UNCLASSIFIED
(Approved for Publication)

CLM - P 158

INVARIANTS IN THE MOTION OF A CHARGED PARTICLE IN A SPATIALLY MODULATED MAGNETIC FIELD

by

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(Submitted for publication in J. Math. Phys.)

ABSTRACT

In this paper we study the effect of a spatial modulation in the magnetic field on invariants such as the magnetic moment $\mu = V_{\perp}^2/B$. In particular we investigate whether an invariant still exists when the wavelength of the modulation is comparable to the gyro radius of the particle. In an axially symmetric magnetic field, with a square wave modulation, the orbit equations reduce to algebraic relations convenient for numerical study. We find from such studies that orbits are of two types; (a) regular orbits which generate an invariant, (b) orbits which are quasi-ergodic. We have also calculated an invariant by perturbation theory with the depth of modulation of the field as a small parameter. For this we develop a modified form of perturbation theory which overcomes the difficulty of infinities arising at resonance between the perturbation and the cyclotron period. This difficulty in fact corresponds to a change in topology of the invariant curves. The invariant calculated from this theory shows very good agreement with the numerically computed orbits of type (a). The transition to quasi-ergodic behaviour cannot be predicted analytically but some indication of it may exist in the complex topology of the invariant curves in the ergodic regions.

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January, 1968. (MB)

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1. INTRODUCTION

The invariants of motion play an important role both in the trapping and containment of charged particles in magnetic confinement systems. For motion in sufficiently smooth fields, an important adiabatic invariant is the magnetic moment $\mu = V_{\perp}^2/B$, where $V_{\perp}$ is the particle velocity transverse to the magnetic field $\underset{\sim}{B}$. In this paper we consider the effect of a spatial modulation, superimposed on an otherwise uniform magnetic field, on the behaviour of such invariants. We are particularly interested in the situation where the wavelength of the modulation may be comparable with the gyromagnetic radius of the particle. In this case μ is no longer a valid invariant and we investigate whether any alternative invariant exists.

Interest in a possible invariant in a modulated magnetic field arose out of studies of the containment properties of magnetic traps incorporating such fields[1,2,3] but here we are concerned only with the invariant itself, not with any possible containment. Accordingly we consider a very simple field, with axial symmetry and without any end effects. The axial component of the magnetic field B_z is $B_0(1+\varepsilon\, g(z))$ where $g(z)$ is the periodic modulation with wavelength λ; there is no radial dependence so that $\nabla \times \underset{\sim}{B} \neq 0$, although $\nabla \cdot \underset{\sim}{B} = 0$.

The motion of a charged particle in this field is studied both numerically (Section 2) and analytically (Section 3). In the numerical studies $g(z)$ is taken to be a step function, alternately ± 1 with discontinuities at $z = n\,\lambda/2$. With this form of magnetic field the orbit equations reduce to a set of transfer matrices so that the orbit can be computed over many field periods with speed and accuracy.

The results of these numerical calculations show that, depending on their initial values, orbits can be classified into two types; (a) regular orbits which correspond to the existence of an invariant of the motion and (b) orbits which fill quasi-ergodically all the available part of phase space not mapped by orbits of type (a).

In the analytic investigation we study invariants using perturbation theory, with ε as a small parameter. Straightforward perturbation theory fails because the perturbation may 'resonate' with the cyclotron period of the unperturbed orbit, leading to the problem of 'vanishing denominators'. However, a modified form of perturbation theory is introduced which overcomes this difficulty and permits us to generate an adiabatic invariant J, as a series in ε. This is valid throughout phase space, even in the region of resonances.

Comparison of the first few terms of the invariant, $J \approx J_0 + \varepsilon J_1$, with the numerical computations shows excellent agreement with the regular orbits calculated numerically.

