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KINETIC THEORY OF PLASMA

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KINETIC THEORY OF PLASMA

by

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PART I

1. Introduction

The object of kinetic theory is to extract from an exact and detailed description of a complex physical system that information needed to describe its gross behaviour.

For example, consider a system composed of many charged particles having mass m_i, charge e_i and velocities $\underline{v}_i$; then in a volume V large enough to contain many particles, we may define a macroscopic density ρ, velocity $\underline{V}$, charge, Q, and current, $\underline{J}$, by:-

$$\frac{1}{V}\Sigma\, m_i = \rho; \qquad \frac{1}{V}\sum_i m_i \underline{v}_i = \rho \underline{V}$$

$$\frac{1}{V}\Sigma\, e_i = Q; \qquad \frac{1}{V}\sum_i e_i \underline{v}_i = \underline{J}$$

It is often convenient to write:-

$$\underline{v}_i = \underline{V} + \underline{c}_i; \quad \underline{J} = Q\,\underline{V} + \underline{j}$$

The macroscopic variables ρ, $\underline{V}$ etc. satisfy equations of motion that may be deduced from the laws of motion for the individual particle, i.e. from:-

$$m\,\underline{v}_i = \underline{F}_i = e_i\,(\underline{E} + \underline{v}_i \times \underline{B}) + \underline{F}_{int}$$

where:-

$$\sum_i \underline{F}_{int} = 0$$

from conservation of momentum. Then $\frac{\partial\rho}{\partial t}$ is determined by equating the rate of change of density to the flux into a volume; i.e.

$$\frac{\partial\rho}{\partial t} + \underline{\nabla}\cdot(\rho\,\underline{V}) = 0 \qquad (1.1.1)$$

The rate of change of momentum is similarly:-

$$\frac{\partial}{\partial t}(\rho\,\underline{V}) + \frac{\partial}{\partial \underline{x}}\cdot\sum_i m\,\underline{v}_i\underline{v}_i = \frac{\partial}{\partial t}(\rho\,\underline{V}) + \frac{\partial}{\partial \underline{x}}\cdot(\rho\,\underline{V}\underline{V}) + \frac{\partial}{\partial \underline{x}}\cdot\Sigma m\underline{c}_i\underline{c}_i$$

$$= \sum_i \underline{F}_i = \sum_i e_i\,(\underline{E} + \underline{v}_i \times \underline{B}) = Q(\underline{E} + \underline{V}\times\underline{B}) + \underline{j}\times\underline{B}$$

and defining the stress tensor $p_{rs} = \sum_i mc_{ir}c_{is}$, and using the continuity equation (1.1.1).

$$\rho(\frac{\partial V}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\underline{V}) = -\,\underline{\nabla}\,\underline{p} + Q\,(\underline{E} + \underline{V}\times\underline{B}) + \underline{j}\times\underline{B} \qquad (1.1.2)$$

In a similar way, we may equate the rate of gain of energy to the rate of doing work, i.e.

$$\frac{d}{dt}\left(\sum_i \tfrac{1}{2} mv_i^2\right) + \underline{\nabla} \cdot \sum_i \underline{v}_i \tfrac{1}{2} mv_i^2 = \sum_i \underline{F}_i \cdot \underline{v}_i = \sum_i e\, \underline{v}_i \cdot \underline{E} = \underline{J} \cdot \underline{E}$$

or

$$\underline{V}\cdot\left(\rho\frac{D\underline{V}}{Dt} + \underline{\nabla}\cdot \mathrm{p}\right) + \frac{D}{Dt}\sum_i \tfrac{1}{2} mc_i^2 + \sum_i \tfrac{1}{2} mc_i^2\, \underline{\nabla}\cdot\underline{V} + (\underline{\nabla}\cdot \mathrm{p})\cdot\underline{v} + \underline{\nabla}\circ \Sigma\underline{c}\tfrac{1}{2}mc^2 = Q\,\underline{V}\cdot\underline{E} + \underline{j}\cdot\underline{E}$$

