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RADIATIVE TRANSFER OF DOPPLER BROADENED RESONANCE LINES

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RADIATIVE TRANSFER OF DOPPLER BROADENED RESONANCE LINES

by

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A B S T R A C T

The radiative transfer of Doppler broadened resonance lines in a plasma which is not in local thermodynamic equilibrium is studied for a simple model of an infinite, uniform plane parallel plasma. A two level atomic model is used in which electrons are transferred between the levels by electron excitation and de-excitation, spontaneous emission and photo-excitation. Stimulated emission is neglected. It is assumed that the source function is independent of frequency and that the spontaneous emission is isotropic and unpolarised. The radiation is emitted in a Doppler broadened profile in which natural and pressure broadening may be neglected. In the calculations the population of the excited level is assumed to be much smaller than the ground level.

The distribution of excited atoms in a steady state with their own resonance radiation has been calculated, and from this the rate of loss of energy, and the line profiles and intensities are obtained. The rate of loss of energy is compared with a model of diffusion of photons in frequency space. The line profiles are self-reversed although the calculations are for a uniform plasma.

The validity of the assumptions is examined and the restrictions they place on the range of application are considered with particular reference to Lyman alpha radiation.

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1. Introduction

The resonance lines emitted by a plasma are important for the means they provide of studying the plasma and because the process of radiative transfer influences physical properties such as the distribution of excited atoms and the radiative energy balance. The problem of radiative transfer is simplest when the plasma is optically thin and the radiation emitted leaves the plasma without any interaction. A more complicated situation, but one where many simplifying assumptions may be made, is when there is local thermodynamic equilibrium and the populations of the excited levels are given by the Boltzmann distribution. This has been extensively studied for its application to stellar atmospheres. (Chandrasekhar 1960).

In high temperature laboratory plasmas and some astronomical plasmas such as the solar chromosphere, these assumptions are often invalid. The populations of the excited levels can be calculated only by considering the transfer of radiation and the atomic processes together, and the result depends on the geometry and optical depth of the plasma. The importance of this type of calculation has been pointed out by Thomas (1957). The general solution of radiative transfer for an atom with many levels is too difficult to be solved at present, but in order to study the main consequences of radiative transfer in a plasma, a simple two level atomic model has been used in a uniform infinite plane parallel plasma. Four processes transferring electrons between the levels are included. These are electron excitation and de-excitation, spontaneous emission and photo-excitation. Stimulated emission has been neglected. The distribution of excited atoms in a steady state with their own resonance radiation has been calculated, and from this the total intensities of the lines with their profiles have been calculated together with the total energy loss from the plasma. The line profiles show self-reversal although the calculations are for a plasma of uniform electron density and temperature. In some conditions the intensity of the line is greater than the corresponding optically thin intensity. The rate of loss of energy is compared with a simple model of the diffusion of photons in frequency space.

2. Basic equations

In an optically thick plasma with a uniform number density n_1 of atoms in the ground level and a uniform electron temperature T_e and density n_e, the number density n_2 of atoms in the excited level will vary throughout the plasma. The distribution of atoms in the excited level is calculated from the atomic processes between the levels and the equation of transfer of radiation.

Using the definitions and nomenolature of Ambartsumyan (1958a) the equation of transfer is

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + \frac{J_\nu}{\chi_\nu} \qquad (1)$$

where I_ν is the specific intensity of radiation at a given point and in a given direction at a frequency ν, τ_ν is the optical depth, J_ν the emission coefficient and χ_ν the absorption coefficient. This equation has an analytical solution of

$$I_\nu = I_\nu^0 e^{-\tau_\nu} + e^{-\tau_\nu} \int_0^{\tau_\nu} \frac{J_\nu}{\chi_\nu} e^{\tau'_\nu} d\tau'_\nu \qquad (2)$$

where I_ν^0 is the intensity incident at the origin of τ_ν.
This equation represents the change in intensity of radiation after passing through a plasma that emits radiation as well as absorbs it.

The coefficients of absorption and emission may be expressed in terms of the Einstein coefficients. For a Maxwellian distribution of atoms with negligible natural and pressure broadening, the absorption coeffficient is

$$\chi_\nu = \frac{n_1 h\nu B_{12} e^{-(\nu-\nu_0)^2/\Delta\nu_D^2}}{\Delta\nu_D \sqrt{\pi}\,\rho c} \qquad (3)$$

where ν_0 is the central frequency of the absorption line, B_{12} is the Einstein absorption coefficient, $\Delta\nu_D$ the e^{-1} frequency of the gaussian profile and ρ the mass density of the plasma.

If it is assumed that the emission coefficient has the same frequency spectrum as the absorption coefficient then

$$J_\nu = \frac{n_2 h\nu}{\Delta\nu_D \sqrt{\pi}\,\rho} \left(\frac{A_{21}}{4\pi} + \frac{I_\nu}{c} B_{21} \right) e^{-(\nu-\nu_0)^2/\Delta\nu_D^2} \qquad (4)$$

The validity of this assumption and others that are made will be discussed later. A_{21} is the Einstein spontaneous transition coefficient and B_{21} is the stimulated emission coefficient.

The ratio of the emission coefficient to the absorption coefficient is known as the source function f. If stimulated emission is neglected

$$f = \frac{J_\nu}{\chi_\nu} = \frac{c}{4\pi} \frac{A_{21}}{B_{12}} \frac{n_2}{n_1} \qquad (5)$$

and it is independent of frequency.

In a steady state the rates of the processes populating and depopulating the excited level must be equal. If n_2 is always small compared with n_1 then

$$\frac{n_2}{n_1} = \frac{\dfrac{B_{12}}{\Delta\nu_D \sqrt{\pi}} \displaystyle\int_0^\infty \rho_\nu e^{-(\nu-\nu_0)^2/\Delta\nu_D^2} d\nu + n_e X_{12}}{A_{21} + n_e Y_{21}} \qquad (6)$$

X_{12} is the rate coefficient for excitation by electron collision defined so that the number of excitations/cc/sec is $n_1 n_e X_{12}$ and Y_{21} is the rate coefficient for de-excitation by electron collision defined so that the number of de-excitations/cc/sec is $n_2 n_e Y_{21}$. The integral term represents photo-excitation and ρ_ν is the radiation energy density at the point in space being considered and is obtained by integrating the intensity over all solid angles

$$\rho_\nu = \frac{1}{c}\int^{4\pi} I_\nu \, d\omega \tag{7}$$

It is convenient to express the variables of these equations in terms of dimensionless parameters. A dimensionless frequency x may be defined by

$$x = \frac{\nu - \nu_0}{\Delta\nu_D} \tag{8}$$

In full thermodynamic equilibrium, intensities are given by the Planck function and the populations by the Boltzmann distribution, and the remaining variables may be expressed as a fraction of their black body value.

If stimulated emission is negligible the black body radiation density ρ_{bb} is

$$\rho_{bb} \approx \frac{8\pi h\nu^3}{c^3} e^{-h\nu/kT_e} \tag{9}$$

The principle of detailed balancing gives the black body source function f_{bb} as

$$f_{bb} = \frac{c}{4\pi} \frac{A_{21}}{B_{12}} \frac{X_{12}}{Y_{21}} \tag{10}$$

It also equals the black body intensity so that

$$f_{bb} = \frac{c}{4\pi} \rho_{bb} \tag{11}$$

Dividing equations 2, 5, 6 and 7 by f_{bb} and putting

$$\frac{f}{f_{bb}} = F \quad \text{and} \quad \frac{\rho_x}{\rho_{bb}} = P_x \tag{12}$$

gives

$$\frac{I_x}{f_{bb}} = \frac{I_x^0}{f_{bb}} e^{-\tau_x} + e^{-\tau_x}\int_0^{\tau_x} F e^{\tau_x'} d\tau_x' \tag{13}$$

This is the equation of transfer in its final form. It has to be solved simultaneously with the following equations:

$$P_x = \frac{1}{4\pi}\int^{4\pi} \frac{I_x}{f_{bb}} d\omega \qquad (14)$$

$$F = \frac{\bar{P} A_{21} + n_e Y_{21}}{A_{21} + n_e Y_{21}} \qquad (15)$$

where $$\bar{P} = \frac{2}{\sqrt{\pi}}\int_0^{\infty} P_x e^{-x^2} dx \qquad (16)$$

$\frac{I_x}{f_{bb}}$ is the intensity expressed as a fraction of the black body intensity. The optical depth at a frequency x is related to the optical depth at the line centre τ_0 by

$$\tau_x = \tau_0 e^{-x^2} \qquad (17)$$

For an optically thin plasma with no radiation incident from outside, the fractional source function F has the value F_0

$$F_0 = \frac{n_e Y_{21}}{A_{21} + n_e Y_{21}} \qquad (18)$$

Equation (15) may now be expressed in terms of F_0

$$F = \bar{P}(1 - F_0) + F_0 \qquad (19)$$

The essence of the problem is to solve equations (13), (14), (16), (17) and (19) to obtain the fractional source function F as a function of the optical depth τ through the plasma described by the parameters of F_0 and total optical thickness.

3. The method of solution

Let the plane parallel plasma have a finite thickness along the z-axis, and be infinite along the x and y axes. By symmetry F is a function of z only and is symmetrical about the centre plane of the plasma. Suppose that

F is specified at a number of values of z throughout the plasma. Then the corresponding normalised radiation excitation rate $\bar{P}$ can be calculated from equations (13), (14), (16), using an interpolation method for F. The values of $\bar{P}$ will not necessarily satisfy the steady state condition, equation (19), if the values of F are arbitrarily chosen. But the relation between the values of $\bar{P}$ obtained in this way, to the values of F assumed at each of the points j is given by

$$\underline{\bar{P}} = \underline{\underline{a}}\,\underline{F} \tag{20}$$

where the coefficients a_{ij} depend only on the optical thickness of the plasma.

If linear interpolation is used for F, this result may be shown as follows. In figure 1, let the plane of constant z upon which the values F_j are specified be separated by an optical depth $\Delta\tau_0$ measured along the z-axis at the centre of the line. At a frequency x and at an angle θ to the z-axis the optical depth between the planes is $\Delta\tau_x$ and

$$\Delta\tau_x = \frac{\Delta\tau_0 e^{-x^2}}{|\cos\theta|} \tag{21}$$

F at a point τ_x' from the jth plane measured toward the j + 1 plane is given by

$$F = F_j + \frac{F_{j+1} - F_j}{\Delta\tau_x}\,\tau_x' \tag{22}$$

Inserting equation (22) into equation (13) and integrating over the interval $\Delta\tau_x$ gives

$$\left(\frac{I_x}{f_{bb}}\right)_{j+1} = \left(\frac{I_x}{f_{bb}}\right)_j e^{-\Delta\tau_x} + F_j\left(1 - e^{-\Delta\tau_x}\right) + \frac{(F_{j+1} - F_j)}{\Delta\tau_x}\left(e^{-\Delta\tau_x} + \Delta\tau_x - 1\right) \tag{23}$$

where $\left(\frac{I_x}{f_{bb}}\right)_{j+1}$ is the intensity arriving at the j + 1th plane from the direction of the jth plane. Since there is no external radiation incident on the plasma, $\left(\frac{I_x}{f_{bb}}\right)_1$ is zero. Thus the intensity incident at each plane can be calculated using equation (23) for each value of j in turn. Since equation (23) is a linear relation between the intensity and F, all the intensities are linear functions of the assumed values of F. The numerical integration formulae are simply sums of the values of the variable multiplied by some numerical constant, so that the total normalised excitation rate $\overline{P}$ obtained by integrating the values of $\left(\frac{I_x}{f_{bb}}\right)_j$ over all frequencies and angles in equations (14), (16) and (17) is also a linear function of the values F_j and equation (20) is true.

For a steady state condition, equation (19) must also be satisfied. Inserting it into equation (20) and eliminating $\overline{P}$ gives a series of equations for the values of F_j. These are represented by the matrix equation

$$-\underline{F_0} = \left[(1-F_0)\underline{\underline{a}} - \underline{\underline{I}}\right]\underline{F} \qquad (24)$$

where $\underline{F_0}$ is a vector with all its components equal to F_0 and $\underline{\underline{I}}$ is a unit matrix.

This equation can be solved by any of the standard methods given the coefficients a_{ij}. The coefficients are calculated essentially in the following way. Differentiating equation (20) gives

$$a_{ji} = \frac{\partial \overline{P_j}}{\partial F_i} \qquad (25)$$

This equation is valid for any choice of F so that the values of F are chosen to be zero except for F_i which is conveniently put at unity. The corresponding values of $\overline{P_j}$ can be obtained from equation (23) in the manner described in the proof of equation (20) and so the coefficients a_{ij} can be obtained. They depend only on the optical thickness $\Delta\tau_0$.

If a polynomial interpolation is used for F, equation (20) is still valid. A better interpolation function would require fewer points to give the same accuracy in the results, and the calculations indicate that a great improvement would be obtained by using at least a quadratic fit. If the quadratic term of the interpolation is the largest neglected, then doubling the number of points and so halving the interval between them will reduce the error of the linear interpolation by four. Figure 2 shows this effect. The accuracy of all the results was estimated by calculating them with half the number of points. The results for an optical depth up to ten are accurate to 1% or 2%; the results at high optical depth and small F_0 may be in error by as much as 10%, particularly at the edge.