2. NUMERICAL COMPUTATIONS

It has been shown by Laing and Robson[4] that within a range of z for which $f(z) = 1 + \varepsilon g(z)$ has a constant value f_i, the orbit equations may be put in the form

$$\frac{d^2r}{d\tau^2} = \frac{r_o^4}{r^3} - r f_i^2 \qquad \ldots (1)$$

$$\frac{d^2z}{d\tau^2} = 0 \qquad \ldots (2)$$

We have used as a dimensionless time variable $\tau = \frac{1}{2}\omega t$, $\omega = eB_o/mc$ and the length r_o is related to the constant canonical momentum p_θ conjugate to the azimuthal coordinate θ about the axis of symmetry.

The general solution of (1) is

$$r^2 = \alpha_i + \beta_i \cos(2 f_i \tau + \varphi_i) \qquad \ldots (3)$$

where

$$\alpha_i^2 - \beta_i^2 = r_o^4 / f_i^2$$

At any time τ, the state of motion of the particle is thus described by only two parameters α_i, φ_i. At a discontinuity in f, both r and $\frac{dr}{d\tau}$ are continuous so that at the boundary between region (i) and region (i + 1).

$$\alpha_i + \beta_i \cos\varphi_i = \alpha_{i+1} + \beta_{i+1} \cos\psi_{i+1}$$

and

$$f_i \beta_i \sin\varphi_i = f_{i+1} \beta_{i+1} \sin\psi_{i+1} \qquad \ldots (4)$$

Here we have set $\tau = 0$ at the boundary between the i^{th} and $(i + 1)^{th}$ regions and ψ_i, φ_i denote the phase at the beginning and end respectively of the i^{th} region, thus

$$\varphi_i = \psi_i + \lambda f_i/u_i \qquad \ldots (5)$$

where $u_i = (dz/d\tau)_i$ and can be determined from (3) and the energy equation. This takes the form

$$\left(\frac{dr}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2 + \left(\frac{r_o^2}{r} - r f_i\right)^2 = v^2 \qquad \ldots (6)$$

where we have written $v = 2V/\omega$. A more detailed account of the procedure for computing the trajectory of a particle is given by Dunnett et al[1].

Instead of using φ and α to represent the orbit the results are presented in terms of the phase φ and the normalized magnetic moment

$$\xi_i = v_\perp^2/v^2 = 1 - \frac{1}{v^2}\left(\frac{dz}{d\tau}\right)^2$$

The value of ξ is constant within a region of uniform magnetic field, and is related to α_i by

$$\xi = 2(f_i/v^2)\,(\alpha_i\, f_i - r_o^2)\ .$$

In the calculation, then, values of $(\xi\,.\,\varphi)$ are computed in successive periods of the magnetic field and the successive points (ξ_i,φ_i) are plotted to give a representation of the 'orbit'. Some typical results are shown in Figs.1-3. These all refer to orbits with $v = 2$, $\lambda = 2\pi$, $r_o = 2$ which were of particular interest in the containment problem studied by Dunnett et al[1]. They represent a particle injected parallel to the axis at a distance $r_o = 2$ from the axis with a velocity satisfying the resonance criterion $V = \omega\lambda/2\pi$, i.e., $v = 2$ for $\lambda = 2\pi$. The three figures are computed with amplitudes of the modulating field which are respectively 0.025, 0.05 and 0.10 of the main field.

From these diagrams it can be seen that orbits are of two distinct types. In some regions of the $(\xi\,,\varphi)$ plane the successive values of $(\xi_i\,,\,\varphi_i)$ along an orbit, lie on a smooth curve. These, type (a), orbits correspond to the existence of an invariant of motion. However for orbits in other regions of $(\xi\,,\varphi)$ the successive values of $(\xi_i\,,\,\varphi_i)$ do not lie on a regular curve but fill, quasi-ergodically, part of the $(\xi\,,\varphi)$ plane. In fact one of these, type (b), orbits eventually fills all the $(\xi\,,\varphi)$ plane except for regions already mapped by type (a) orbits and some regions of large ξ. (These excluded large ξ regions correspond to particle reflection at the field discontinuities). It should be stressed that with type (a) orbits each initial value $(\xi_o\,,\,\varphi_o)$ generates only one invariant curve, (except that it may generate a chain of 'islands', as shown). However with the type (b) quasi-ergodic orbits, any single initial state $(\xi_o\,,\varphi_o)$ will ultimately map the whole area. It is in this sense that we refer to the motion as quasi-ergodic.