and defining the internal energy:-

$$U = \sum_i \tfrac{1}{2}mc_i^2 = \frac{3}{2}\, nkT$$

and the heat flux:-

$$\underline{q} = \Sigma\, \underline{c}\, \tfrac{1}{2}mc^2$$

$$\frac{DU}{Dt} + U\,\underline{\nabla}\cdot\underline{V} + (\underline{\nabla}\cdot\underline{p})\underline{V} + \underline{\nabla}\cdot\underline{q} = \underline{j}\cdot\underline{E} \qquad (1.1.3)$$

For an ionized gas, the charged particles are of two sorts; electrons and ions, and the interdiffusion of these particles gives rise to a current; the electrons and ions moving with speeds:-

$$\underline{V} + \Delta\,\underline{V}_-,\ \underline{V} + \Delta\underline{V}_+ .$$

By considering the two gases separately the quantity $\Delta\underline{V}_-$ is found to satisfy:-

$$\frac{D\underline{V}}{Dt} + \frac{D\Delta\underline{V}}{Dt} + \Delta\underline{V}\cdot\underline{\nabla}\Delta\underline{V} + \Delta\underline{V}\cdot\underline{V} + \frac{1}{nm_-}\underline{\nabla}\cdot \mathrm{p}_- = \frac{e_-}{m_-}\left(\underline{E} + \underline{V}\times\underline{B} + \Delta\underline{V}\times\underline{B}\right) + \frac{\underline{F}_-}{m_-}$$

If this and similar equation for ΔV_+ are multiplied by a product of density and charge there results:-

$$\frac{D\underline{j}}{Dt} + \frac{e_-}{m_-}\underline{\nabla}\cdot\underline{p}_- + \frac{e_+}{M_+}\underline{\nabla}\cdot\underline{p}_+ - \frac{Q}{\rho}\underline{\nabla}\cdot \mathrm{p}_T + \left[\frac{Q^2}{\rho} - \left[n_-\left(\frac{e^2}{m_+}\right) + n_+\left(\frac{e}{M_+}\right)^2\right].\right.$$

$$\left.\cdot\left[\underline{E} + \underline{V}\times\underline{B}\right] + \frac{Q}{\rho}\left(\underline{j}\times\underline{B}\right) - \left[n_-\left(\frac{e^2}{m_-}\right)\Delta\,\underline{V}_- + n_+\left(\frac{e}{M_+}\right)^2\Delta\,\underline{V}_+\right]\times\underline{B} = 0\right.$$

$$\frac{m_-}{ne^2}\frac{D\underline{j}}{Dt} + \frac{m_-}{ne_-}\left[\underline{\nabla}\cdot\underline{p}_- - \underline{j}\times\underline{B}\right] + \left[\underline{E} + \underline{V}\times\underline{B}\right] - \eta\underline{j} = 0 \qquad (1.1.4)$$

where:-

$$\eta\, j = \frac{m_-}{ne^2}\left[\frac{ne_-}{m_-}F_- + \frac{ne_+}{m_+}F_+\right]$$

A more formal treatment of the relation between the microscopic and the macroscopic can be effected by employing a distribution function f; a quantity which describes the statistical evolution of the system, in which case the underlying microscopic dynamics of the system is embraced in an equation of motion for f, the equation of transport.