Some early results for F_0 nearly unity were obtained by iteration and the calculation converged satisfactorily. The convergence for small F_0 is very slow. The improvement in F for each iteration is rather less than F_0.

4. <u>The results</u>

The distribution of F was calculated with an IBM 7090 for a range of optical thicknesses and values of F_0 . The results for F_0 ranging from 0.5 to 0.001 are shown in figures 3 to 7. The abscissa represents position in the plasma, the origin being the outside edge and the right hand extreme being the centre of the plasma. The ordinate is the fractional source function F which is plotted for various total optical thicknesses of the plasma measured at the line centre. F equal to unity represents a Boltzmann population. If the plasma is optically thin, F equals F_0 . When the optical thickness becomes large, the population approaches the Boltzmann population over a large portion of the plasma. The smaller is F_0, the larger is the optical thickness necessary for this. The variation in F is then concentrated in a small portion of space and the results obtained with equally spaced points become inaccurate.

Equation (24) shows that when F_0 is very small, its contribution to the coefficients is negligible and then for a given optical thickness of the plasma, the shape of the solution for F is independent of F_0 and its magnitude directly proportional to F_0. It is not easy to show from a general analysis when F_0 is sufficiently small, but examination of the numerical results for F_0 of 0.01, 0.001 and 0.0001 shows that if F at the centre of the plasma is less than 0.1, the shape is constant within an accuracy of 5%. The results for F_0 of 0.0001 are not shown here. Since F increases with optical thickness of the plasma, the larger the optical thickness the smaller F_0 must be to satisfy the criterion.

5. Solution and results for a semi-infinite plasma

In a semi-infinite plasma, the source function and all intensities have their black body value except for a small region near the edge. The source function in this region may be calculated by considering a finite plane parallel plasma illuminated on one side by an isotropic black body intensity. The equations may be solved as before, but now there is a contribution from this incident intensity and the equation for $\bar{P}$ becomes

$$\underline{\bar{P}} = \underline{\underline{a}}\,\underline{F} + \underline{c}\,\frac{I^0}{f_{bb}} \qquad (26)$$

$\frac{I^0}{f_{bb}}$ is the black body intensity and is unity. Combining this equation with equation (19) gives

$$-\underline{F_0} - (1-F_0)\,\underline{c} = \left[(1-F_0)\,\underline{\underline{a}} - \underline{\underline{I}}\right]\underline{F} \qquad (27)$$

The results of the calculations for a range of values of F_0 are shown in figures 8 to 11. The ordinate is the fractional source function and the abscissa is the optical depth measured at the centre of line from the edge of the plasma.

The calculations for a finite plane parallel plasma have shown that for large optical thicknesses, the source function is near to its black body value over most of the plasma, variation in F is then concentrated in a small portion of space and the results obtained with equally spaced points become inaccurate. But if the optical thickness of the central region is sufficiently large, the intensities arriving at the edge region will be black body over the range of frequencies in which the effective part of the absorption coefficient is situated, and the source function at the edge of a large finite region will be the same as for a semi-infinite plasma. Thus the source function for a large plane parallel plasma may be represented by two edge regions calculated from a semi-infinite plasma and a black body region between. There is a region of overlap between the values of the source function calculated this way and those calculated earlier, and the agreement is good. The accuracy of the results for F_0 of 0.5 is about 2%. For F_0 of 0.001 the results are 10% low in the centre and perhaps up to 30% low at the edge.

6. The rate of loss of energy

The total rate of energy radiated from the surface of the plasma is obtained by integrating the intensity $I(\nu, \theta)$ leaving the surface over all frequencies and angles. For a plane parallel plasma

$$\frac{dE}{dt} = \int_0^\infty \int_0^{\pi/2} 2\pi \, I(\nu, \theta) \cos\theta \sin\theta \, d\theta \, d\nu \ \text{ergs cm}^{-2}\text{sec}^{-1} \tag{28}$$

For an optically thin plasma

$$I(\nu, \theta) = \frac{f \chi_0 \rho L}{\cos\theta} \, e^{-(\nu - \nu_0)^2 / \Delta\nu_D^2} \tag{29}$$

where χ_0 is the absorption coefficient at the centre of the line and L is the geometrical thickness of the plasma.

The fractional source function has been calculated across an optically thick plasma and the equation of transfer may be solved to give the intensity leaving the surface at all frequencies and angles. The total rate of loss of energy from both surfaces is conveniently expressed as a fraction W of the rate at which energy would be lost if the plasma were optically thin. So

$$W = \frac{2}{\sqrt{\pi}\, F_0 \tau} \int_0^\infty \int_0^{\pi/2} \frac{I(x,\theta)}{f_{bb}} \sin\theta \cos\theta \, d\theta \, dx \tag{30}$$

where $\frac{I(x,\theta)}{f_{bb}}$ is the intensity calculated to be leaving the surface of the plasma expressed as a fraction of the black body intensity.

The fraction W is shown in figure 12 plotted against optical thickness for various values of F_0 .

F_0 is the fractional source function when the plasma is optically thin, but equation (18) shows that it is also the fraction of excited atoms which undergo collisional de-excitation and consequently a fraction $1-F_0$ of the photons absorbed are reëmitted. In an optically thin plasma atoms are excited by electron collision only and all the photons escape. In an optically thick plasma electron excitation is again the source of photons. Photo-excitation and spontaneous emission simply represent the absorption of one photon and the subsequent emission of another. If F_0 is very small all the photons created by electron excitation will eventually escape and the rate of loss of energy is the same as if the plasma were optically thin, and W is unity. At each absorption a fraction F_0 of all the photons are converted back to electron kinetic energy by de-excitation collisions. As the optical depth increases, so does the number of times a photon is absorbed, and the probability of it reaching the surface before suffering de-excitation becomes small.

It has been suggested by Zanstra (1949) and Osterbrock (1962) that for a doppler broadened line the change in frequency upon reëmission of the photon provides a more important escape mechanism than diffusion through space, since if a photon is emitted in the wings of the line where the optical depth of the plasma is small it will escape.

Assume that a photon will escape if it is emitted at a frequency greater than x_1 and it will be absorbed if it is emitted at a frequency lower than x_1. If the optical depth at the centre of the line from the edge to the centre of the plasma is τ_0, then at a frequency x_1 the optical depth τ_1 is

$$\tau_1 = \tau_0 e^{-x^2} \tag{31}$$

It has been assumed in the calculations of the fractional source function that the emission coefficient has a gaussian profile, so that the probability that a photon is emitted between x and $x + dx$ is $\frac{1}{\sqrt{\pi}} e^{-x^2} dx$ irrespective of the frequency at which it was absorbed. So the probability q that a photon is emitted at a frequency greater than x_1, from the line centre is

$$q = \frac{2}{\sqrt{\pi}} \int_{x_1}^{\infty} e^{-x^2} dx = 1 - \operatorname{erf} x_1 \tag{32}$$

If atoms excited by electron collisions emit photons, a fraction q will escape and 1-q will be absorbed. Of those absorbed a fraction $1 - F_0$ will be reëmitted and of these a fraction q will again escape. The total fraction W of photons emitted as the result of an electron excitation which escapes is therefore

$$W = q + (1-F_0)(1-q)q + (1-F_0)^2(1-q)^2 q + \cdots$$
$$= \frac{q}{1-(1-F_0)(1-q)} \qquad (33)$$

When F_0 is sufficiently small, W is close to unity and all the photons eventually escape, when F_0 is nearly unity, the higher terms of the series are all very small and W equals q. Figure 12 shows W calculated from this simple model when τ_1 is unity.

These curves give the rate of loss of energy as a fraction of the rate of loss which would exist if no photons were absorbed. The total rate of loss of energy from both sides of a plane parallel plasma of thickness L cms is

$$\frac{dE}{dt} = n_1 n_e X_{12}(1-F_0) L W h\nu_0 \text{ ergs cm}^{-2}\text{ sec}^{-1} \qquad (34)$$

7. Line profiles and total intensities

The line profiles emitted normally to the plasma boundary have been calculated from the spatial distribution of the fractional source function using the equation of transfer. The profiles are shown in figures 13 to 18. The ordinate is the intensity expressed as a fraction of the black body intensity and the abscissa is the dimensionless frequency x. These profiles show self-reversal although the calculations assume that the plasma has a uniform density and temperature. This is most pronounced for small F_0 where the radiation processes dominate the collisions, and in these conditions the intensity at the centre of the line can never rise to the black body intensity however large the optical depth may be. When F_0 is unity, the profiles show no self-reversal, and the central intensity rises to the black body value at an optical depth of 5.

The area of the profiles gives the total intensity of the lines. These total intensities are shown in figure 19 and they are normalised so that a gaussian line profile expressed in dimensionless frequency having a central intensity at the black body intensity has a value of unity. The curve for F_0 of unity is essentially that calculated by Van der Held (1931).

These total normalised intensities may be compared with those that would be emitted if the plasma were optically thin. In the same units the total normalised intensity for an optically thin plasma is $F_0 \tau_0$. Comparison of this intensity with those in figure 19 for an optically thick plasma gives the paradoxical result that the intensity emerging normally from an optically thick plasma may be greater than the intensity calculated by assuming the plasma absorbs no photons. It is illustrated most clearly at an optical depth of 10 and F_0 of 0.001. The calculated total normalised intensity is 0.0195 whilst the product $F_0 \tau_0$ is 0.010. It has been explained that for small F_0 and modest optical depths all the photons generated by electron excitation eventually escape. But the plasma used for these calculations is infinite along two of its axes, and this restricts the solid angle in which the photons can escape and the intensity must be increased to maintain the equilibrium.

8. The physical assumptions and their validity

A number of assumptions have been made in the calculations described above. These assumptions are justified only under certain physical conditions, and the restrictions which they place on the validity of these calculations are described in this section and illustrated by reference to Lyman α radiation. The assumptions that have been made are

(a) the source function is independent of frequency

(b) spontaneous emission is isotropic and unpolarised.

(c) stimulated emission is negligible and the density of excited atoms is much less than the density of ground state atoms.

(d) natural broadening is negligible compared with doppler broadening.

(e) pressure broadening is negligible compared with doppler broadening

These assumptions will be considered in turn.

(a) The source function is independent of frequency.

This is the most important assumption that has been made, and it means that the profiles of the absorption coefficient and the emission coefficient are the same. The gaussian profile of the absorption coefficient results from the Maxwellian velocity distribution of the ground state atoms. For the emission coefficient to have the same profile the excited atoms considered by themselves must also have a Maxwellian distribution of the same kinetic temperature as the ground state atoms. If the probability of excitation is independent of the velocity of the ground state atoms, then the excited atoms created will be a completely random sample of the ground state atoms and will therefore have the same velocity distribution. Excitation which is independent of the atom velocity occurs with inelastic electron collisions, where the velocity of the electrons is very much greater than the velocity of the atoms, and also with photo-excitation by isotropic white light or by a constant isotropic intensity over a band of frequencies much greater than the effective part of the absorption coefficient for the line. In these two forms of excitation the source function is unconditionally independent of frequency.

When the ground state atoms are excited by the absorption of a narrow band of frequencies such as those in a spectral line, the slower atoms are excited preferentially. The atoms at the instant they are excited, will not have a Maxwellian velocity distribution. But they will be redistributed into a Maxwellian distribution if they undergo many collisions with other atoms, excited or ground state, during their lifetime. So that when atoms are excited by a narrow band of frequencies the source function is independent of frequency only if the time between collisions of the excited atoms with other atoms is short compared with the lifetime of the excited state.

Little information is available for the cross section of elastic collisions between hydrogen atoms. McDowell (1958) has calculated the cross sections for hydrogen ions and hydrogen atoms. They are not very dependent on energy and the rate coefficient for a Maxwellian distribution has been estimated from the cross section at the mean energy. The density at which the mean time between collisions equals the lifetime of the excited state of the Lyman α line depends on temperature and this is shown in figure 20. The values of the parameter F_0 are also shown for Lyman α and they have been calculated from the excitation coefficients of Seaton (1962).

The condition obtained for the source function to be independent of frequency is probably rather pessimistic. No consideration has been given to whether small departures of the source function from this assumption would cause serious errors.

(b) The spontaneous emission is isotropic and unpolarised.

If the excitation is isotropic then the number of atoms in the different states of the same level will be equal. Under these conditions, dipole radiation between level nj and $n'j'$ is isotropic and unpolarised. If the excitation is by some anisotropic process such as the absorption of a unidirectional beam of light, then the subsequent emission may also be anisotropic because the excitation may establish different numbers of atoms in the different states of the same level. However, collisions between the atoms will distribute them evenly between the states, and if the states are broadened by Stark effect during the collisions so that the states merge, the assumption of isotropic unpolarised emission is justified.

(c) Stimulated emission is negligible and the density of excited atoms is much less than the density of ground state atoms.

Stimulated emission is negligible if

$$A_{21} \gg B_{21}\rho_{\nu} \qquad (35)$$

where ρ_{ν} is the radiation density, A_{21} is the Einstein coefficient for spontaneous emission and B_{21} is the Einstein coefficient for stimulated emission. The intensity at any frequency cannot exceed the black body intensity corresponding to the electron temperature of the plasma. Using the Planck black body intensity to calculate the radiation density and the relation between the Einstein coefficients, the inequality (35) reduces to

$$e^{h\nu/kT} \gg 1 \qquad (36)$$

This also represents the restriction that the density of excited atoms is small compared with the density of ground state atoms, since the excited atom density cannot exceed the Boltzmann population and the ratio of the statistical weights of the two levels is usually of the order of unity.

