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A TRANSPORT EQUATION FOR THE MULTIPLE SCATTERING OF ELECTROMAGNETIC WAVES BY A TURBULENT PLASMA

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by

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A B S T R A C T

Calculations of the scattering of electromagnetic waves by a turbulent plasma are usually based upon either a weak scattering or a random-walk approximation. The multiple scattering process is considered here by a method which is independent of the magnitude of the turbulent fluctuations. A solution is obtained in the form of a transport equation whose familiar properties describe the long-range behaviour of the scattering. The diffraction effects are contained in the kernel of the transport equation which is equivalent to the scattering cross-section per unit volume of the plasma.

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1. INTRODUCTION

The propagation of electromagnetic radiation through a turbulent plasma is of considerable interest in both naturally occurring and laboratory produced plasmas. For example, fluctuations in the electron density of the ionosphere are known to be responsible for the 'twinkling' effect observed in radio stars and the scattering of microwaves by turbulent instabilities has been observed in high-current discharges such as ZETA. (Edwards & Stott, 1965; Stott, 1967a)

Radiation propagating through a turbulent plasma is scattered by random localised variations in the refractive index produced by the fluctuating electron density. The scattering is in general a non-linear process since the radiation amplitude at any point in the plasma is composed partly of waves scattered from other regions of the plasma, in addition to the incident wave. The problem may be linearised and a straightforward solution obtained if the refractive index fluctuations are of sufficiently small magnitude to produce very weak scattering. Calculations of this nature (e.g. Tatarskii 1961, Chernov 1960) are useful in ionospheric scattering but are not applicable to the recent experiments which have been carried out in laboratory plasmas at frequencies close to the plasma frequency. If the turbulent scale length is large compared to the radiation wavelength, the simplifying approximation of ray-optics may be made. A ray-optics solution has been given by Wort (1966) by tracing the trajectories of individual ray-paths through a randomly spaced set of parallel, cylindrical plasma filaments which is taken as a model of two-dimensional turbulence. Some of the more drastic assumptions of this model, but not the restriction to ray-optics, have been removed by the recent numerical work of Rusbridge (1968).

A wave-optics solution for the problem of multiple scattering has been given by Tatarskii (1964) and independently by Stott (1967b). The electromagnetic wave equation is taken as the basis for a Green function series solution in which the randomly varying part of the refractive index is the expansion parameter. Averaging term by term results in an approximate geometric series which can be summed to give an expression for the mean Green function. The imaginary part of the singularities of the mean Green function in k-space represents the apparent attenuation experienced by an initially coherent wave as it is converted by scattering into incoherent radiation. However it is difficult to calculate the distribution of the incoherent radiation especially for experimentally realistic scattering geometries. There are further limitations to the Green function method stemming from the random phase approximation which implies that the scattering time be short compared to the free time between scatterings. It is also necessary to make mathematically unsupported assumptions regarding the continuity of the analytic properties of the Green function throughout the expansion and resummation procedure.

In this paper we consider the scattering of electromagnetic waves in a turbulent plasma using a method which avoids these difficulties. A generalised phase-space expansion technique, which removes the time-scale restriction, is employed to derive a transport equation for the energy density of the scattered electromagnetic field. The advantage of this formulation is that it separates the short-range interactions, ie wave-optics effects, from the long-range, ray-optics effects. The former are treated correctly in terms of diffraction theory and are contained in the kernel, the equivalent of the scattering cross-section in a particle transport equation, whilst the long-range effects are conveniently handled by the usual properties of the transport equation.

2. PLASMA TURBULENCE AND DENSITY FLUCTUATIONS

The term 'turbulent plasma' will be used here simply to refer to a plasma which supports density fluctuations of a scale-length longer than the Debye length yet small compared to the overall dimensions of the plasma and with a time-scale which is much shorter than the plasma lifetime. It will be assumed that a statistical description of the turbulence is both adequate and meaningful.

The local electron density will be written as: $n = \bar{n} + n(\underline{r},t)$ where $\bar{n}$ is the mean electron density, which is assumed stationary in space and time, and $n(\underline{r},t)$ is a randomly varying quantity with zero mean value.

We will consider initially a Gaussian functional probability distribution for $n(\underline{r},t)$, since a more generalised distribution can be included easily at a later stage.

$$P\left([n(\underline{r},t)]\right) = N \exp\left\{- \int n(\underline{r},t) \left(Q_n(\underline{r},\underline{r}';t,t')\right)^{-1} n(\underline{r}',t')d\underline{r}d\underline{r}'dtdt'\right\} \qquad \ldots (2.1)$$

N being a normalisation such that the total probability is unity.

Integration over the functional space (Gel'fand and Yaglom 1960) of $n(\underline{r},t)$ gives

$$\langle n(\underline{r},t)n(\underline{r}',t')\rangle = \int n(\underline{r},t)\ n(\underline{r}',t')\ P\left([n]\right) \delta n = Q_n(\underline{r},\underline{r}',t,t') \qquad \ldots (2.2)$$

$Q_n(\underline{r},\underline{r}',t,t')$ is the density correlation function. We will use triangular brackets as in (2.2) to denote ensemble averages.

It is convenient to introduce a typical scale-length a and time τ without insisting at this stage on any particular functional form for the correlation.

For homogeneous turbulence

$$Q_n(\underline{r},\underline{r}',t,t') = Q_n(\underline{r}-\underline{r}',t-t').$$

We see that on taking ensemble averages by integration, all the odd moments of $n(\underline{r},t)$ are zero, whilst the even moments can be expressed as permutations of the binary correlation $Q_n(\underline{r},\underline{r}',t,t')$. Thus for odd k

$$\langle n(\underline{r}_1,t_1) \;.....\; n(\underline{r}_k,t_k)\rangle = 0$$

and for even k

$$\langle n(\underline{r}_1,t_1) \;...\; n(\underline{r}_k,t_k)\rangle = \sum_{\text{perm}} Q_n(\underline{r}_1,\underline{r}_2,t_1,t_2) \;...\; Q(\underline{r}_\ell,\underline{t}_k,t_\ell,t_k) \qquad ... (2.3)$$

The dielectric constant of a plasma in an oscillating electromagnetic field of frequency ω is given by the Appleton-Hartree equation (Budden 1961).

$$\xi = 1 - X\left\{1-iZ - \frac{\frac{1}{2}Y_T^{\,2}}{1-X-iZ} \pm \left[\frac{\frac{1}{4}Y_T^{\,2}}{1-X-iZ} + Y_L^{\,2}\right]^{\frac{1}{2}}\right\}^{-1} \qquad ... (2.4)$$

where

$$X = 4\pi ne^2/m\omega^2 = \omega_p^2/\omega^2 = n/n_c$$

$$Z = \nu/\omega$$

$$Y = \omega_c/\omega$$

ω_p is the plasma frequency, ω_c the electron cyclotron frequency and ν the electron collision frequency. Y_L and Y_T are components parallel and transverse to the direction of the steady magnetic field $\underline{B}_o$. n_c is the critical density, i.e. the density at which the plasma frequency ω_p is equal to ω.