The region occupied by type (a) 'regular' orbits decreases, and that occupied by type (b) 'quasi-ergodic' orbits increases, as the strength of the field modulation ε is increased. In Fig.1, where $\varepsilon = 0.025$ almost all possible orbits are regular and possess a valid invariant. In Fig.3, where $\varepsilon = 0.1$, almost the whole plane is mapped by a quasi-ergodic orbit and only a small class of orbits possess a valid invariant. In all cases there is a sharp transition between regions where an invariant exists and those where quasi-ergodic behaviour pertains.

This kind of behaviour is similar to that found in several other problems, notably in the quest for a third invariant of galactic motion[5]. It is worth noting, however, that in our present problem the motion for $\varepsilon = 0$ is unbounded whereas interest usually centres around motion which is periodic when $\varepsilon = 0$.

We note in passing that if μ were an invariant the orbits (ξ_i , φ_i) would be on horizontal straight lines ξ = constant.

3. THE INVARIANT

The results of the numerical calculations indicate that, at least for small ε, an invariant may exist even when the usual magnetic moment invariant is no longer valid. In this section we show how this invariant may be calculated by a perturbation expansion in ε. The essential problem is to devise a perturbation expansion which remains valid even when the modulated magnetic field gives a perturbation in resonance with the unperturbed cyclotron orbit, i.e., when its effect is no longer 'small'. In conventional perturbation calculations this resonance gives rise to vanishing denominators in the coefficients of the ε power series. We shall show how this difficulty may be overcome, in principle to any finite order in ε.

Perturbation Theory

The Hamiltonian describing the motion of a charged particle in the magnetic vector potential $\underset{\sim}{A}$ is given by

$$H = (\underset{\sim}{p} - e\underset{\sim}{A}/c)^2/2m$$

In our case, using cylindrical coordinates, the vector potential has only one non-zero component, $A_\theta = r\, B_z/2$. Using C for r_0^2, but otherwise that same notation as in Section 2, we may then write

$$H = \tfrac{1}{2}\left(p_r^2 + p_z^2 + [C/r - r\,(1 + \varepsilon g(z))]^2\right) \qquad \dots (7)$$

As we shall evaluate the invariant only to first order in ε, we express H approximately in the form $H_o + \varepsilon H_i$

$$H_o = \tfrac{1}{2}\left[p_r^2 + p_z^2 + (C/r - r)^2\right],$$

$$H_i = (r^2 - C)\; g(z) \qquad \dots (8)$$

H_o is thus the Hamiltonian for motion in a uniform magnetic field.

The object is to generate a new constant of the motion J which is to replace the constant p_z, or equivalently the constant ξ which exists when $\varepsilon = 0$. The system is still conservative and $H = H_o + \varepsilon H_i$ is a constant of the motion so that any other constant of motion J has the property $[J, H] = 0$, where $[\ ,\ ]$ is the Poisson bracket. Setting $J = J_o + \varepsilon J_1 + \ldots$, $H = H_o + \varepsilon H_1$ and expanding the Poisson bracket one obtains a set of recurrence equations, of which the first two are

$$[J_o, H_o] = 0$$

$$[J_1, H_o] + [J_o, H_1] = 0 \qquad \ldots (9)$$

We first perform a simplifying canonical transformation[6] $(r, p_r) \to (Q, P)$ so that $\frac{1}{2}[p_r^2 + (C/r - r)^2] \to P$. This is effected by the generating function