This restriction may be relaxed at very low electron densities since the intensities will be well below black body intensity and the density of excited atoms well below the Boltzmann population.

If stimulated emission for Lyman α is not to exceed 20% of the spontaneous emission then the electron temperature must not exceed 7×10^{4} °K.

(d) Natural broadening is negligible compared with doppler broadening.

At large distances from the centre of the profile the contribution to the absorption coefficient of the wings of the natural broadening profile is greater than that of the wings of the doppler profile. The frequency at which the two are equal has been calculated together with the absorption

TABLE 1

The dimensionless frequency at which the doppler and natural broadening profiles are equal and the maximum optical depth for which natural broadening may be neglected for various ratios of the e^{-1} doppler width frequency to the damping constant.

$\frac{\Delta\nu_D}{\delta}$	$\frac{\Delta\nu_e}{\Delta\nu_D}$	τ_{max}
1×10^9	6.73	2.3×10^{18}
3.2×10^8	6.55	2.1×10^{17}
1×10^8	6.37	2.0×10^{16}
3.2×10^7	6.18	1.9×10^{15}
1×10^7	5.98	1.8×10^{14}
3.2×10^6	5.78	1.7×10^{13}
1×10^6	5.57	1.5×10^{12}
3.2×10^5	5.36	1.4×10^{11}
1×10^5	5.13	1.3×10^{10}
3.2×10^4	4.89	1.2×10^9
1×10^4	4.63	1.1×10^8
3.2×10^3	4.37	9.4×10^6
1×10^3	4.08	8.2×10^5
3.2×10^2	3.76	7.0×10^4
1×10^2	3.41	5.7×10^3
3.2×10^1	3.02	4.5×10^2
1×10^1	2.54	3.2×10^1

TABLE 2

The maximum optical depth for which natural broadening may be neglected for Lyman α radiation for various atom temperatures.

T^oK	τ_{max}
10	5.1×10^3
10^2	6.2×10^4
10^3	7.5×10^5
10^4	8.6×10^6
10^5	9.8×10^7
10^6	1.1×10^9

coefficient of the doppler profile. If the total optical depth at this frequency is small, the intensity of the profile may be calculated from the source function using the optically thin approximation. For large optical thicknesses, the fractional source function is unity for most of the volume so that an upper value for the intensity of the profile for this cross-over frequency may be calculated. If this is small, say less than 0.05 of the black body intensity, the significant part of the profile calculated from pure doppler broadening will be correct. So natural broadening may be neglected provided that the optical depth does not exceed τ_{max} which is defined by

$$\tau_{max}\, e^{-\left(\frac{\Delta\nu_e}{\Delta\nu_D}\right)^2} = 0.05 \qquad (37)$$

where $\Delta\nu_e$ is the frequency at which the two profiles are equal and $\Delta\nu_D$ is the e^{-1} frequency of the doppler profile.

$\frac{\Delta\nu_e}{\Delta\nu_D}$ and τ_{max} are given in Table 1 for various ratios of the doppler broadening to natural broadening. δ is $\gamma/4\pi$ where γ is the lifetime of the level.

For Lyman α the relation may be expressed as a maximum optical depth against temperature. This is shown in Table 2.

The main effect of adding natural broadening to a doppler broadened profile is to increase the wings considerably. Since the total area of the absorption coefficient must remain the same, the centre of the profile is reduced, but only very slightly. The rate of excitation of atoms by absorption is determined mainly by the central region of the absorption profile. If the addition of natural broadening reduces the absorption coefficient at the centre by 5%, then the total rate of photo-excitation will be reduced by less than this. The rate of emission, proportional to the total area of the emission coefficient, is unchanged, so that the source function calculated from a pure doppler broadened profile will be less than 5% in error. Tables of the combined absorption profile have been published (Ambartsumyan 1958b). The absorption coefficient at the centre is reduced by 5% when $\delta/\Delta\nu_D$ is 0.04. For Lyman α this corresponds to a temperature of 1 $^{\circ}$K, so that above this temperature a good approximation of the effect of natural broadening may be made by calculating the line profiles using the correct combined expression for the emission coefficient from the fractional source function obtained for pure doppler broadening.

(e) Pressure broadening is negligible compared with doppler broadening.

The argument used for natural broadening may be applied to pressure broadening. The pressure broadened profiles of Lyman α calculated by Griem, Kolb and Shen (1959) were used to find the frequency at which the absorption coefficient of the pressure broadened profile equals the doppler profile. The optical depth for the line centre for which the intensity of the cross over frequency is 5% of the black body intensity was calculated for various temperatures and densities. Figure 21 shows the maximum optical depth at which the pressure broadening of Lyman α may be neglected for various ion densities and temperatures. For a given optical depth these curves give a maximum ion density against temperature. This limit for an optical depth of unity is shown in figure 20 to give an indication of where the pressure broadening of Lyman α becomes important.

9. Conclusion

These calculations show how the excitation in a plane parallel plasma not in local thermodynamic equilibrium increases with the total optical thickness of the plasma. At low densities when electron excitation and de-excitation are small, a large optical thickness is required before the excited atom density and line intensities reach their black body level. The excited atom density at the edge is not enhanced so much as at the centre, and since the intensity at the centre of the line profile is determined by the conditions at the edge of the atmosphere, the line profile is self-reversed and the line centre can never achieve a black body intensity. If the optical depth is large enough, part of the wings of the line profile will reach a black body intensity. At low electron densities all the photons generated by electron excitation eventually leave the plasma, since they can only be prevented from leaving by converting the energy into kinetic energy by an electron de-excitation collision. In a plasma infinite in some direction the solid angle in which photons can leave if the plasma is optically thick is restricted and consequently including the absorption of photons can increase the calculated intensity emitted by the plasma.

Of the assumptions made in the calculations, that of a constant source function is the most restrictive if it is to be satisfied unconditionally. The effect of a source function varying in frequency has not been considered. In the calculation of the source function quite large additions of natural broadening have little effect, but the effect on the line profile emitted from the plasma is great.

Acknowledgements

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(a) Chapter 2
(b) Appendix 1, page 616

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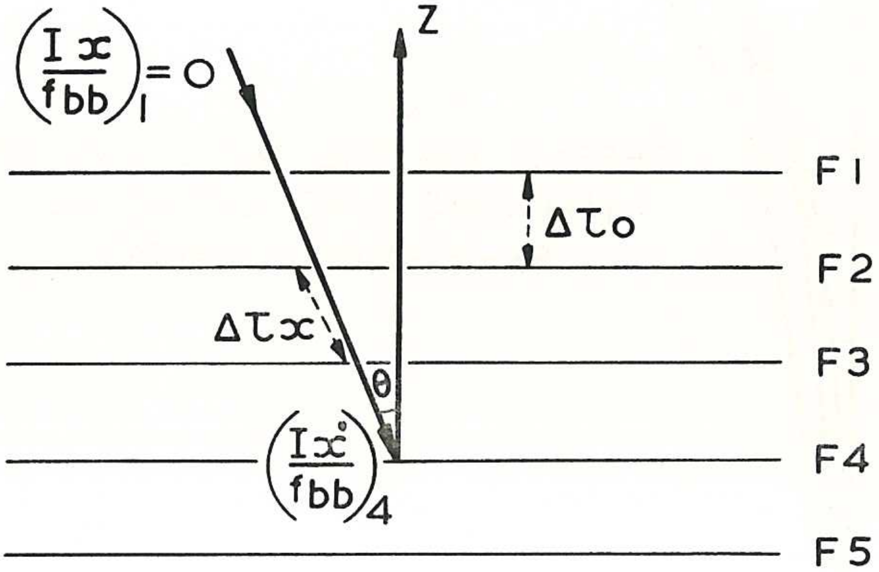


FIGURE 1.
SOLUTION OF THE EQUATION OF
TRANSFER THROUGH THE ATMOSPHERE.

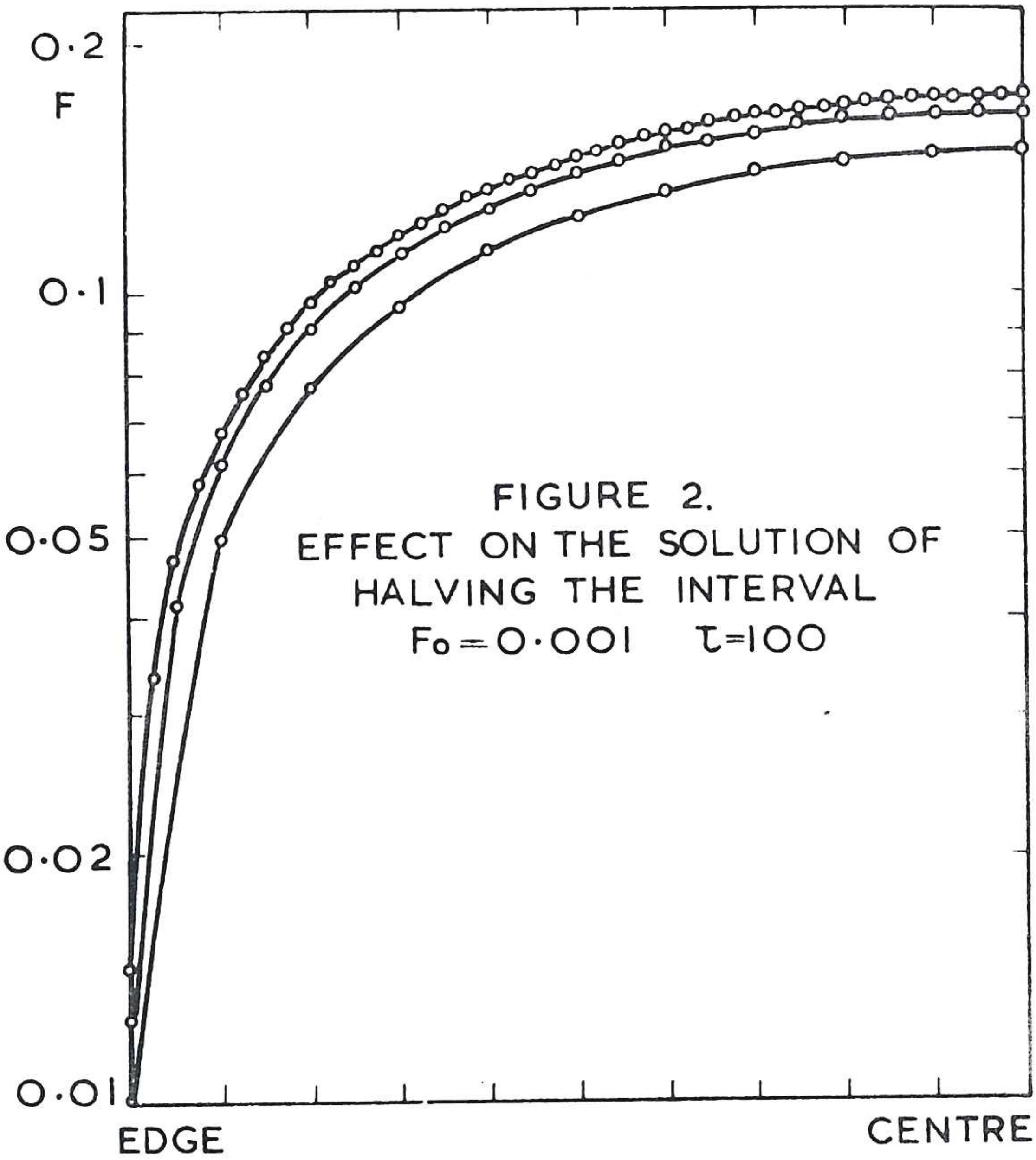


FIGURE 2.
EFFECT ON THE SOLUTION OF HALVING THE INTERVAL
$F_o = 0{\cdot}001 \quad \tau = 100$

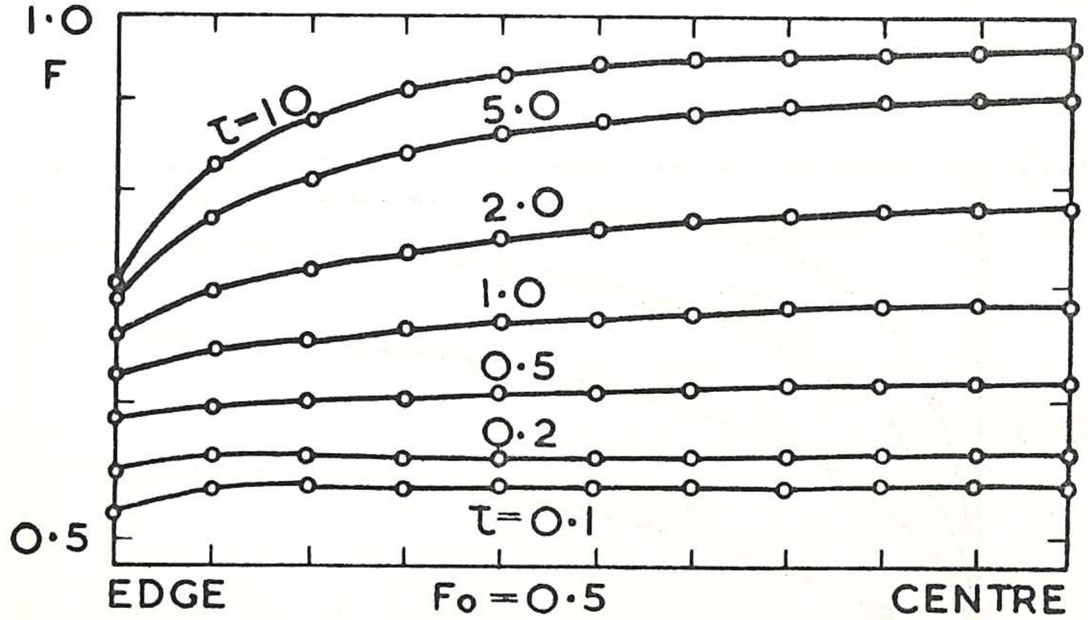


FIGURE 3.