The dielectric constant is a function of the electron density. Thus density fluctuations produce corresponding local variations in the dielectric constant and hence the refractive index, resulting in the randomisation of the phase of a propagating wave front. We shall obtain a solution in terms of the dielectric correlation function, which is denumerable in terms of the density correlation by means of (2.4), although the dependence is non-linear in general. In many cases however, the Appleton-Hartree equation can be simplified. For example, if the collision and cyclotron frequencies are small compared to ω, i.e. $Z \ll 1$, $Y \ll 1$, the dielectric constant is linearly dependent upon the density.

$$\xi = 1 - X = 1 - n/n_c$$

Then

$$\bar{\xi} = 1 - \bar{n}/n_c$$

and

$$Q_\xi = n_c^{-2} Q_n .$$

3. THE GENERALISED EXPANSION

We will illustrate the general principles of the method of solution which will be employed by considering briefly the analogous problem of Brownian motion. Consider a single classical particle moving with a velocity $U(t)$ in a medium of viscosity J under the influence of a fluctuating force $f(t)$. The equation of motion is

$$\frac{dU}{dt} = - JU + f(t).$$

Since the particle is a discrete entity the probability $p(u,t)$ of finding $U(t)$ equal to some value u at a time t is a δ-function

$$p(u,t) = \delta(u-U(t)).$$

Now

$$\frac{\partial p}{\partial t} = \frac{\partial U}{\partial t}\frac{\partial}{\partial U} \delta(u-U(t))$$

$$= - \frac{\partial U}{\partial t}\frac{\partial}{\partial u} \delta(u-U(t))$$

$$= - \frac{\partial}{\partial u} (- Ju + f(t)) \delta(u-U(t))$$

i.e.

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial u} (-Ju + f(t)) p = 0. \quad \ldots (3.1)$$

This is Liouville's equation for a single particle and since it is linear in p it must be true also for an ensemble of particles.

We may now specify the randomly varying force $f(t)$ by means of the functional probability

$$F([f]) = N \exp\left\{-\tfrac{1}{2}\iint f(t_1)\, g^{-1}(t_1 - t_2)\, f(t_2)\, dt_1 dt_2\right\}$$

where N is a normalisation such that the total probability is unity. Then clearly

$$\langle p \rangle = F([u])$$

$$= N \exp\left\{-\tfrac{1}{2}\iint \left(\frac{\partial u}{\partial t_1} + Ju\right) g^{-1}(t_1 - t_2) \left(\frac{\partial u}{\partial t_2} + Ju\right) dt_1 dt_2\right\}.$$

If the time scale of $g(t)$ is sufficiently short, as for example if $g(t) = \Upsilon e^{-\chi t}$ and $\chi \gg J$, $\langle p \rangle$ is a Gaussian

$$\langle p \rangle = \left(\frac{\Upsilon J}{\pi(J+\chi)}\right)^{\frac{1}{2}} \exp\left\{-\frac{(J+\chi)}{2\Upsilon J}\, u^2\right\}$$

which is a result well-known in the theory of Brownian motion.

It can be shown easily that the mean distribution function satisfies the Fokker-Planck equation

$$\left\{\frac{\partial}{\partial t} + \frac{\partial}{\partial u}\left(-Ju - \Upsilon(J+\chi)^{-1}\frac{\partial}{\partial u}\right)\right\} \langle p \rangle = 0. \qquad \ldots (3.2)$$

In a more exact formulation of Brownian motion, the scattering force $f(t)$ is velocity dependent and no longer completely externally defined. This force $f(u,t)$ is not completely random since u is a functional of f but a convenient method of solution is to take the force as being approximately random and to obtain a solution as a series expansion about the equilibrium $\langle p_0 \rangle$. As pointed out by Edwards (1965), this approximation is independent of the magnitude of the fluctuations and must be distinguished from the usual form of weak coupling random phase approximation. One can derive the Fokker-Planck equation of Brownian motion in the form

$$\left\{ \frac{\partial}{\partial t} + \frac{\partial}{\partial u} \left(\Omega u + S \, . \, \frac{\partial}{\partial u} \right) \right\} \langle p \rangle = 0 \qquad \ldots (3.3)$$

$\Omega = J + R$ where J is the viscous friction and R, which is referred to as the dynamic friction, is a term occurring when the force f is internally dependent upon the variable u.

Returning to the electromagnetic case, it is a fairly straightforward matter to obtain from Maxwell's equations the analogous equation to (3.1). We may then proceed along the lines suggested above to a Fokker-Planck equation (Stott 1967b) but this is by no means so easy as it was for the Brownian motion model. The difficulties are apparently caused by the basic feature of Liouville's equation which separates the space and time variables. This is a useful thing to do in kinetic theory where they really are separable quantities, but is inconvenient in the electromagnetic case where space and time are closely inter-related. These and other difficulties can be avoided by using the Lagrangian formulation of statistical mechanics invented by Edwards (1965) for the analogous quantum-mechanical problem of electrons in a disordered system.

The Lagrangian equation of motion is

$$\frac{\partial}{\partial x_i} \left(\frac{\partial \mathcal{L}}{\left(\frac{\partial A_\alpha}{\partial x_i} \right)} \right) - \frac{\partial \mathcal{L}}{\partial A_\alpha} = 0 \qquad \ldots (3.4)$$

$\mathcal{L}$ being the Lagrangian density and x_i the space-time 4-vector; $x_i = (\underline{r}, ict)$. A_α are the components of the vector potential $\underline{A}$, defined in the Coulomb gauge, div $\underline{A} = 0$. There is also a similar equation to (3.4) for the complex conjugate A_α^*.

Then

$$\underline{E} = \frac{1}{c}\frac{\partial A}{\partial t} \quad \text{and} \quad \underline{E}^* = \frac{1}{c}\frac{\partial A^*}{\partial t}$$

$$\underline{H} = \text{curl } \underline{A} \quad \text{and} \quad \underline{H}^* = \text{curl } \underline{A}^* .$$

The Lagrangian density for the electromagnetic field (Landau and Lifshitz 1959) is

$$\mathcal{L} = \frac{1}{8\pi}\left(\underline{E}.\underline{D}^* - \underline{B}.\underline{H}^*\right)$$

or in terms of the vector potential

$$\mathcal{L} = (8\pi)^{-1}\left\{\frac{1}{c^2}\,\xi(\underline{r},t)\,.\,\frac{\partial}{\partial t}\,\underline{A}\,.\,\frac{\partial}{\partial t}\,\underline{A}^* - \text{curl } \underline{A}.\text{curl } \underline{A}^*\right\} \qquad \ldots (3.5)$$

where $\xi(\underline{r},t)$ is the dielectric constant.