There are several sorts of distribution function f, ranging from the Liouville function $F(\underline{x}_1, \underline{x}_2 - - - \underline{x}_N, \underline{v}_1 - - - \underline{v}_N)$ to the Boltzmann single particle function $f(\underline{x},\underline{v})$. The Liouville function is a function of the complete set of micro co-ordinates, and satisfies the equation:-

$$\frac{\partial F}{\partial t} + [\, H(\underline{x}_1 - - - \underline{x}_N, \underline{v}_1 - - - \underline{v}_N), F \,] = 0 \qquad (1.1.5)$$

which is completely equivalent to the microscopic dynamics, H being the complete Hamiltonian. This can be written introducing the acceleration field $\underline{A}_i\ (x_1 - - - xN)$

$$\frac{\partial F}{\partial t} + \sum_i (\underline{v}_i \cdot \frac{\partial F}{\partial \underline{x}_i} + \underline{A}_i\ (\underline{x}_1 - - \underline{x}_N) \cdot \frac{\partial F}{\partial \underline{v}}) = 0$$

The equivalence of Liouville's equation and the equations of motion is established by observing that if the system is given as in the state specified by:-

$$\underline{x}_1(o),\ \underline{x}_2\ (o) - - - \underline{x}_N(o),\ \underline{v}_1\ (o),\ \underline{v}_2\ (o) - - - v_N(o)$$

so that:-

$$F(o) = \pi\ \delta\ (\underline{x}_i - \underline{x}_i\ (o)\ )\ \delta(\underline{v}_i - \underline{v}_i(o)\ )$$

its subsequent evolution is described by:-

$$F(t) = \pi\ \delta(\underline{x}_i - X_i(t)\ )\ \delta(\underline{v}_i - \underline{V}_i(t)\ ) \qquad (1.1.6)$$

where $\underline{X}_i(t)$ and $\underline{V}_i(t)$ are the relevant solutions of the equations of motion, i.e.

$$\underline{X}_i(t) = \underline{x}_i(o) + \int_o^t dt' \underline{V}_i(t')$$

$$\underline{V}_i(t) = \underline{v}_i(o) + \int_o^t dt' \underline{A}_i\ (\underline{X}_i(t'),\ \underline{V}_i(t');\ \Sigma \underline{X}_{not\ i}(t'),\ \underline{V}_{not\ i}(t')\ ) \qquad (1.1.7)$$

The Boltzmann function on the other hand satisfies an equation of the form:-

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{x}} + \underline{A}_o(x,t) \cdot \frac{\partial f}{\partial \underline{v}} = \frac{\partial f}{\partial t}\ \text{int} \qquad (1.1.8)$$

The transport equation for the Boltzmann function may be obtained by repeated integration of the Liouville equation, for f itself is defined by:-

$$f(\underline{x}_1, \underline{v}_1) = V \int F(\underline{x}_1 - - \underline{x}_N, \underline{v}_1 - - \underline{v}_N)\ d^3x_2 - - d^3x_N\ d^3v_2 - - d^3v_N$$

If there were no interaction between particles so that the acceleration field $\underline{A}_i$ could be factored as $\underline{A}_1\ (x_2, v_2)$ - - etc., then a closed equation for f could be obtained by integrating over the Liouville equation as:-

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{x}} + \underline{A} \cdot \frac{\partial f}{\partial \underline{v}} = 0 \qquad (1.1.9)$$

the collisionless Boltzmann, or Vlasov equation.

If, however, there exists an inter-particle potential $\phi(x_i, x_j)$ the final integral cannot be evaluated in terms of $\underline{x}_1$, $\underline{v}_1$ and f alone, instead it becomes:-

$$-\frac{V}{m_1} \Sigma_j \frac{\partial}{\partial \underline{v}_1} \cdot \int \frac{\partial}{\partial x_1} \phi(\underline{x}_1, \underline{x}_j)\, F(\underline{x}_1 - - - \underline{x}_N)\, d^3v_2 - d^3v_N d^3x_2 - d^3x_N$$

or, introducing the two particle function:-

$$f(1,2) = V^2 \int F(\underline{x}_1, \underline{x}_2 - - \underline{x}_N)\, d^3x_3 - - d^3x_N\, d^3v_3 - - d^3v_N$$

$$\frac{\partial f_1}{\partial t}\bigg|_{int} = \frac{N}{Vm_1} \frac{\partial}{\partial \underline{v}_1} \cdot \int \frac{\partial \phi}{\partial \underline{x}_1}^{(1,2)} f(1,2)\, d^3x_2\, d^3v_2$$