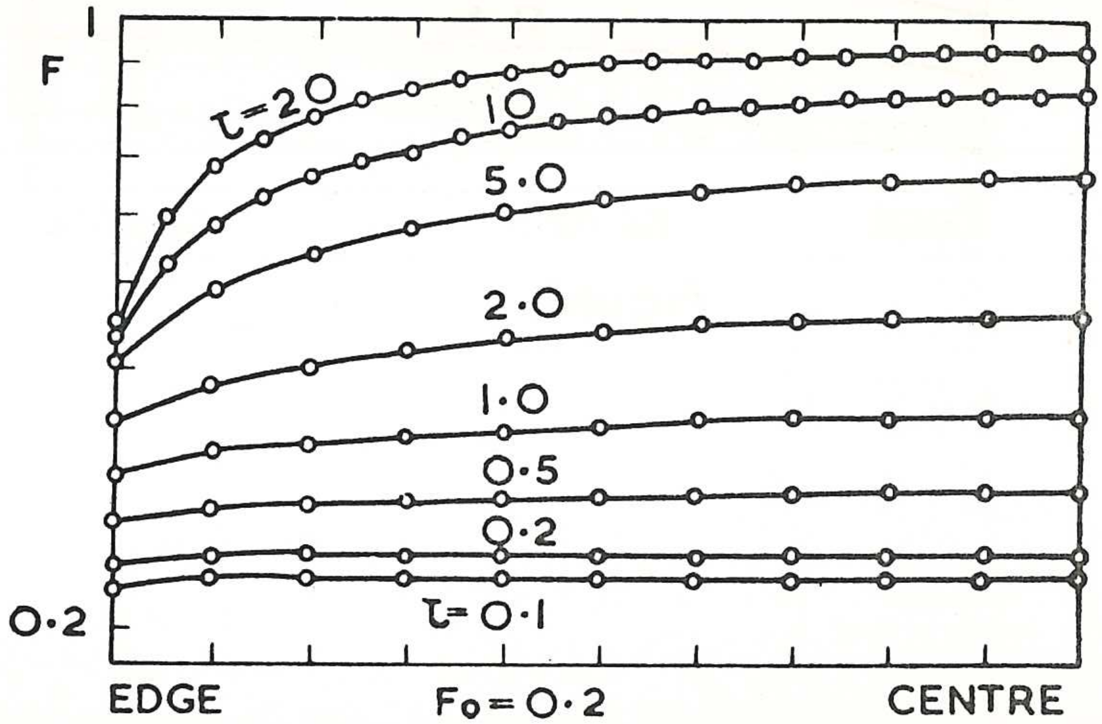


FIGURE 4.

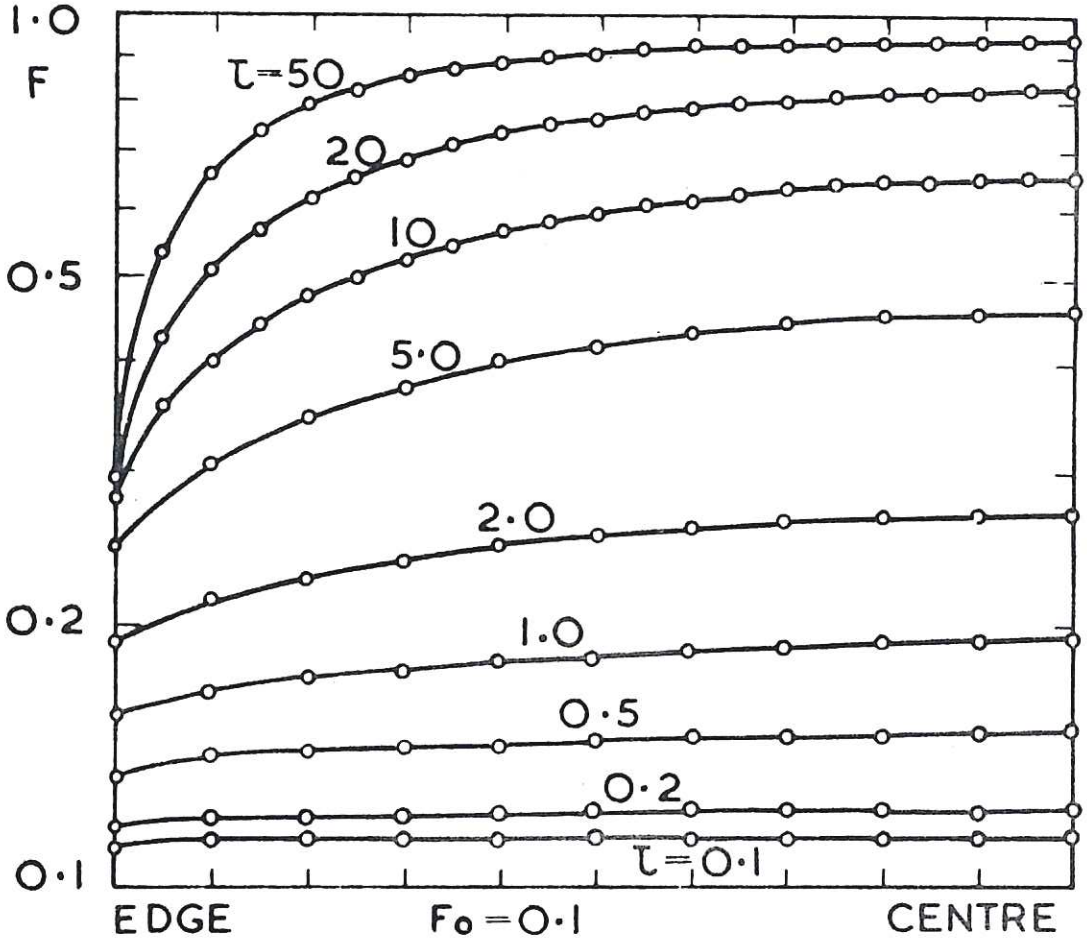


FIGURE 5.

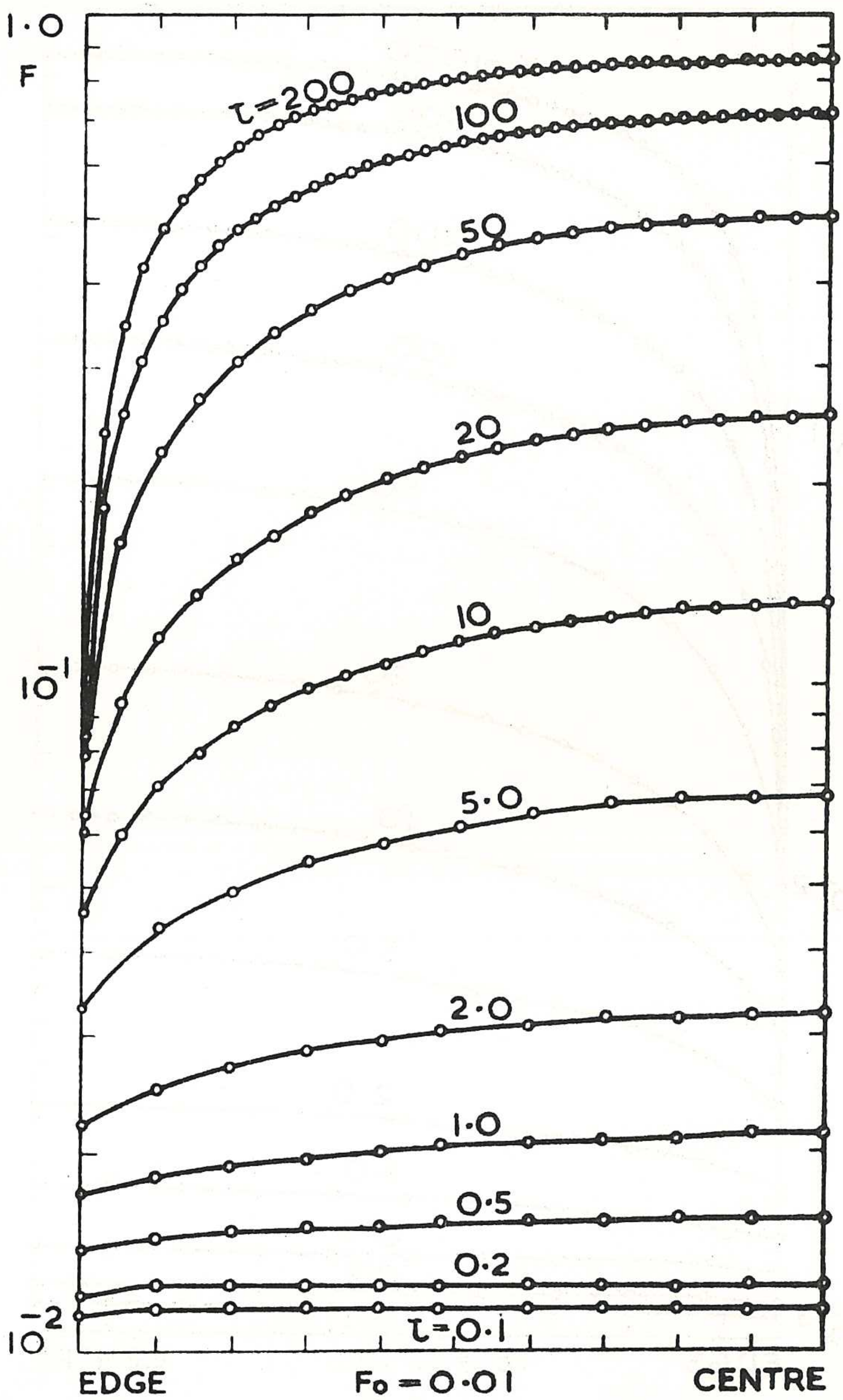

1·0
F
10⁻¹
10⁻²
τ=200
100
50
20
10
5·0
2·0
1·0
0·5
0·2
τ=0·1
EDGE
Fo = 0·01
CENTRE

FIGURE 6.