We introduce the probability function for A_α and A^*_α; $P([A_\alpha, A^*_\alpha])$ and following Edwards (1965) the Lagrangian equivalent of (3.1) is,

$$\left\{\frac{\partial}{\partial A_\alpha}\left(\frac{\partial}{\partial x_i}\,.\,\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial A_\alpha}{\partial x_i}\right)} - \frac{\partial\mathcal{L}}{\partial A_\alpha}\right) - \frac{\partial}{\partial A^*_\alpha}\left(\frac{\partial}{\partial x_i}\,.\,\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial A^*_\alpha}{\partial x_i}\right)} - \frac{\partial\mathcal{L}}{\partial A^*_\alpha}\right)\right\}.P([A_\alpha, A^*_\alpha]) = 0 \qquad \ldots (3.6)$$

Substituting for $\mathcal{L}$ from (3.5)

$$\left\{\frac{\partial}{\partial A_\alpha}\left(\xi(\underline{r},t)\,\frac{\partial^2}{\partial t^2}\,A_\alpha - \text{curl.curl.}A_\alpha\right)\right.$$

$$\left. - \frac{\partial}{\partial A^*_\alpha}\left(\xi\,(\underline{r},t)\,\frac{\partial^2}{\partial t^2}\,A^*_\alpha - \text{curl.curl.}A^*_\alpha\right)\right\} P([A_\alpha, A^*_\alpha]) = 0. \qquad \ldots (3.7)$$

We will use the superscript k to denote the 4-dimensional Fourier transform, thus

$$A^k = A(\underline{k}, \tfrac{\omega}{c}) = \int A(\underline{r},t)\,\exp\,(-\,i\omega t + i\underline{k}.\underline{r})\,d\underline{r}\,dt$$

and

$$A^{k*} = A^*(\underline{k}, \tfrac{\omega}{c}) = \int A^*(\underline{r},t)\,\exp\,(i\omega t - i\underline{k}.\underline{r})\,d\underline{r}\,dt.$$

Fourier transformation of (3.7) gives

$$\left[\sum_{\substack{\alpha\beta\\k}}\left\{\frac{\partial}{\partial A^k_\alpha}\left(M^k_{\alpha\beta}\,A^k_\beta + \delta_{\alpha\beta}\,\frac{\omega_k^2}{c^2}\,\sum_j{}'\,\xi^{k-j}\,A^j_\beta\right)\right\}\right. \qquad \ldots (3.8)$$

$$\left. - \sum_{\substack{\alpha\beta\\k}}\left\{\frac{\partial}{\partial A^{k*}_\alpha}\left(M^k_{\alpha\beta}\,A^{k*}_\beta + \delta_{\alpha\beta}\,\frac{\omega_k^2}{c^2}\,\sum_j{}'\,\xi^{k-j}\,A^{j*}_\beta\right)\right\}\right] P([A_\alpha, A^*_\alpha]) = 0 .$$

$$M^k_{\alpha\beta} = \delta_{\alpha\beta} \frac{\omega_k^2}{c^2} \bar{\xi} - k^2 D^k_{\alpha\beta}$$

and

$$D^k_{\alpha\beta} = \left(\delta_{\alpha\beta} - k_\alpha k_\beta k^{-2}\right) .$$

The term $\bar{\xi} = [\xi^{k-j}]_{k=j}$ has been included in $M^k_{\alpha\beta}$ and the prime on the second summation sign in (3.8) is used to indicate that the term $k = j$ is omitted from the summation. The subscripts α, β etc. are used to denote the polarisation components of the vector potential and the usual tensor conventions regarding summation over repeated indices will be employed.

The Brownian motion model suggests a Gaussian form for the equilibrium distribution

$$P_o([A,A^*]) = N \exp\left\{-\sum_{\alpha k} A^k_\alpha (C^k)^{-1} A^{k*}_\alpha\right\} . \qquad \text{... (3.9)}$$

The gauge condition provides the constraint

$$\text{div } \underline{A} = \underline{k}.\underline{A}^k = 0$$

and hence the correlation

$$\begin{aligned} \langle A^k_\alpha A^{k*}_\beta \rangle &= \int A^k_\alpha A^{k*}_\beta P_o([A]) \, \delta(\underline{k}.A^k) \, \delta A \\ &= C^k D^k_{\alpha\beta} \qquad \text{... (3.10)} \\ &= C^k_{\alpha\beta} . \end{aligned}$$

We expect P_o to be the solution of the Fokker-Planck equation

$$\left[\sum_{\substack{\alpha\beta \\ k}} \frac{\partial}{\partial A^k_\alpha} \left\{ \Omega^k_{\alpha\beta} A^k_\beta + S^k_{\alpha\beta} \frac{\partial}{\partial A^{k*}_\beta} \right\} - \sum_{\substack{\alpha\beta \\ k}} \frac{\partial}{\partial A^{k*}_\alpha} \left\{ \Omega^{k*}_{\alpha\beta} A^{k*}_\beta + S^{k*}_{\alpha\beta} \frac{\partial}{\partial A^k_\beta} \right\} \right] P_o([A,A^*]) = 0 . \qquad \text{... (3.11)}$$

Substitution of (3.9) into (3.11) requires that

$$\Omega^k_{\alpha\beta} - \sum_\gamma S^k_{\alpha\gamma} (C^k_{\gamma\beta})^{-1} = 0 \qquad \text{... (3.12)}$$

and

$$\Omega^{k*}_{\alpha\beta} - \sum_{\gamma} S^{k*}_{\alpha\gamma} (C^{k}_{\gamma\beta})^{-1} = 0.$$

We see that

$$\sum_{\gamma} D^{k}_{\alpha\gamma} . D^{k}_{\gamma\beta} = D^{k}_{\alpha\beta}$$

which suggests that

$$\Omega^{k}_{\alpha\beta} = \Omega^{k} D^{k}_{\alpha\beta} \qquad \ldots (3.13)$$

and

$$S^{k}_{\alpha\beta} = S^{k} D^{k}_{\alpha\beta} . \qquad \ldots (3.14)$$

We will write

$$\Omega^{k}_{\alpha\beta} = M^{k}_{\alpha\beta} + R^{k}_{\alpha\beta} \qquad \ldots (3.15)$$

defining $R^{k}_{\alpha\beta}$; the electromagnetic analogue of the dynamic friction term in Brownian motion.

