an arbitrary equilibrium configuration and this in turn provides a framework for the stability analysis of realistic equilibria.

The results we have described already allow a comparison of arbitrary equilibria with the plane slab model, both in terms of the physical processes and of the dispersion equation itself. We have shown that the important resonance in the plane slab model, between a plane wave and a particle moving uniformly at its phase velocity, is replaced in general equilibria by resonance between a normal mode of oscillation of the system and the quasi-cyclic particle motion which must exist in general confined equilibria. A pair of basic criteria sufficient to ensure the stability of an arbitrary equilibrium against all low-frequency oscillations has also been derived. These lead to simple conditions on the distribution function F which will ensure stability; many of these have previously been obtained individually but our derivation emphasises their inter-relation. We have also derived a formal dispersion equation for oscillations in general equilibria in a form which allows direct comparison with, and reduces in the appropriate limit to, that obtained in simpler idealised models.

To conclude this discussion of stability of general equilibria we describe briefly how the results of this paper will be used in a subsequent report to discuss micro-instabilities in real confinement systems. So far our results are not restricted to any particular shape of confinement system nor to any particular plasma distribution - both geometry and velocity distributions are arbitrary. However, if we specialise to an axi-symmetric system, and to a near Maxwellian velocity distribution

$$F_o = \left(\frac{m}{2\pi T}\right)^{3/2} n(\psi)e^{-Km/T}$$

then the most important stability criterion (30), related to the power transfer, can be expressed in the form

$$P = -\sqrt{\frac{\pi}{8}} \sum_{i,e} \frac{e^2}{m} \left(\frac{m}{T}\right)^{5/2} \int d\psi \; n(\psi) \; \omega(\omega + kV_J) \int d\mu \frac{|R|^2}{\left|\frac{\partial T}{\partial K}\right|} e^{-\frac{mK}{T}}$$

where

$$V_J = \frac{T_i}{e_i} \frac{1}{n} \frac{dn}{d\psi} (\underset{\sim}{e}_1 \times \underset{\sim}{\nabla}\psi \cdot \underset{\sim}{\nabla}\theta) \quad \text{and} \quad k = \frac{\partial S_o}{\partial \theta} \; .$$

From this expression it is not difficult to determine the stability of the system if one knows what value to use for (real) ω, and the appropriate value can be determined as follows. The expression (21) or (22), together with Poisson's equation, leads to a complicated integro-differential equation for ω, involving $\psi_o e^{iS_1}$ and ∇S_o, but this can be simplified by noting that for a wide class of microinstabilities the quantity M_o is $\ll 1$ for electrons and $\gg 1$ for ions. Then a straightforward expansion in $M_o \ll 1$ for the electrons yields

$$\rho_e = \frac{e^2 n}{T_e} \left[-\psi + \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{m}{T}\right)^{3/2} (\omega + kV_{Je}) \int \frac{B}{q} \, d\mu dK \frac{\oint \psi \frac{d\ell}{q}}{\oint \omega^* \frac{d\ell}{q}} e^{-\frac{m}{T}K} \right]$$

For the ions one must deal with several rapidly oscillating functions of the form $\exp i M$, but we again note that after one circuit round a closed field line $\exp i M$ increases by $\exp i M_o$; consequently if a partial integration in ℓ is performed on the numerator then in the integrated parts the oscillating function $(\exp i M_o - 1)$ is <u>exactly</u> cancelled and only slowly varying quantities remain. In fact in this way, by repeated partial integration, one generates a formal series in

$$\frac{V_{thi}}{\omega^*} \frac{\partial}{\partial \ell}$$

where V_{thi} is the ion thermal speed. Thus the ion contribution to the perturbed charge is transformed to a differential expression

$$\rho_i \sim \frac{e^2 n}{T_i} \left\{ -\psi + \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{m}{T}\right)^{3/2} (\omega + kV_{Ji}) \int \frac{B}{q} \, d\mu dK \; e^{-\frac{m}{T}K} \frac{J_o}{\omega^*} \left[\psi J_o - q \frac{\partial}{\partial \ell} \frac{q}{\omega^*} \frac{\partial}{\partial \ell} \left(\frac{\psi J_o}{\omega^*}\right) \right] \right\}$$

so that ω (real) is determined by

$$\alpha \frac{\partial^2 \psi}{\partial \ell^2} + \beta \frac{\partial \psi}{\partial \ell} + \gamma \psi = \Lambda \langle \psi \rangle$$

whose coefficients are given in terms of the magnetic field and ∇S_o, ω, $\frac{1}{n}\frac{\partial n}{\partial \psi}$. Numerical solution of this equation is not difficult and forms the basis for a detailed investigation of the stability properties of a real confinement system.

In principle, one computes $\nabla S_0 (\omega, \psi)$ and ω is then determined by the W.K.B. phase integral over ψ. However within our approximation of small λ the required result can be found by putting $\nabla S_0 = 0$ and first computing a local eigenvalue $\omega(\psi)$ - then the required value corresponds to a stationary value of $\omega(\psi)$.

We have already carried out such calculations, in conjunction with D.L. Fisher and B. McNamara, for a simple octopole and a simple quadrupole. These, and other results, will be reported in a subsequent paper.

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