$$W = \int^r dr\, [2P - (C/r - r)^2]^{\frac{1}{2}} \qquad \ldots (10)$$

which gives

$$r = P + C + A \sin 2Q \qquad \ldots (11)$$

where

$$A = [(P + C)^2 - C^2]^{\frac{1}{2}}$$

The transformed Hamiltonian then becomes

$$H \to P + \tfrac{1}{2} p_z^2 + \varepsilon\Omega \qquad \ldots (12)$$

where

$$\Omega = (P + A \sin 2Q)\, g(z)$$

The recurrence equations (9) become, writing $p_z = p$ for brevity

$$\frac{\partial J_o}{\partial Q} + p \frac{\partial J_o}{\partial z} = 0 \qquad \ldots (13)$$

$$\frac{\partial J_1}{\partial Q} + p \frac{\partial J_1}{\partial z} + [J_o, \Omega] = 0 \qquad \ldots (14)$$

Equation (13) indicates that J_o is an arbitrary function of P, p, $pQ - z$. However dependence on the last of these is ruled out by requiring that J_o be periodic both in Q and in z for all p. In fact we need regard J_o only as an arbitrary function of p. Then, after a change of variables to $\alpha = pQ - z$, $\beta = z$, equation (14) may be written

$$\frac{\partial J_1}{\partial \beta} = \frac{1}{p} \frac{\partial J_o}{\partial p} \frac{\partial \Omega}{\partial z}$$

so that

$$J_1 = \frac{1}{p} \frac{\partial J_o}{\partial p} \int^\beta d\beta' \left[P + A \sin \frac{2}{p}(\alpha + \beta')\right] \frac{dg(\beta')}{d\beta'} \qquad \ldots (15)$$

Now $g(\beta)$ is a periodic function of period λ, so that the integral is unbounded in β whenever $p = \lambda/\pi n$. In the neighbourhood of shuch points our expansion in ε must apparently break down - the well known vanishing denominator problem.

This failure of perturbation theory near a resonance has a simple interpretation. It is a manifestation of the fact that at such a point the topology of the true invariant curves, J = constant, differs from that of the curves J_0 = constant. Generally, if ε is small the contours of $(J_0 + \varepsilon J_1)$ = constant can be topologically different to those of J_0 = constant only if J_1 is large. Thus the appearance of a large J_1 is simply the response of perturbation theory to the change in topology. Consequently a valid perturbation expansion <u>can</u> be obtained if J_0 is chosen so that a <u>small</u> J_1 <u>can</u> make the topology of the $(J_0 + \varepsilon J_1)$ curves differ from that of the J_0 curves. This is the case if $\partial J_0/\partial p$ vanishes at the points concerned.

We illustrate this by evaluating J_1 for the square-wave modulation used in the numerical calculation, $g(\beta') = \pm 1$. Then

$$\frac{dg}{d\beta'} = 2 \sum_{-\infty}^{\infty} (-1)^n \, \delta\,(\beta' - n\lambda/2)$$

and so

$$J_1 = \frac{2}{p}\left(\frac{\partial J_0}{\partial p}\right)\left\{\frac{1+(-1)^N}{2} + \frac{A}{2\cos(\lambda/2p)}\left[\sin\,\frac{(2\alpha-\lambda/2)}{p} + (-1)^N \sin\,\frac{2\alpha+(N+\frac{1}{2})\lambda)}{p}\right]\right\} + C(\alpha, p, P). \quad \ldots (16)$$

In (16), β has been chosen in the interval $[N\lambda/2, (N+1)\lambda/2]$ so that $\beta = N\lambda/2 + \Upsilon$, $0 < \Upsilon < \lambda/2$. The integration 'constant' C may conveniently be chosen so that terms independent of N disappear; then if J_1 is evaluated at the mid point of an interval,

$$J_1 = (-1)^N \frac{1}{p}\frac{\partial J_0}{\partial p}\left[P + \frac{A}{\cos(\lambda/2p)}\sin 2Q\right] \quad \ldots (17)$$

we now observe that, as expected, J_1 is unbounded at $p = o$, and at the zeros of $\cos(\lambda/2p)$ unless $\frac{\partial J_0}{\partial p}$ is chosen so that it vanishes at these points. A suitable choice for J_0 is

$$J_0 = p^3\left(\sin\,(\lambda/2p) - 6\,p/\lambda\,\cos\,(\lambda/2p)\right)$$

giving finally

$$\begin{aligned} J &\simeq J_0 + \varepsilon J_1 \\ &= p^3\left(\sin\,(\lambda/2p) - 6\,p/\lambda\,\cos\,(\lambda/2p)\right) \\ &\mp \tfrac{1}{2}\,\varepsilon\,\lambda\left(1 + 48\,(p/\lambda)^2\right)\left(P\cos\,(\lambda/2p) + A\sin 2Q\right). \end{aligned} \quad \ldots (18)$$