The operator in (3.11) is Hermite's operator generalised for the functional space of A^{k}_{α} and $P([A])$ may therefore be expanded as a series of functional Hermite polynomials (Erdelyi et al. 1953) multiplied by the basic Gaussian.

i.e.

$$P([A,A^*]) = P_o + P_1 + P_2 + \text{etc.}$$
$$= \sum_{n}{}' a_n \prod_{k} H_{n_k, n_{k*}} (A^k, A^{k*}) \, P_o . \qquad \ldots (3.16)$$

The functional Hermite polynomials are mutually orthogonal and are defined by

$$H_{n_k, n_{k*}} (A^k, A^{k*}) = \left[(n_k + 1)! \right]^{-\frac{1}{2}} \left[(n_{k*} + 1)! \right]^{-\frac{1}{2}}$$
$$. \left(C^k \frac{\partial}{\partial A^{k*}} - A^k \right)^{n_k} \left(C^k \frac{\partial}{\partial A^k} - A^{k*} \right)^{n_{k*}} P_o \qquad \ldots (3.17)$$

where $\underline{n} = (n_k \ldots\ldots, n_{k*} \ldots\ldots)$ is a vector in the Hilbert space of the polynominals.

We proceed with the expansion in a systematic manner by first rearranging equation (3.8)

$$\left\{ \sum_{\substack{\alpha\beta \\ k}} \frac{\partial}{\partial A_\alpha^k} \left(\Omega_{\alpha\beta}^k \; A_\beta^k + S_{\alpha\beta}^k \; \frac{\partial}{\partial A_\beta^{k*}} \right) - \text{complex conjugate} \right\} P$$

$$+ \left\{ \sum_{\substack{\alpha\beta \\ k}} \frac{\partial}{\partial A_\alpha^k} \cdot \delta_{\alpha\beta} \frac{\omega_k^2}{c^2} \sum_j{}' \xi^{k-j} \; A_\beta^j \; . \; - \text{c.c} \right\} P$$

$$- \left\{ \sum_{\substack{\alpha\beta \\ k}} \frac{\partial}{\partial A_\alpha^k} \left(R_{\alpha\beta}^k \; A_\beta^k + S_{\alpha\beta}^k \; \frac{\partial}{\partial A_\alpha^{k*}} \right) - \text{c.c} \right\} P = 0. \qquad \ldots (3.18)$$

If we assign the nominal order ξ^2 to R and S we can make the expansion in ascending orders of ξ. Thus to the zeroth order we have (3.11) and to the first order

$$HP_1 = \left\{ - \sum_{\substack{\alpha\beta \\ k}} \frac{\partial}{\partial A_\alpha^k} \cdot \delta_{\alpha\beta} \frac{\omega_k^2}{c^2} \sum_j{}' \; \xi^{k-j} \; A_\beta^j - \text{c.c} \right\} P_o$$

$$= \left\{ - \sum_{\substack{\alpha\beta \\ k_j}}{}' A_\alpha^{k*} \left(C^k \right)^{-1} \delta_{\alpha\beta} \frac{\omega_k^2}{c^2} \xi^{k-j} \; A_\beta^j \; - \text{c.c} \right\} P_o . \qquad \ldots (3.19)$$

This suggests that P_1 has the form

$$P_1 = \sum_{\substack{\alpha\beta \\ k_j}}{}' A_\alpha^{k*} \; L_{\alpha\beta}^{k,j} \; (C^k)^{-1} \; \frac{\omega_k^2}{c^2} \; \xi^{k-j} \; A_\beta^j \; P_o \; . \qquad \ldots (3.20)$$

Substituting in (3.19)

$$L_{\alpha\beta}^{kj} = \left(\Omega_{\alpha\beta}^k - \Omega_{\alpha\beta}^{j*} \right)^{-1} . \qquad \ldots (3.21)$$

Continuing to the next order in ξ with the series solution of (3.18)

$$H P_2 = \left\{ - \sum_{\substack{\alpha\beta \\ k}} \frac{\partial}{\partial A^k_\alpha} \delta_{\alpha\beta} \frac{\omega^2_k}{c^2} \xi^{k-j} A^j_\beta - c.c \right\} P_1$$

$$+ \left\{ \sum_{\substack{\alpha\beta \\ k_j}}{}' \frac{\partial}{\partial A^k_\alpha} \left(R^k_{\alpha\beta} A^k_\beta + S^k_{\alpha\beta} \frac{\partial}{\partial A^{k*}_\beta} \right) - c.c \right\} P_o \qquad \ldots (3.22)$$

We have defined the mean value of the electromagnetic correlation $\langle A^k_\alpha A^{k*}_\beta \rangle$ using only P_o and therefore the higher order probability terms must not contain any second order Hermite polynominals

i.e.
$$\int A^k_\alpha A^{k*}_\beta (P_1 + P_2 + \ldots \text{etc.}) \, \delta A \, \delta A* = 0. \qquad \ldots (3.23)$$

This condition will now be applied to the series solution of (3.18) to determine successive approximations for the quantities R^k and S^k which will then be used together with the consistency equation (3.12) to obtain an equation for C^k.

We will assume a Gaussian distribution for the dielectric fluctuations $\xi(\underline{r},t)$ and consider the extension of the theory to a more generalised form in the Appendix.

$$F([\xi]) = N \exp \left\{ - \xi(\underline{r},t) \, Q^{-1}_\xi(\underline{r}-\underline{r}',t-t') \, \xi(\underline{r}',t') \right\} \qquad \ldots (3.24)$$

The dielectric correlation is

$$\langle \xi(\underline{r}_1,t_1) \, \xi(\underline{r}_2,t_2) \rangle = Q_\xi(\underline{r}_1-\underline{r}_2 \, ; t_1-t_2) \, . \qquad \ldots (3.25)$$

On averaging over ξ, $\langle P_1 \rangle = 0$. The lowest order approximation is thus obtained by applying the condition (3.25) to the second-order expansion (3.22);

$$R^k_{\alpha\beta} - S^k_{\alpha\beta}(C^k)^{-1} = \sum_j{}' L^{kj}_{\alpha\beta} \frac{\omega^2_k}{c^2} \frac{\omega^2_j}{c^2} Q^{k-j}_\xi$$

$$+ \sum_{\gamma j} L^{kj}_{\alpha\gamma} \frac{\omega^2_k}{c^2} \frac{\omega^2_j}{c^2} Q^{k-j}_\xi C^j_{\gamma\beta} \qquad \ldots (3.26)$$

and a similar equation for the complex conjugates.