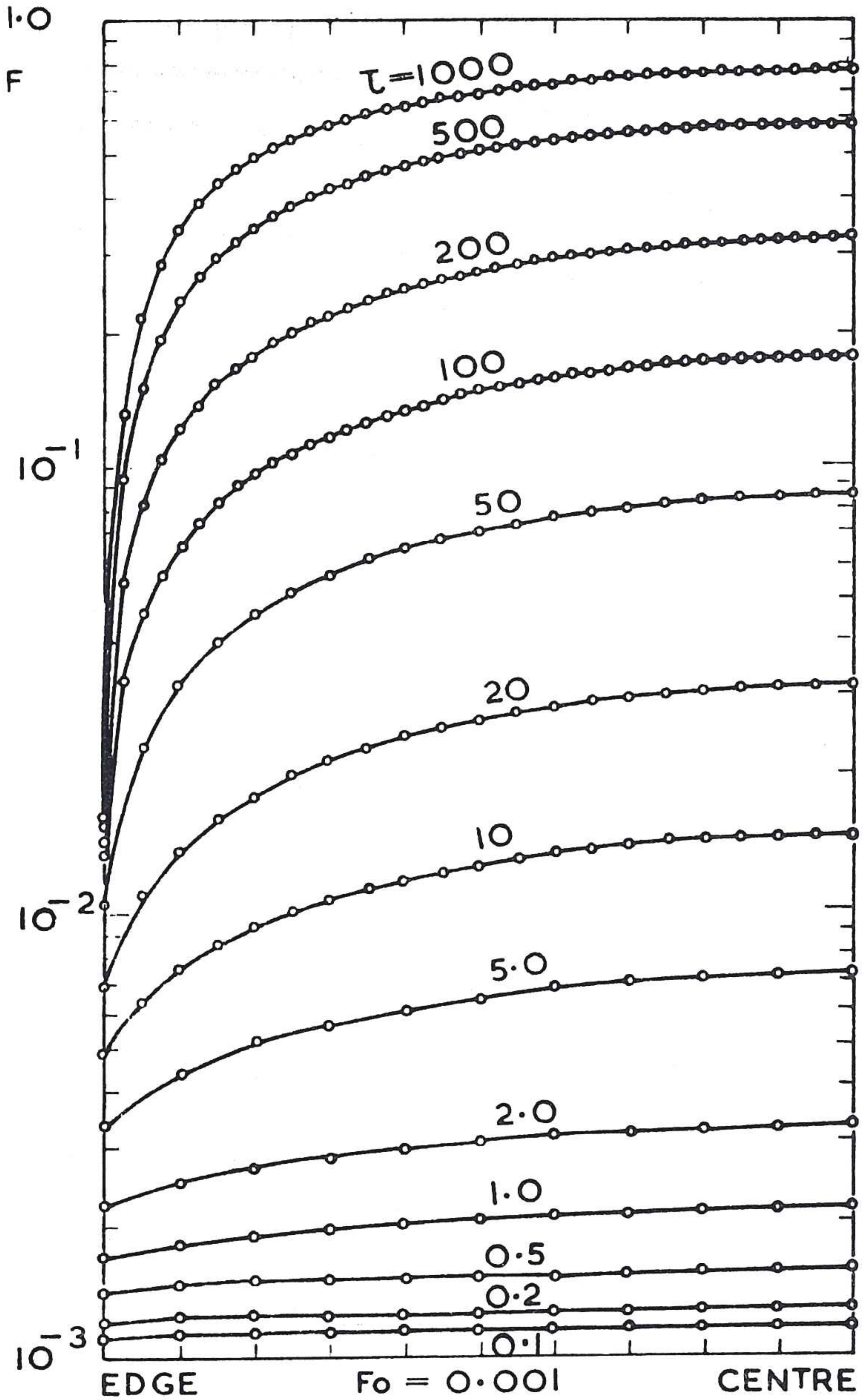

1·0
F
τ=1000
500
200
100
10⁻¹
50
20
10
10⁻²
5·0
2·0
1·0
0·5
0·2
0·1
10⁻³
EDGE
Fo = 0·001
CENTRE

FIGURE 7

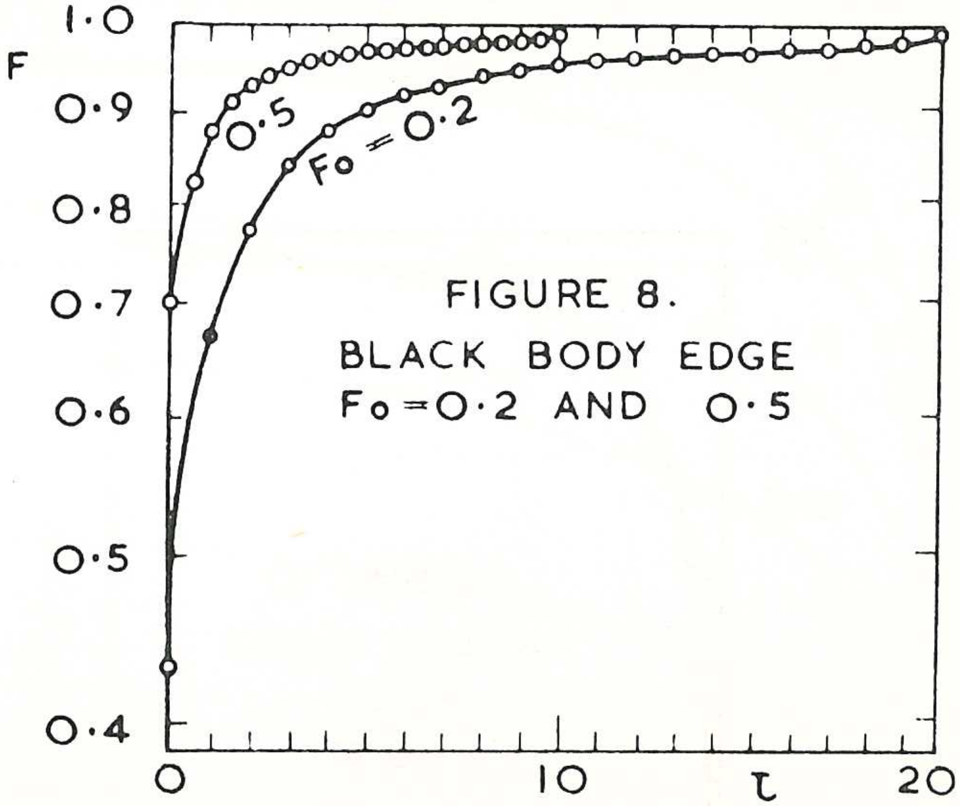


FIGURE 8.
BLACK BODY EDGE
Fo = 0·2 AND 0·5

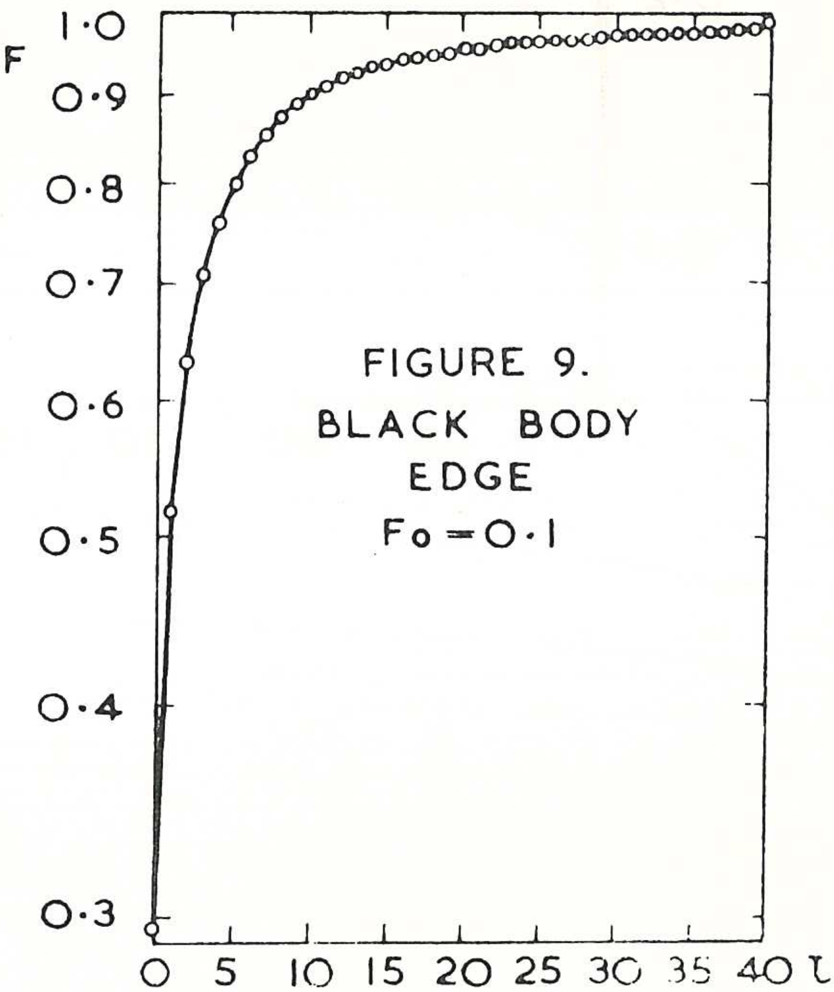


FIGURE 9.
BLACK BODY
EDGE
Fo = 0·1

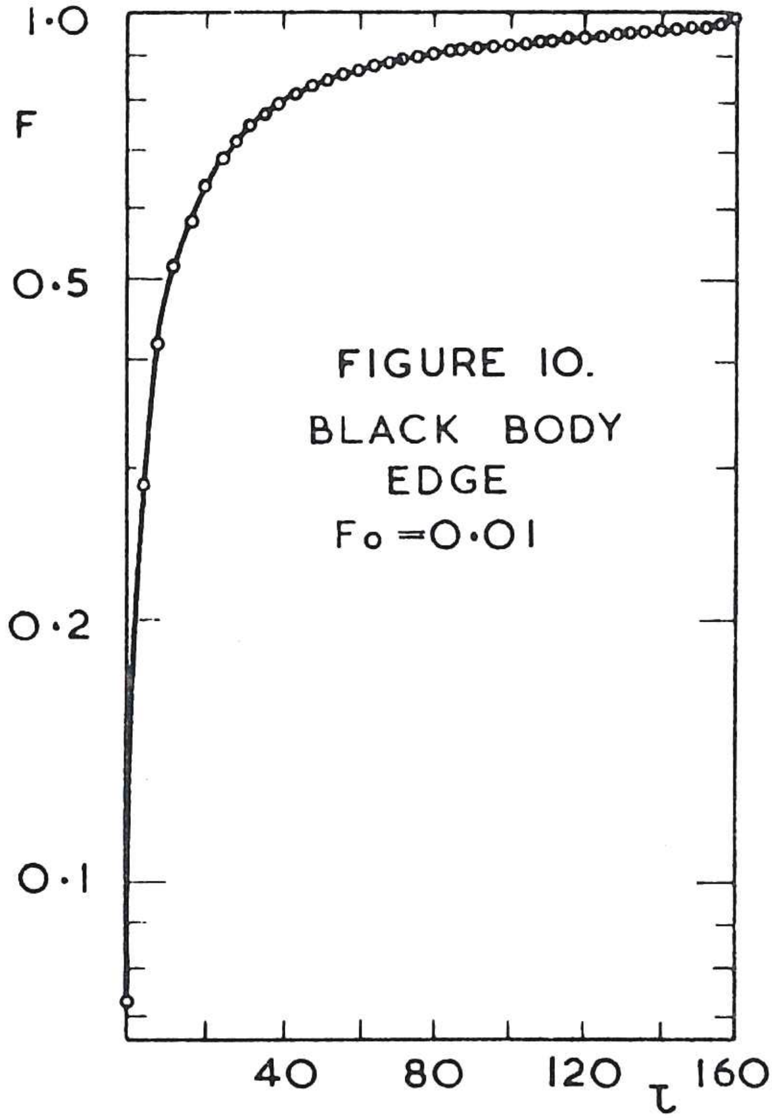