Of course other functions J_0 may be chosen, provided $\frac{\partial J_0}{\partial p}$ vanishes at the appropriate points, and these would apparently lead to a different function for $J_0 + \varepsilon J_1$, $(= \tilde{J}$ say). However it can be shown that there would then be a functional relationship between J and $\tilde{J}$, so that the curves $\tilde{J}$ = constant would be identical with the curves J = constant.

For comparison with the numerical computations J must be expressed in terms of the variables ξ, φ. The only dynamical variable in J_0 is p, and this is exactly $v(1-\xi)^{1/2}$. In J_1, we need only zero order (in ε) relations between P, Q and ξ, φ. These are

$$P = \tfrac{1}{2} v^2 \xi \quad , \qquad Q = \frac{\pi}{4} - \tfrac{1}{2} \varphi \quad .$$

Consequently J can be written

$$J = v^3(1-\xi)^{3/2} \sin \frac{\lambda}{2v(1-\lambda)^{1/2}} - \frac{6v^4}{\lambda} (1-\xi)^2 \cos \frac{\lambda}{2v(1-\xi)^{1/2}} \mp \tfrac{1}{2} \varepsilon \lambda \left[1 + 48v^2 (1-\xi)/\lambda^2\right]$$

$$\times \left[\tfrac{1}{2} v^2 \xi \cos \frac{\lambda}{2v(1-\xi)^{1/2}} + \left((\tfrac{1}{2}v^2\xi)^2 + v^2 \xi C \right)^{1/2} \cos \varphi \right] \quad \ldots (19)$$

Curves J = constant are shown in Figs.4-6 for the same parameters ($v = 2, C = 4, \lambda = 2\pi$ and $\varepsilon = 0.025, 0.05, 0.10$) as were used in the numerical computations of Figs.1-3. It will be seen that there is close agreement between curves derived from the invariant (19) and the computed orbits of type (a).

4. SUMMARY AND DISCUSSION

We have investigated the motion of charged particles in a spatially modulated magnetic field, in particular to see whether any invariant exists when the modulations give resonance with the cyclotron period and destroy the magnetic moment invariant. A study of numerically computed orbits in a modulated field shows that they fall into two classes (a) regular orbits, corresponding to the existence of an invariant and (b) quasi-ergodic orbits. Provided the modulation ε of the field is not too great the regular orbits predominate.

For small ε, then, these numerical studies indicate that an invariant exists, even in the resonant situation. We have, therefore, computed this invariant as a power series in ε. For this we developed a form of perturbation theory which avoids the usual problem of vanishing denominators near resonance and shows that this phenomena really indicates a change in topology of the invariant curves. The invariant calculated in this way is given by Equation (18), and shows very good agreement with the numerically computed orbits of type (a). This agreement confirms that, when an invariant exists it can be calculated by our modified form of perturbation theory.

However perturbation theory can give no direct indication of the transition to quasi-ergodic behaviour. An indirect indication may lie in the chain of 'islands' which the numerical studies show near the boundary between regular and ergodic behaviour. These islands represent further changes in topology which would be reproduced if our perturbation theory were carried to higher order and we may speculate that the increasingly complex behaviour of the analytic curves represent the onset of quasi-ergodic behaviour.