Together with (3.12) we have immediately the lowest order equation for C^k

$$\sum_{\gamma} M^k_{\alpha\gamma} C^k_{\gamma\beta} - \sum_{\gamma j} L^{kj}_{\alpha\gamma} \omega_k^2 \omega_j^2 c^{-4} Q_\xi^{k-j} \left(C^k_{\gamma\beta} - C^j_{\gamma\beta}\right) = 0 \qquad \ldots (3.27)$$

Now

$$\sum_{\gamma} M^k_{\alpha\gamma} C^k_{\gamma\beta} = \sum_{\gamma} \left\{\bar{\xi} \frac{\omega^2}{c^2} \delta_{\alpha\gamma} - k^2 D^k_{\alpha\gamma}\right\} D^k_{\gamma\beta} C^k$$

$$= \left\{\bar{\xi} \frac{\omega^2}{c^2} - k^2\right\} C^k_{\alpha\beta}$$

$$= \left\{k_o^2 - k^2\right\} C^k_{\alpha\beta}$$

and thus

$$\left\{\bar{\xi} \frac{\omega^2}{c^2} - k^2\right\} C^k_{\alpha\beta} - \sum_j{}' (\Omega^k - \Omega^{j*})^{-1} \frac{\omega_k^2 \omega_j^2}{c^4} Q_\xi^{k-j} \left(C^k_{\alpha\beta} - C^j_{\alpha\beta}\right) = 0. \qquad \ldots (3.28)$$

Fourier transformation of (3.30) gives an equation for the propagation through the plasma of the mean electromagnetic correlation

$$\left\{\frac{\bar{\xi}}{c^2} \frac{\partial^2}{\partial t_1^2} - \nabla_1^2\right\} \langle A_\alpha(\underline{r}_1, t_1) A^*_\beta (\underline{r}_2, t_2) \rangle$$

$$- \int (\Omega^k - \Omega^{j*})^{-1} \frac{\omega_k^2 \omega_j^2}{c^4} Q_\xi(k-j) \exp\left\{i\, \underline{k}(\underline{r}_1 - \underline{r}_2) - i \frac{\omega_k}{c} (t_1 - t_2)\right\} \times$$

$$\times \left\{\exp\left(-i\underline{k}(\underline{r}_1' - \underline{r}_2') + i \frac{\omega_k}{c} (t_1' - t_2')\right) - \exp\left(- i\underline{j}(\underline{r}_1' - \underline{r}_2') + i \frac{\omega_j}{c} (t_1' - t_2')\right)\right\}$$

$$\times \langle A_\alpha(\underline{r}_1', t_1') A^*_\beta (\underline{r}_2', t_2') \rangle\, d_j^4\, d\underline{r}_1'\, d\underline{r}_2'\, dt_1'\, dt_2' = 0\,. \qquad \ldots (3.29)$$

Returning to (3.26) we will identify

$$R^k_{\alpha\beta} = - \sum_j{}' L^{kj}_{\alpha\beta} \omega_k^2 \omega_j^2 c^{-4} Q_\xi^{k-j} \qquad \ldots (3.30)$$

and

$$S^k_{\alpha\beta} = - \sum_{\gamma j}{}' L^{kj}_{\alpha\gamma} \omega_k^2 \omega_j^2 c^{-4} Q_\xi^{k-j} C^j_{\gamma\beta}\,. \qquad \ldots (3.31)$$

We note that

$$\sum_{\gamma k} R^k_{\alpha\gamma} C^k_{\gamma\beta} - \sum_k S^k_{\alpha\beta} = 0.$$

This is not a unique choice, but one which is strongly suggested both by the form of (3.26) and of Ω^k which agrees with the Green function expansion in the limit of weak fluctuations (Stott 1967b). We note that R^k and S^k are of order Q_ξ, i.e. ξ^2, confirming our earlier assumption as to the ordering of (3.18).

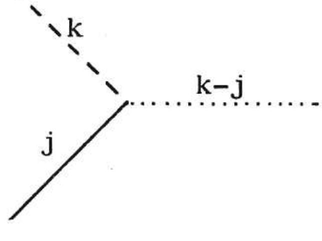
Continuing now with the series expansion of (3.18) the application of the condition (3.23) to the terms involving higher powers of ξ gives higher order corrections to the equation for C^k. The general term of the expansion is

$$\begin{aligned} &\left\{ \sum_{\substack{\alpha\beta \\ k}} \frac{\partial}{\partial A_\alpha^k} \left(\Omega^k_{\alpha\beta} A^k_\beta + S^k_{\alpha\beta} \frac{\partial}{\partial A^{k*}_\beta} \right) - c.c \right\} P_{n+2} \\ + &\left\{ \sum_{\substack{\alpha\beta \\ k}} \frac{\partial}{\partial A_\alpha^k} \, \delta_{\alpha\beta} \, \frac{\omega_k^2}{c^2} \sum_j{}' \, \xi^{k-j} A^j_\beta - c.c \right\} P_{n+1} \\ - &\left\{ \sum_{\substack{\alpha\beta \\ k}} \frac{\partial}{\partial A_\alpha^k} \left(R^k_{\alpha\beta} A^k_\beta + S^k_{\alpha\beta} \frac{\partial}{\partial A^{k*}_\beta} \right) - c.c \right\} P_n = 0 \, . \end{aligned} \qquad \ldots (3.32)$$

As the algebraic solution rapidly becomes rather tedious, we will make use of a diagrammatic technique invented by Edwards (1965).

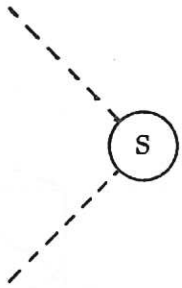
We will use a full line to represent A_α^k (or A_α^{k*}), a broken line to represent $\partial/\partial A_\alpha^k$ and a dotted line for the interaction ξ^{k-j}. We will write the diagrams across the page from right to left as in the normal algebraic convention of time ordering and will therefore adopt the convention that broken lines will always act to the left. The basic term of the expansion is the vertex

$$\frac{\partial}{\partial A_\alpha^k} \, \xi^{k-j} \, A_\alpha^j =$$



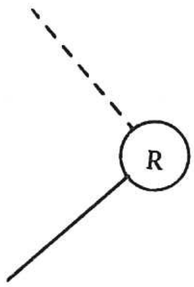
The averaging process is carried out diagramatically by adding a series of these terms and joining together the free ends of pairs of interaction lines into a join which we will mark with a cross. The full lines may either join together to give a correlation term $C^k_{\alpha\beta} = \langle A^k_\alpha A^{k*}_\beta \rangle$ or may be annihilated by a broken line. Each diagram of the averaged series will have two remaining unjoined photon lines. These may be either both broken lines, in which case we identify the diagram as an S-like term

i.e.

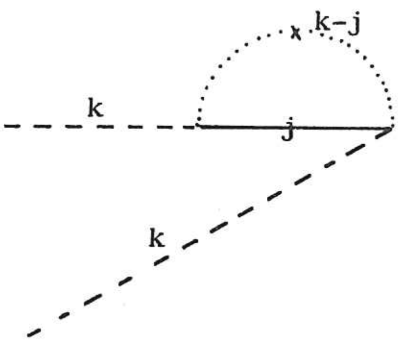


or one broken and one full line which corresponds to an R-like term

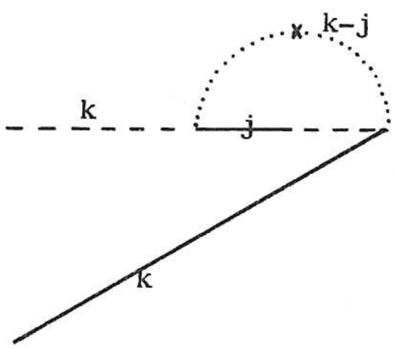
i.e.