FIGURE 10.
BLACK BODY
EDGE
$F_o = 0{\cdot}01$

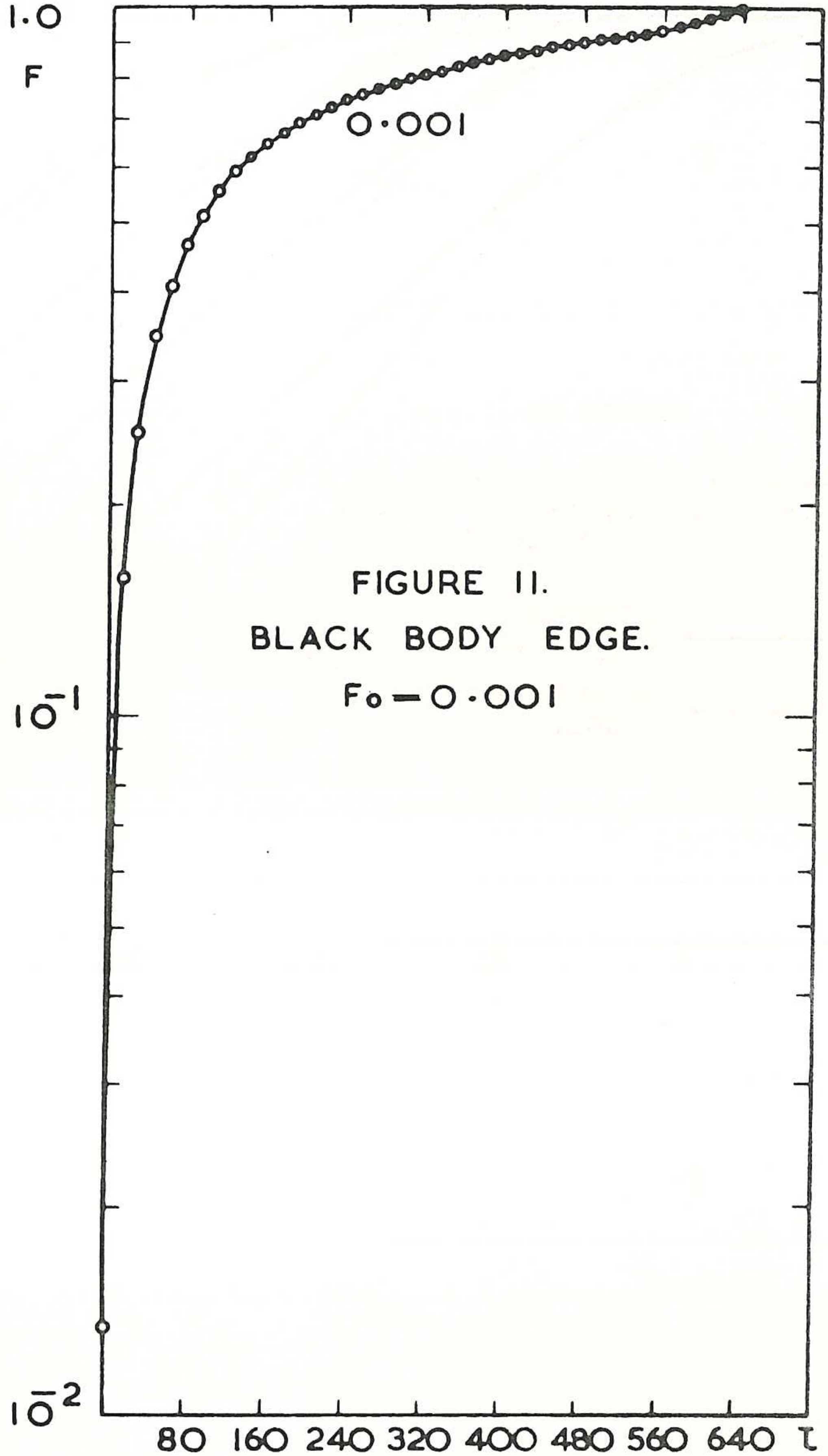


FIGURE 11.
BLACK BODY EDGE.
$F_o = 0 \cdot 001$

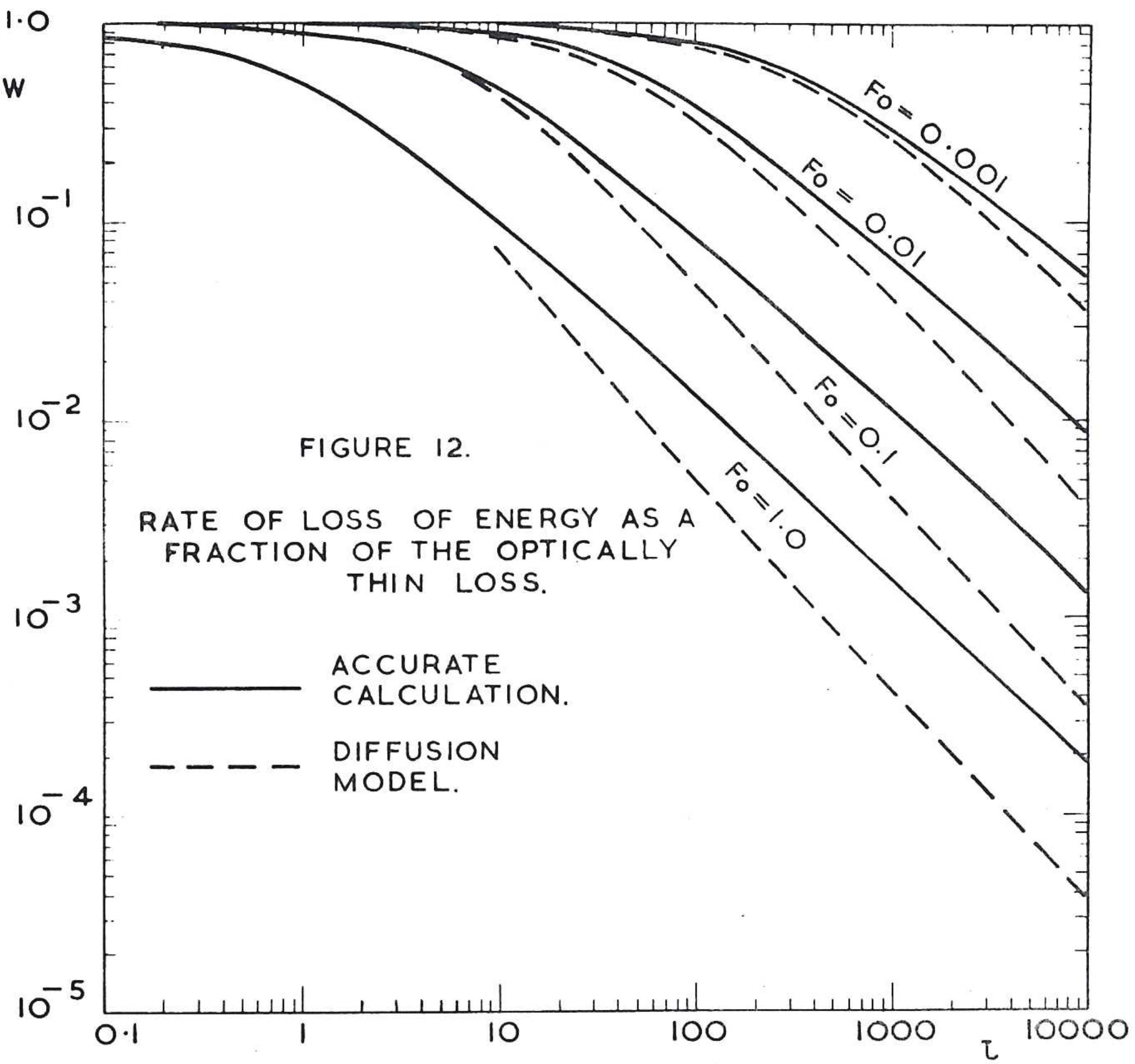
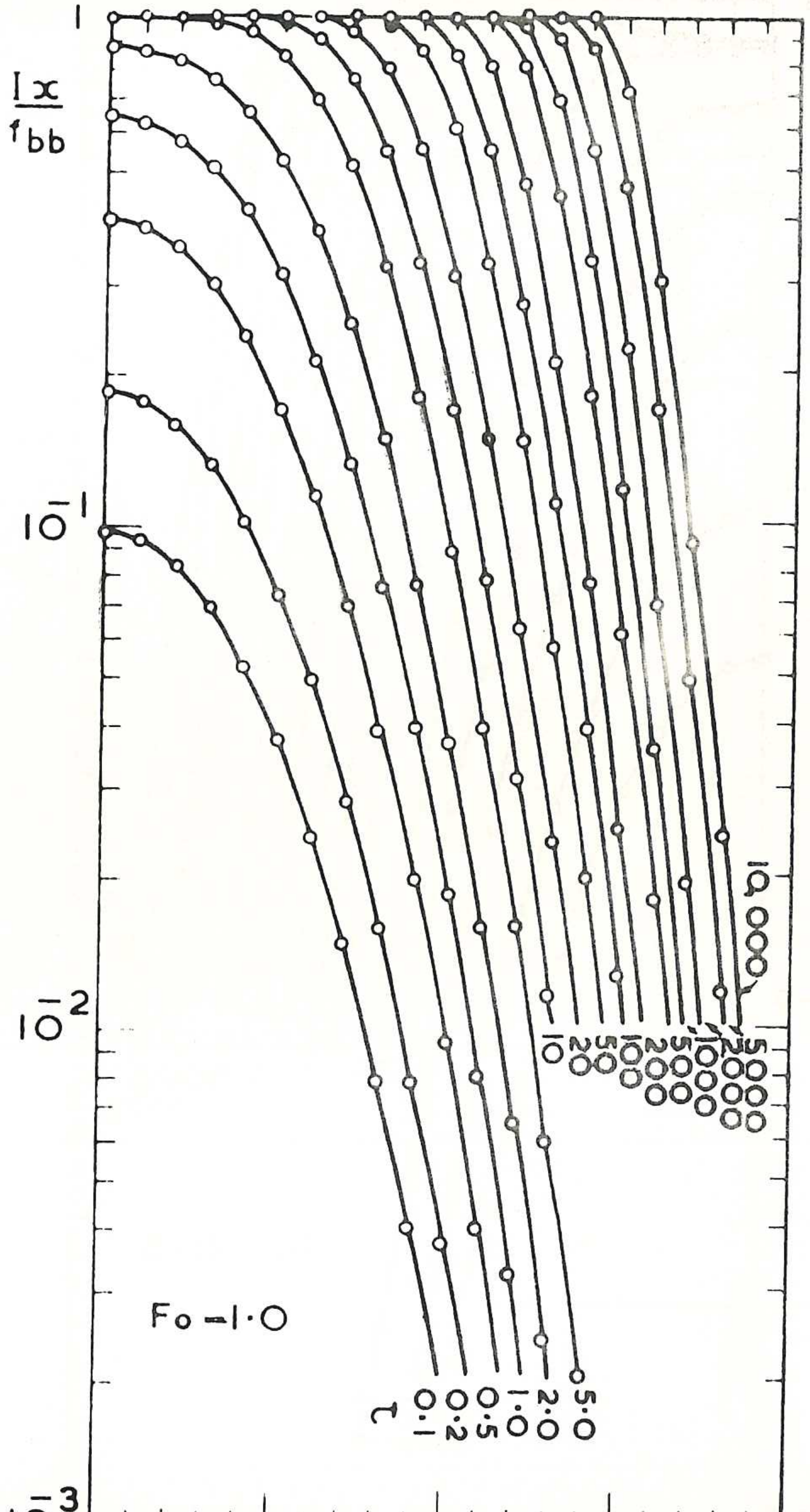


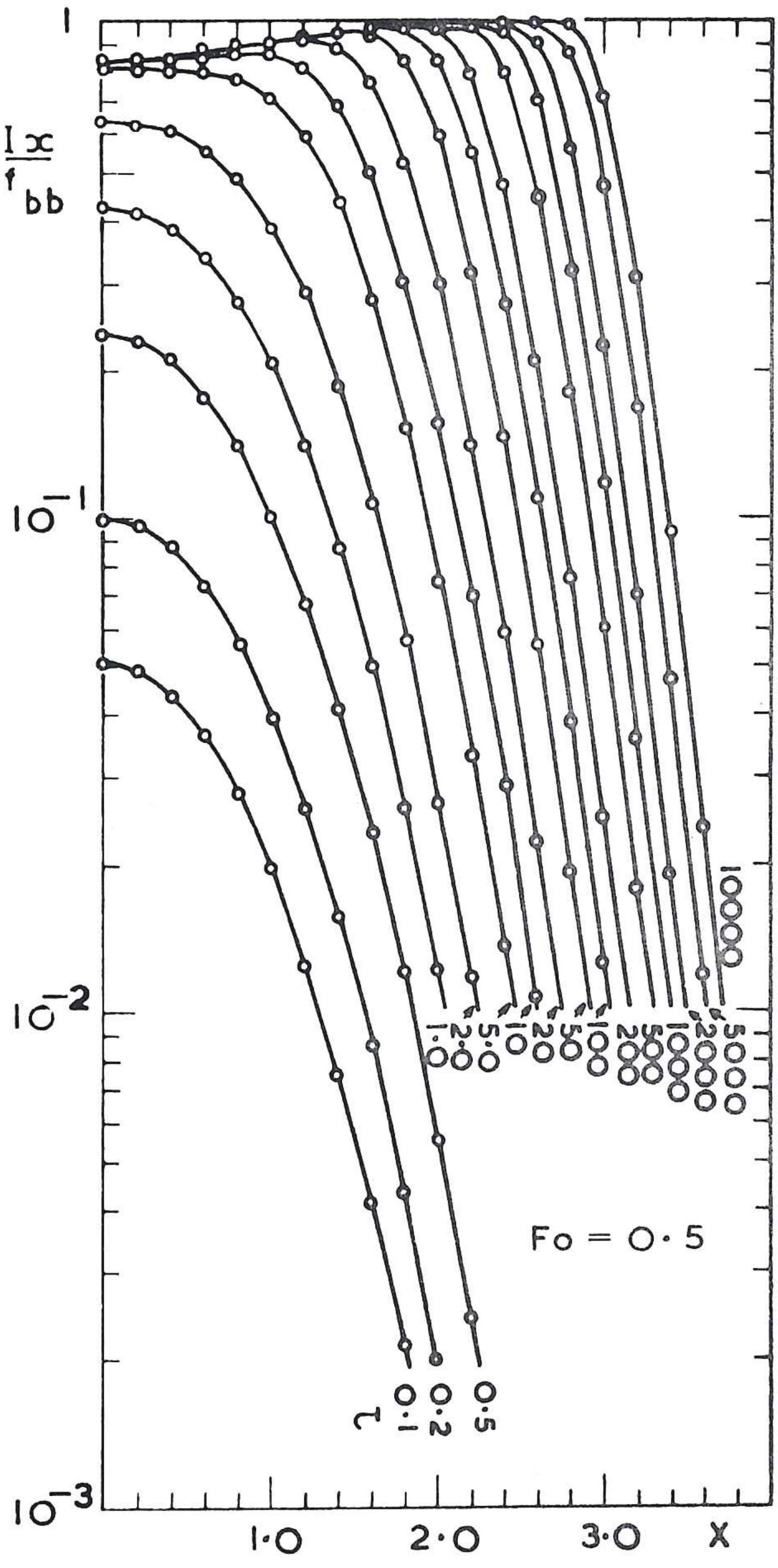
FIGURE 12.

RATE OF LOSS OF ENERGY AS A FRACTION OF THE OPTICALLY THIN LOSS.