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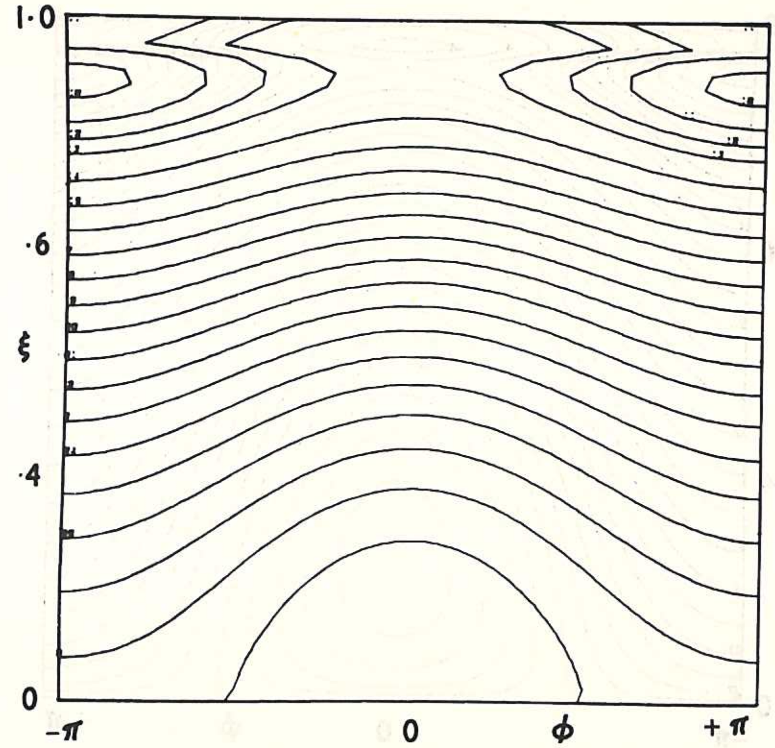

Fig. 1 Numerically computed orbits, $\epsilon = 0.025$ (CLM-P 158)

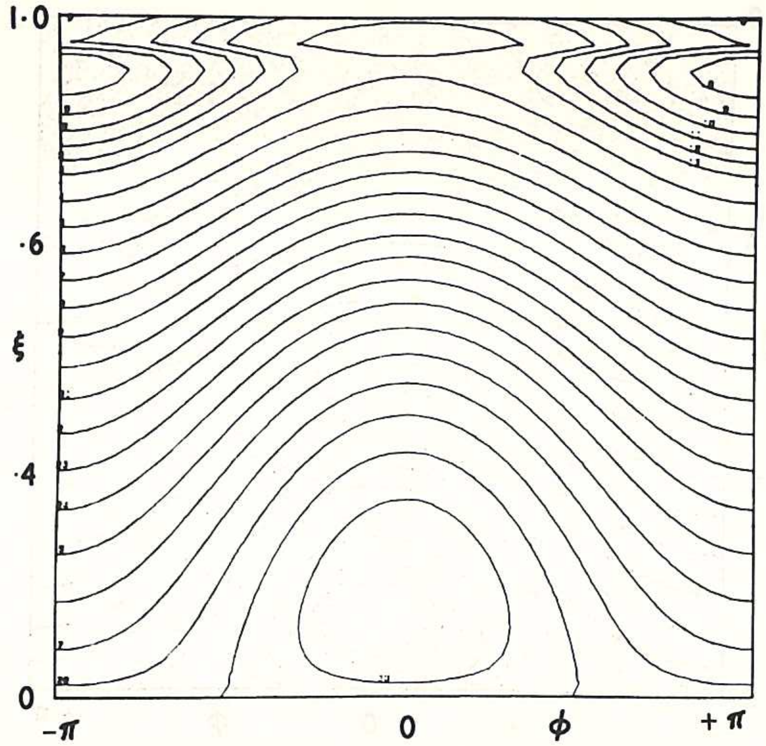

Fig. 2 Numerically computed orbits, $\epsilon = 0.05$ (CLM-P 158)