The second order diagrams, i.e. P_2, are



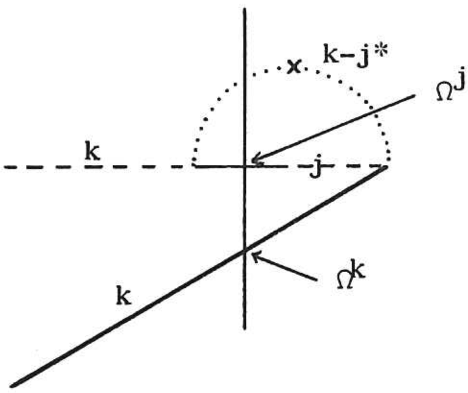
and



... (3.33)

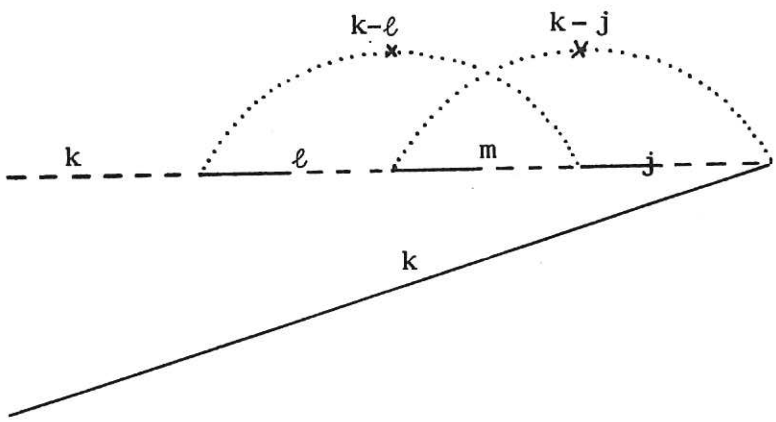
and the central portions of these diagrams are equivalent to the expressions which we have already obtained for $S^k_{\alpha\beta}$ and $R^k_{\alpha\beta}$.

The denominator may be included in a systematic way by drawing an imaginary vertical line in between each pair of vertices and adding an Ω of appropriate Fourier index for each photon line which is cut. For example the second order diagram contributes a denominator $(\Omega^k - \Omega^{j*})^{-1}$



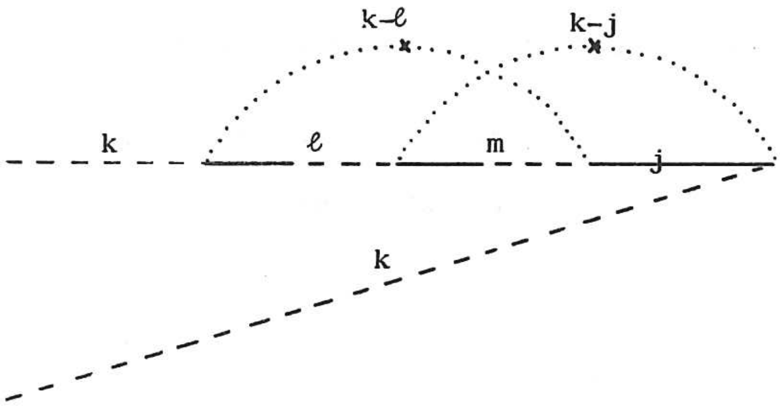
... (3.34)

The irreducible diagrams of the next order, i.e. P_4 , are



... (3.35)

which is a contribution to $R^k_{\alpha\beta}$



and ... (3.36)

which is a contribution to $S^k_{\alpha\beta}$.

Continuing in this way with the expansion, we see that the generalised form of (3.28) is

$$(k_o^2 - k^2)C^k - \sum_j{}' \chi^{k-j} (C^k - C^j) = 0 \qquad \text{... (3.37)}$$

χ^{k-j} is the sum of all the irreducible diagrams like (3.33), (3.35) and (3.36). The first few terms are

$$\begin{aligned}\chi^{k-j} &= (\Omega^k - \Omega^{j*})^{-1} \omega_k^2 \omega_j^2 c^{-4} Q_\xi^{k-j} \\ &+ \Sigma_{\ell m} (\Omega^k - \Omega^{\ell *})^{-1} \omega_k^2 \omega_\ell^2 c^{-4} Q_\xi^{k-\ell} (\Omega^k - \Omega^{j*})^{-1} \qquad \text{... (3.38)} \\ &\cdot \omega_k^2 \omega_j^2 c^{-4} Q_\xi^{k-j} (\Omega^k - \Omega^{m*})^{-1} \delta(m + k - j - \ell) + \ldots\end{aligned}$$

This equation is not complete in itself of course since Ω^k involves R^k and generalising (3.30)

$$R^k = \sum_j{}' \chi^{k-j} \qquad \text{... (3.39)}$$

We can argue that the main effect of the dependence of Ω^k on R^k will be to displace slightly the position of the pole in k-space at $\Omega^k = 0$, away from the value $k = \pm \omega/c \sqrt{\xi}$ which it would have if there were no scattering. This effect will not be too dependant on the value of k except very close to $\xi = 0$, and we can thus set up an ordering procedure in R^k and form an approximate solution to (3.38). A simple example will be given in section 5.

We shall see in the following section that the formulation developed above leads naturally to a transport equation for the multiply scattered electromagnetic field, which will satisfy the usual sum-rules for energy and wave momentum conservation. We note also that the time-scale restriction, imposed by expansion, has been avoided.

4. A PHOTON TRANSPORT EQUATION

In the above analysis, the short-range effects of the scattering have been correctly treated by wave-optics and we now proceed to the photon limit which provides a convenient basis for a description of the diffusion-like behaviour of the long-range effects of the multiple scattering.