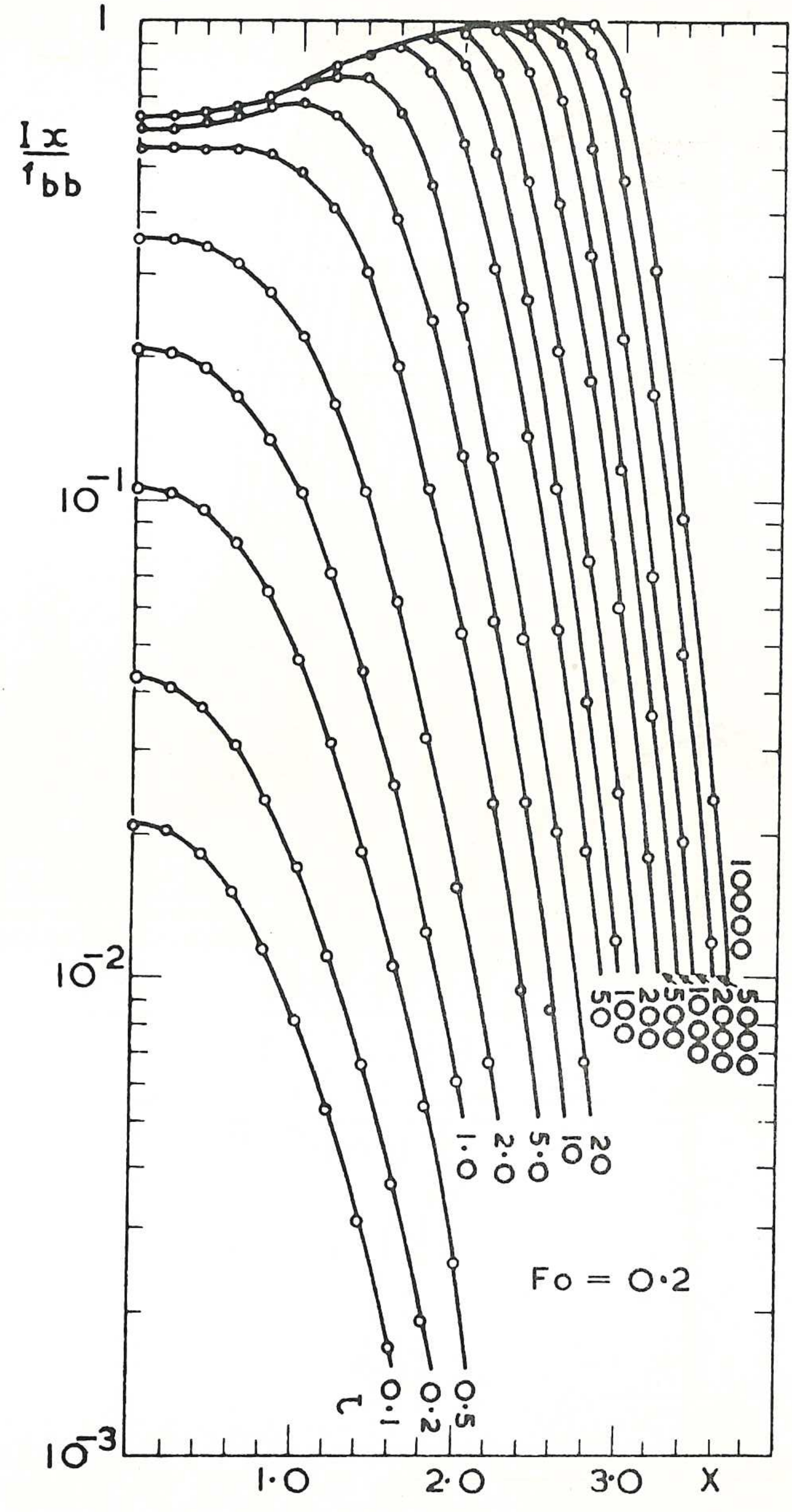
1
Ix / I_bb
10^-1
10^-2
10^-3
1·0
2·0
3·0
X
Fo =1·0
τ
0·1
0·2
0·5
1·0
2·0
5·0
10
20
50
100
200
500
1000
2000
5000
10,000
20
50
100
200
500
1000
2000
5000
10000
20000
50000

FIGURE 13.



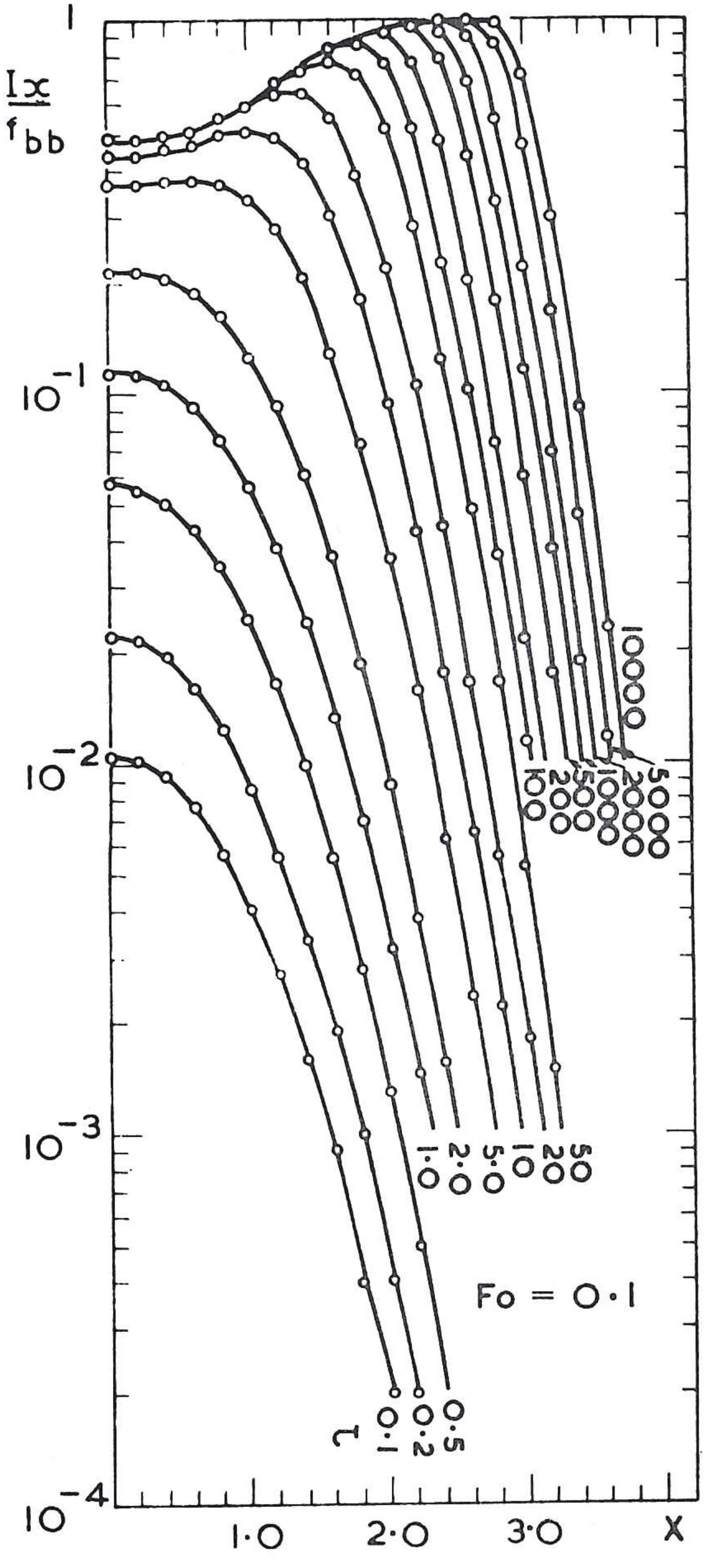
1
Ix / f bb
10^-1
10^-2
10^-3
1·0
2·0
3·0
X
τ
0·1
0·2
0·5
1·0
2·0
5·0
10
20
50
100
200
500
1000
2000
5000
10000
Fo = 0·5

FIGURE 14.



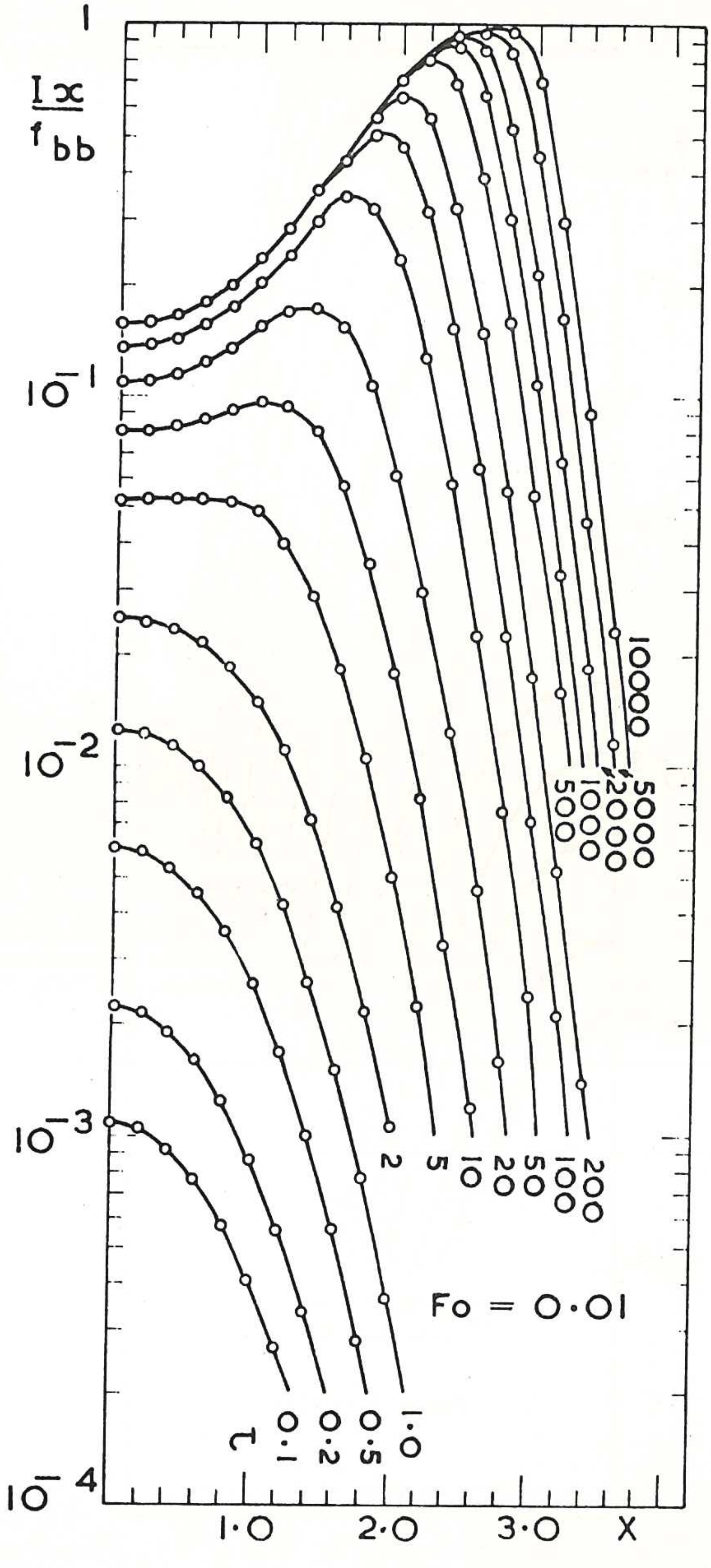
Ix
Ibb
1
10⁻¹
10⁻²
10⁻³
1·0
2·0
3·0
X
τ
0·1
0·2
0·5
1·0
2·0
5·0
10
20
50
100
200
500
1000
2000
5000
10000
Fo = 0·2

FIGURE 15.



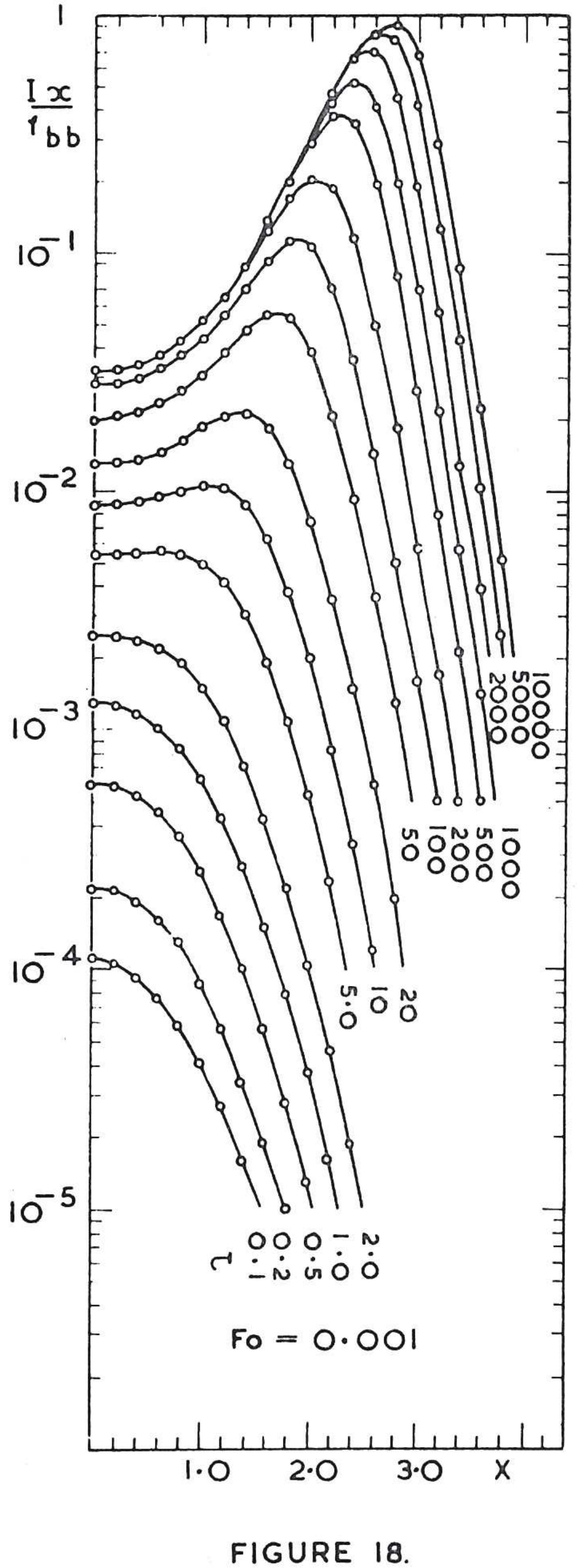
Ix / f_{bb}
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
1·0
2·0
3·0
X
τ
0·1
0·2
0·5
1·0
2·0
5·0
10
20
50
100
200
500
1000
2000
5000
10000
20000
50000
Fo = 0·1

FIGURE 16.