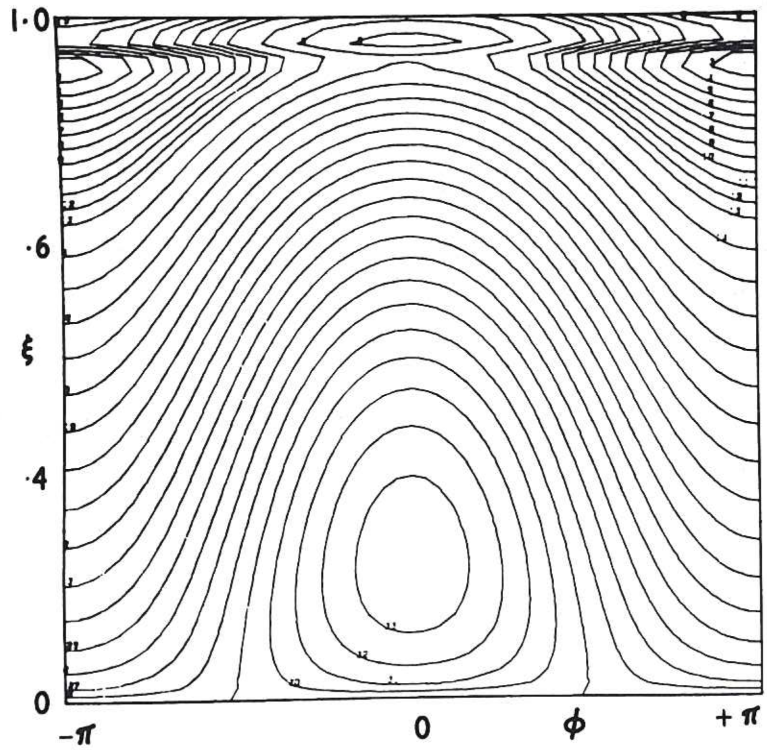

Fig. 3 Numerically computed orbits, $\epsilon = 0.10$ (CLM-P 158)

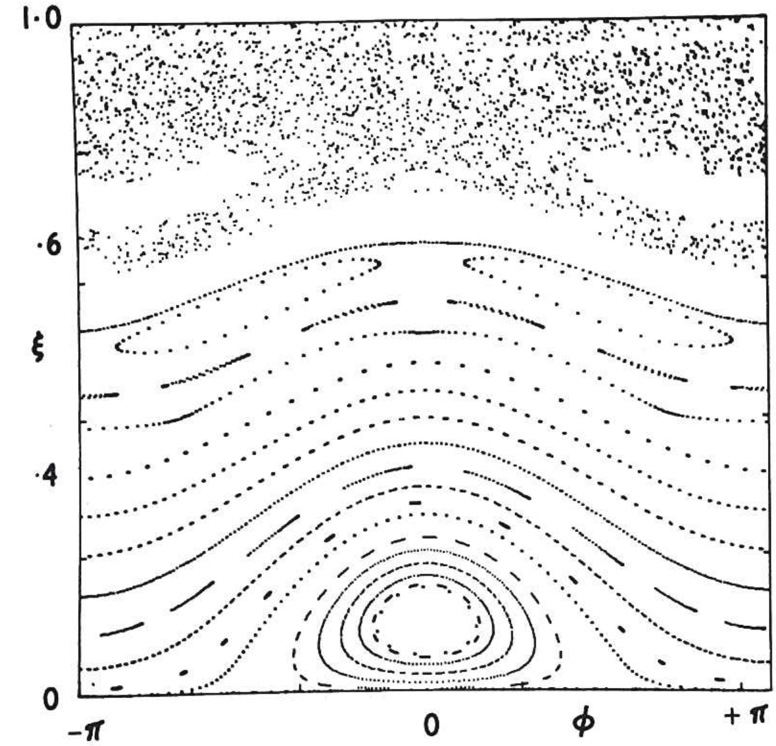

Fig. 4 Invariant curves $J_0 + \epsilon J_1 =$ constant, $\epsilon = 0.025$ (CLM-P 158)