The mean electromagnetic field correlation is

$$C_{\alpha\beta}(\underline{r}_1,\underline{r}_2,t_1,t_2) \;=\; \langle A_\alpha(\underline{r}_1,t_1)\, A^*_\beta(\underline{r}_2,t_2)\rangle \;.$$

Introducing a change of space and time variables,

$$\underline{R} = \underline{r}_1 - \underline{r}_2 \quad ; \quad \underline{r} = \tfrac{1}{2}(\underline{r}_1 + \underline{r}_2)$$

$$T = t_1 - t_2 \quad ; \quad t = \tfrac{1}{2}(t_1 + t_2) \;.$$

The correlation may be written in the form

$$C_{\alpha\beta}(\underline{r},\underline{R},t,T) = \langle A_\alpha(\underline{r}+\tfrac{1}{2}\underline{R},t+\tfrac{1}{2}T)\, A^*_\beta(\underline{r}-\tfrac{1}{2}\underline{R},t-\tfrac{1}{2}T)\rangle .$$

We Fourier transform with respect to the variables $\underline{R}$ and T

$$C_{\alpha\beta}(\underline{r},t,\underline{k},\omega) = \int C_{\alpha\beta}(\underline{r},\underline{R},r,T)\, e^{i\underline{k}\,\underline{R}}\, e^{-i\omega T}\, d\underline{R}\, dT \qquad \ldots (4.1)$$

and in the limit $|T| \to 0$ we define

$$f(\underline{r},\underline{k},t) = \sum_\beta \int C_{\alpha\beta}(\underline{r},t,\underline{k},\omega)\, d\omega \,.\, \delta_{\alpha\beta}$$

$$= \sum_\beta \delta_{\alpha\beta} \int \langle A_\alpha(\underline{r}+\tfrac{1}{2}\underline{R},t)\, A^*_\beta(\underline{r}-\tfrac{1}{2}\underline{R},t)\rangle\, e^{i\,\underline{k}\underline{R}}\, d\underline{R} \qquad \ldots (4.2)$$

Following the usual procedure of taking the classical limit in quantum mechanics, we will identify $f(\underline{r},\underline{k},t)$ as the probability of finding a photon with momentum $\underline{k}$ at a point $\underline{r}$ at time t. We will verify later that $f(\underline{r},\underline{k},t)$ does indeed have the correct properties required of a classical probability distribution function.

In coordinate space, (3.37) becomes

$$\left\{ \frac{\xi}{c^2}\left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial t_2}\right)\left(\frac{\partial}{\partial t_1}-\frac{\partial}{\partial t_2}\right) - \left(\nabla_{r_1}+\nabla_{r_2}\right)\left(\nabla_{r_1}-\nabla_{r_2}\right)\right\} C_{\alpha\beta}$$

$$-\frac{i}{2}\,\mathrm{Im}\int \chi^{k-j}\left(C^k_{\alpha\beta}-C^j_{\alpha\beta}\right) d^4j\, d^4k\, e^{-i\underline{k}(\underline{r}_1-\underline{r}_2)}\, e^{i\omega_k(t_1-t_2)} = 0 \,.$$

Thus

$$\left\{ \frac{\xi}{c^2}\cdot\frac{\partial}{\partial t}\cdot\frac{\partial}{\partial T} - \nabla_r \cdot \nabla_R \right\} C_{\alpha\beta}(\underline{r},t,\underline{R},T)$$

$$- i\,\mathrm{Im}\int \chi^{k-j}\left(C^k_{\alpha\beta}-C^j_{\alpha\beta}\right) d^4j\, d^4k\, e^{-i\underline{k}\cdot\underline{R}}\, e^{i\omega_k T} = 0. \qquad \ldots (4.4)$$

Fourier transformation of $\underline{R}$ and T gives the photon transport equation,

$$\left\{ \frac{\xi}{c^2}\,\omega\,\frac{\partial}{\partial t} + \underline{k}\cdot\nabla\right\} f(\underline{r},\underline{k},t) + \int \Gamma(\underline{k}-\underline{j})\left\{f(\underline{r},\underline{k},t) - f(\underline{r},\underline{j},t)\right\} d_{\underline{j}} = 0$$

where

$$\Gamma(\underline{k}-\underline{j}) = \mathrm{Im}\,.\,\chi(\underline{k}-\underline{j})\,. \qquad \ldots (4.5)$$

Transport equations of this form are well known in many branches of physics and the kernel, $\Gamma(\underline{k}-\underline{j})$ is equivalent to the mean scattering cross-section per unit plasma volume. The total scattering cross-section is obtained by integration over all $\underline{j}$ and equation (4.5) can be expressed conveniently by measuring space and time in the respective units of mean free path and time

i.e.
$$r_{mfp} = \left(\int \Gamma(\underline{k}-\underline{j})\, d\underline{j} \right)^{-1}$$

and
$$t_{mfp} = \bar{\xi} \frac{\omega_k}{c|k|} r_{mfp} .$$

The normalised transport equation is

$$\left(\frac{\partial}{\partial t} + \hat{\underline{k}}.\nabla \right) f(\underline{r},\underline{k},t) + \int \alpha(\underline{k}-\underline{j}) \left\{ f(\underline{r},\underline{k},t) - f(\underline{r},\underline{j},t) \right\} d\underline{j} = S(\underline{r},\underline{k},t) \quad \ldots (4.6)$$

where $\hat{\underline{k}}$ is a unit vector in the direction of $\underline{k}$ and $S(\underline{r},\underline{k},t)$ is a photon source which has been included for completeness. $\alpha(\underline{k}-\underline{j})$ is the scattering cross-section normalised so that

$$\int \alpha(\underline{k}-\underline{j})\, d\underline{j} = 1 .$$

The photon number density $\rho\,(\underline{r},t)$ and current density $\underline{J}(\underline{r},t)$ are given by the appropriate moments of $f(\underline{r},\underline{k},t)$.

$$\rho(\underline{r},t) = \int f(\underline{r},\underline{k},dt)\, d\underline{k}$$

$$\underline{J}(\underline{r},t) = \int f(\underline{r},\underline{k}.dt)\, \hat{\underline{k}}.d\underline{k} .$$

Integrating equation (4.6)

$$\frac{\partial}{\partial t} \rho(\underline{r},t) + \nabla \underline{J}(\underline{r},t) = S(\underline{r},t) = \int S(\underline{r},\underline{k},t)\, d\underline{k}$$

and the first moment of the transport equation is

$$\frac{\partial}{\partial t} \underline{J}(\underline{r},t) + \nabla\rho(\underline{r},t) + \underline{J}(\underline{r},t) = \int \hat{\underline{k}}.S(\underline{r},\underline{k},t)\, d\underline{k} \ .$$

If the photon source is zero within the plasma volume and $\underline{J}$ has only a slowly varying time dependence, we may eliminate $\underline{J}$ to obtain the familiar diffusion equation

$$\frac{\partial}{\partial t} \rho(\underline{r},t) + \nabla^2 \rho(\underline{r},t) = 0 \ .$$

5. CALCULATION OF THE SCATTERING CROSS-SECTION

The cross-section for a single scattering event occurring in a unit volume of plasma is given by the imaginary part of the expression (3.38), which is the sum of the series of irreducible scattering diagrams discussed earlier. The individual terms represent successively higher-order contributions to the scattering and each term involves denominators of the form

$$(\Omega^k - \Omega^{j*})^{-1} = (k_o^2 - j_o^2 - (k^2-j^2) + R^k - R^{j*})^{-1}$$

where

$$k_o^2 = \omega^2\xi/c^2 \ .$$

The singularities of these denominators in the complex space of k and j impose the conservation of energy and momentum between the initial and final states j and k. The R^k, which represent the modifications in energy and momentum induced by the background of turbulent plasma, are themselves given by a non-linear equation (3.39). This coupled set of equations is difficult to solve in general but there are some situations which enable us to limit the number of terms involved and which permit concise solutions.