1
Ix
f bb
10⁻¹
10⁻²
10⁻³
10⁻⁴
1·0
2·0
3·0
X
Fo = 0·01
10000
5000
2000
1000
500
200
100
50
20
10
5
2
τ
0·1
0·2
0·5
1·0

FIGURE 17.



$\frac{I_x}{I_{bb}}$
1
10⁻¹
10⁻²
10⁻³
10⁻⁴
10⁻⁵
1·0
2·0
3·0
X
τ
0·1
0·2
0·5
1·0
2·0
5·0
10
20
50
100
200
500
1000
2000
5000
10000
Fo = 0·001

FIGURE 18.

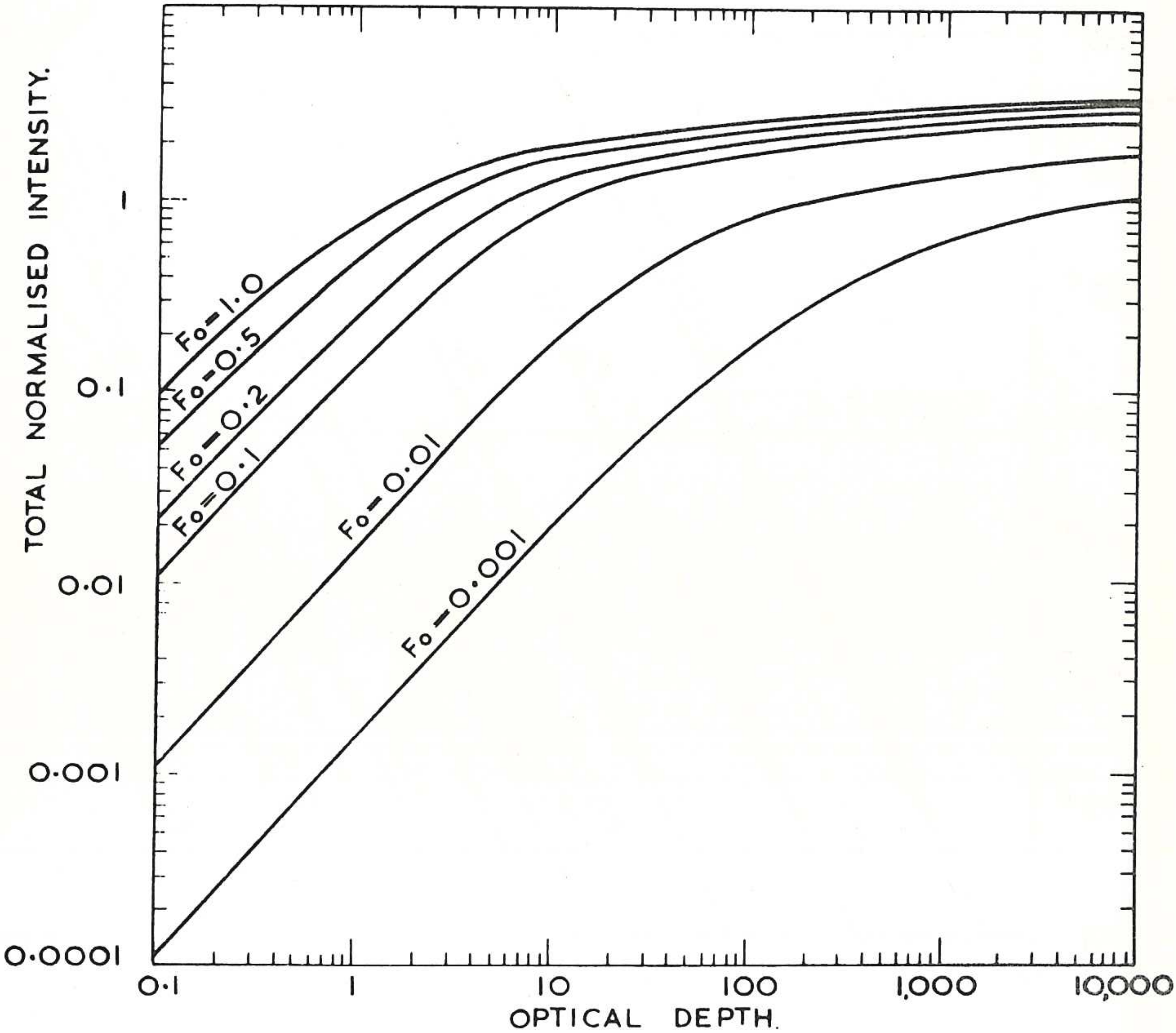

TOTAL NORMALISED INTENSITY.
1
0·1
0·01
0·001
0·0001
0·1
1
10
100
1,000
10,000
OPTICAL DEPTH.
Fo=1·0
Fo=0·5
Fo=0·2
Fo=0·1
Fo=0·01
Fo=0·001

FIGURE 19.

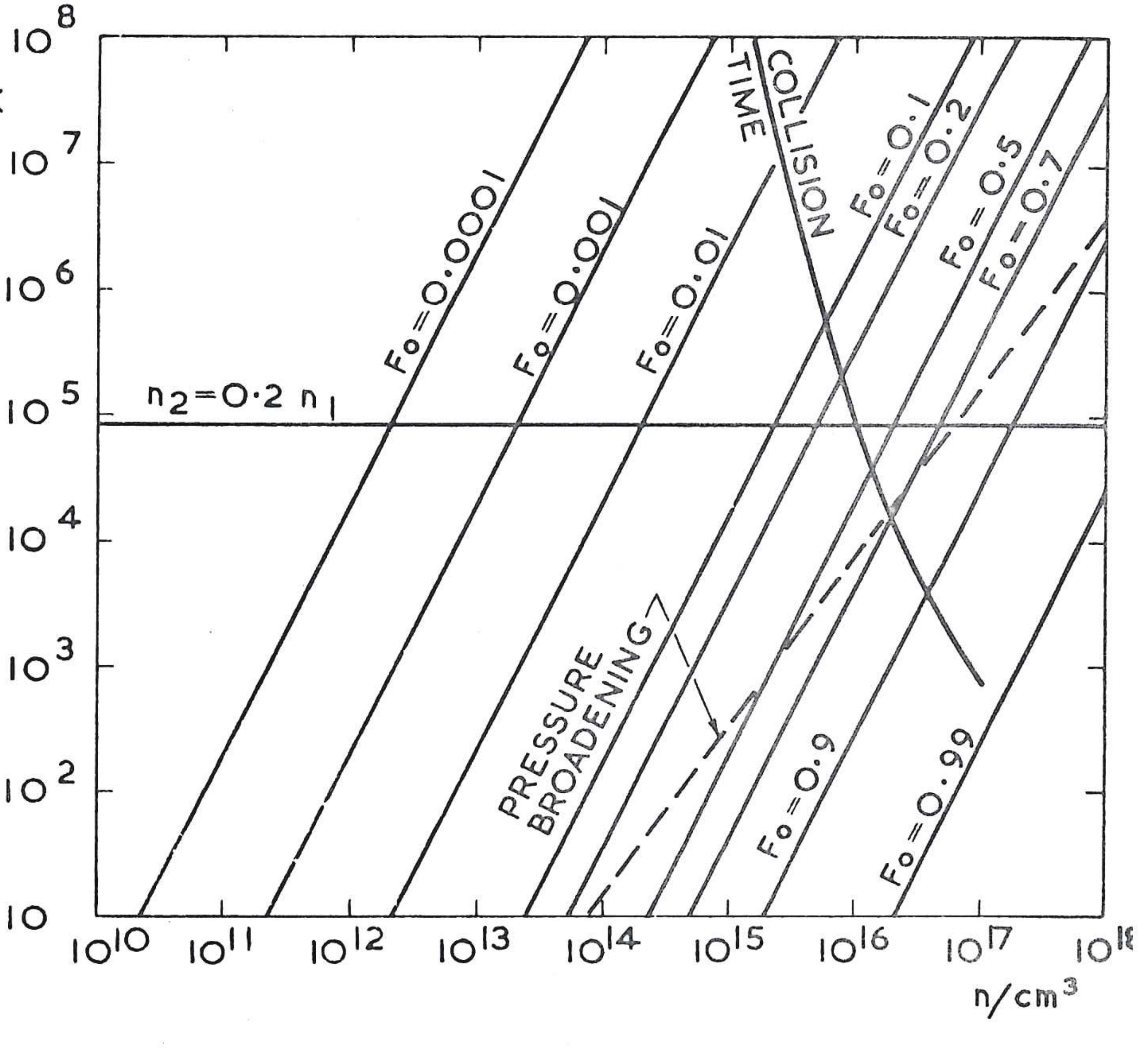
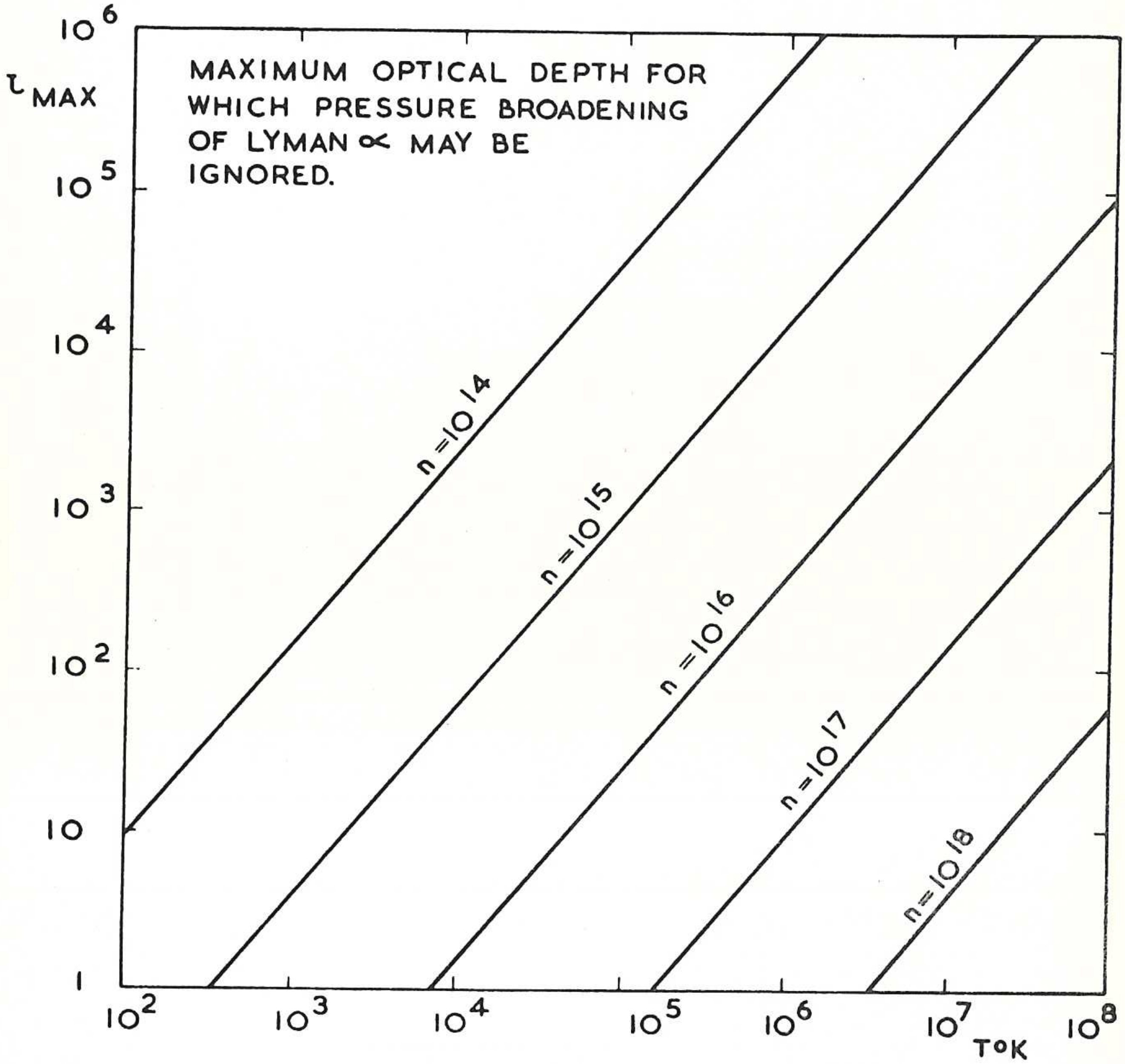


FIGURE 20.

RESTRICTIONS ON THE RANGE OF VALIDITY.



MAXIMUM OPTICAL DEPTH FOR WHICH PRESSURE BROADENING OF LYMAN ∝ MAY BE IGNORED.
τ MAX
10 6
10 5
10 4
10 3
10 2
10
1
10 2
10 3
10 4
10 5
10 6
10 7
10 8
T°K
n = 10 14
n = 10 15
n = 10 16
n = 10 17
n = 10 18

FIGURE 21.