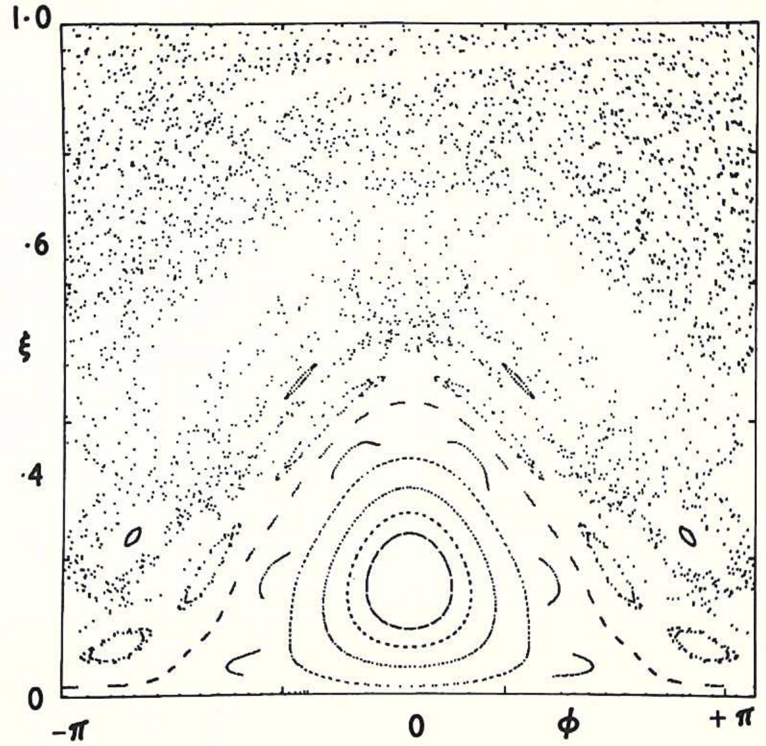

Fig. 5 Invariant curves $J_0 + \epsilon J_1$ = constant, $\epsilon = 0.05$ (CLM-P 158)

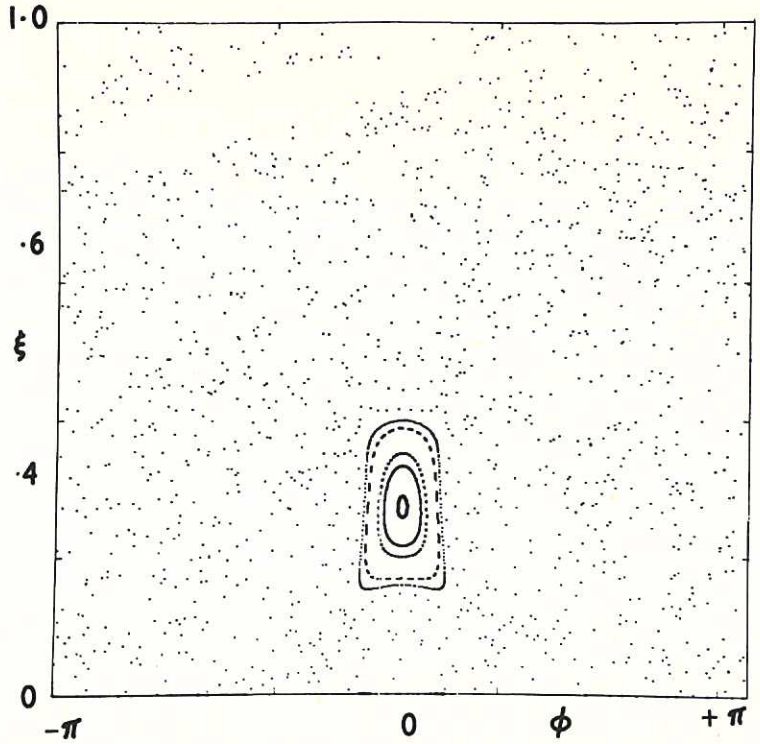

Fig. 6 Invariant curves $J_0 + \epsilon J_1$ = constant, $\epsilon = 0.10$ (CLM-P 158)