We will consider a plasma in which the electron cyclotron and collision frequencies are both small compared to the propagating frequency, since this conveniently simplifies the Appleton-Hartree equation (2.4).

For

$$\omega_c \ll \omega \quad \text{and} \quad \nu \ll \omega \; ;$$

$$\bar{\xi} = 1 - \bar{n}/n_c$$

and

$$Q_\xi = Q_n/n_c^2 \; .$$

The dielectric constant is scalar in this case and there are no polarisation effects.

We will take a stationary, isotropic density fluctuation of the form

$$Q_n(r) = n_o^2 \, q(r/a)$$

where $q(r/a)$ is unity at $r = 0$ and decreases monotonically to zero as $|r|$ increases.

The terms of the series (3.38) contribute to the cross-section in ascending powers of $(n_o/n_c)^2 \, (ka)^3$. In the limit that this parameter is small

i.e.

$$p = (n_o/n_c)^2 \, (ka)^3 \ll 1$$

only the contribution of the leading term will be important. The dynamical friction terms R^k and R^{j*} are also of order p and we may therefore approximate

$$\Omega^k = M^k = k^2 - k_o^2 \; .$$

To the first order approximation we then have

$$\Gamma(\underline{k}-\underline{j}) = \pi \frac{\omega^4}{c^4} \left(\frac{n_o}{n_c} \right)^2 q(\underline{k}-\underline{j}) \; \delta(|k|-|j|) \; .$$

For example, if the correlation is Gaussian, the scattering cross-section per unit volume is

$$\Gamma(\theta) = (8\sqrt{\pi})^{-1} \left(\frac{n_o}{n_c}\right)^2 \frac{\omega^4}{c^4} a^3 \exp\left\{- a^2 \frac{\omega^2}{c^2}\left(1 - \frac{\bar{n}}{n_c}\right) \sin^2 \frac{\theta}{2}\right\}$$

and the mean-free path between successive scatterings is

$$r_{mfp} = 4\pi^{\frac{1}{2}} \left(\frac{n_c}{n_o}\right)^2 \frac{c^2}{\omega^2} \frac{1}{a} \left(1 - \frac{\bar{n}}{n_c}\right).$$

A coherent wave front propagating through the plasma density fluctuations will thus be attenuated with an e-folding length equal to r_{mfp}. As we might expect for the lowest order approximation, this agrees well with other scattering theories, for example that of Tatarshii (1964).

To extend the range of our solution to higher orders of the expansion parameter p, we would proceed along the lines of successive approximations alternating between equations (3.15) and (3.38), but this would be a tedious process and will not be attempted here.

It is frequently more realistic to regard the plasma density fluctuations as a randomly arranged set of discrete plasma 'blobs' rather than a truly turbulent spectrum. We then write the local plasma density as

$$n(\underline{r}) = \sum_{R_\alpha} \eta(\underline{r} - \underline{R}_\alpha)$$

where $\eta(\underline{r})$ is the density profile of a typical plasma blob and $\underline{R}_\alpha$ ... etc. are a set of vectors which we assume to be uniformly distributed with a number density N. We can then construct the density correlations, for example.

$$\langle n(\underline{r}_1)n(\underline{r}_2) \rangle = N \int \eta^2(\underline{k})\, e^{i\underline{k}(\underline{r}_1-\underline{r}_2)}\, d\underline{k}\,.$$

The sum of the series of terms represented by the diagrams like (3.40) is then equal to the Born approximations for the scattering cross-section of a single plasma blob. Interconnected terms like (3,35) may be interpreted as a scattering commencing at one blob before the previous scattering is completed. If the blobs really are discrete we would expect such events to be infrequent and the contribution from these terms to be small. The scattering cross-section is then given simply by

$$\Gamma(\underline{k}-\underline{j}) = N \, \Upsilon(\underline{k}-\underline{j})$$

where $\Upsilon(\underline{k}-\underline{j})$ is the cross-section of a single blob. This is the usual result obtained in transport theory for a set of discrete scatters.

6. CONCLUSION

A transport equation for the multiple scattering of electromagnetic radiation by a turbulent plasma has been derived by a mathematically rigorous method involving the expansion of the Lagrangian probability function for the electromagnetic field. The advantage of this formulation over other multiple scattering theories is that the solution of the transport equation, which is well known in other branches of physics, can take account of the boundary conditions and geometry appropriate to any specific experimental configuration. The kernel of the transport equation, which is equivalent to the scattering cross-section of the usual particle transport equation, has been given in terms of a set of equations involving the plasma correlation function. As with all multiple interaction problems the solution of this set of equations for the general case is a rather formidable task but simple solutions are possible for certain limiting cases and these provide a basis for extending the range of validity of the solution. Numerical results of computations based on this theory and a comparison with experimental observations of microwave scattering in turbulent laboratory plasma will be the subject of a further publication.

7. ACKNOWLEDGEMENT

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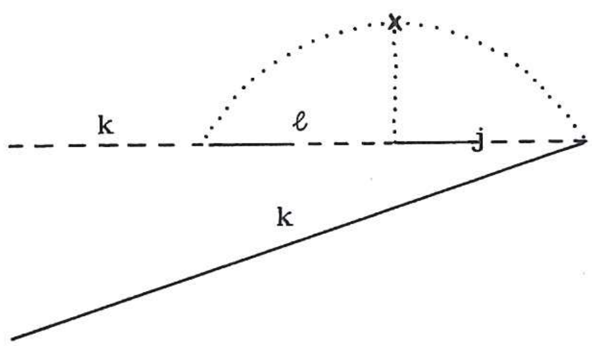
APPENDIX

NON-GAUSSIAN DIELECTRIC FLUCTUATIONS

It is a straightforward, but tedious matter, to extend the treatment to a non-Gaussian probability distribution for $\xi(\underline{r},t)$. For a generalised distribution (3.24) becomes

$$F(\xi) = N \exp \Big\{ - \xi(\underline{r},t)\, Q^{-1}_{\xi^2} (\underline{r},\underline{r}',t,t')\, \xi(\underline{r}',t') - \xi(\underline{r},t)\, \xi(\underline{r}',t')\, \xi(\underline{r}'',t'')\, Q^{-1}_{\xi^3} (\underline{r},\underline{r}',\underline{r}'',t,t'\ t'') - \dots \text{ etc.} \Big\} .$$

We then have contributions to the scattering from irreducible diagrams such as



The contribution from these terms is important for plasmas with non-Gaussian density fluctuations or where magnetic effects distort the dielectric constant for Gaussian density fluctuations.