and the general equation of transport for f becomes:-

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{x}} + \underline{A}_o \cdot \frac{\partial f}{\partial \underline{v}} - \frac{n}{m_1} \frac{\partial}{\partial \underline{v}} \cdot \int \frac{\partial}{\partial \underline{x}_1} \phi(x_1, x_2)\, f(1,2)\, d^3x_2\, d^3v_2$$

or:-

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{x}} + \underline{A}_o \cdot \frac{\partial f}{\partial \underline{v}} + I(f) = 0 \qquad (1.1.10)$$

The first major problem of kinetic theory is to find an approximate form for I(f); the second being that of solving 1.1.10 for a given form of I and deducing the moments required for a macroscopic description of phenomena.

For diffuse gases in which a strong but localized interaction occurs between the particles, a course-grained equation for f may be obtained in which the interaction term I is represented by the rate of change of f produced by impulsive collisions between particles; ($\underline{g} = \underline{v}_1 - \underline{v}_2$, θ = scattering angle):-

$$\frac{\partial f}{\partial t}\bigg|_{int} = \int d\,\Omega \int d^3v_2\, g\, \sigma(g,\theta)\, [\, f(\overline{\underline{v}}_1)\, f(\overline{\underline{v}}_2) - f(\underline{v}_1)\, f(\underline{v}_2)\,]$$

where $\overline{v}_1$ $\overline{v}_2$ are related to v_1, v_2 and θ, being in fact the negatives of the velocities resulting when a collision between $\underline{v}_1$ and $\underline{v}_2$ occurs with a scattering angle θ. Our later work will be devoted to showing that, with certain corrections, this result is a valid approximation for an ionized gas where the forces of interaction are weak, but long ranged. At present we will concentrate on the second problem, that of solving Boltzmann's equation and determining the transport coefficients.

2. Hydrodynamic Equations from the Transport Equation

As a preliminary to any attempt to solve the Boltzmann equation we will use it to form the hydrodynamic equations. To do this we use the definitions of § 1, which, expressed in terms of the distribution function f becomes the following moments of f:-

$$\rho = \int f\ m\ d^3v; \quad \rho\ \underline{V} = \int f\ m\ \underline{v}\ d^3v; \quad \underline{c} = \underline{v} - \underline{V}$$

$$\underline{p} = \int f\ m\ \underline{c}\ \underline{c}\ d^3v; \quad \frac{3}{2}\ kT = \int f\ \tfrac{1}{2}\ m\ c^2\ d^3v; \quad \underline{q} = \int f\ \tfrac{1}{2}\ mc^2 \underline{c}\ d^3v$$

Since the B.E. forms a representation of the dynamics of the system, the macroscopic equations for the moments may be formed therefrom, i.e. from:-

$$\int m\ \left\{\frac{\partial f}{\partial t} + \underline{v}\ .\ \frac{\partial f}{\partial \underline{x}} + \underline{A}\ .\ \frac{\partial f}{\partial \underline{v}} - I(f)\ \right\}\ d^3v = 0$$

$$\frac{\partial \rho}{\partial t} + \text{div.}\ (\rho \underline{V}) = 0 \qquad (1.2.1)$$

since $\int m\ I(f)\ d^3v = 0$ from mass conservation.

From:-

$$\int d^3v\ m\ \underline{v}\ \{ \qquad \} = 0$$

$$\rho\ \frac{D\underline{V}}{Dt} + \underline{\nabla}\ .\ \underline{p} - \underline{F} = 0 \text{ where } \underline{F} = \int m\ \underline{A}\ f\ d^3v \qquad (1.2.2)$$

and (1.2.1) has been used. Finally from:-

$$\int d^3v\ \tfrac{1}{2}\ mv^2\ \{ \qquad \}$$

$$\frac{DU}{Dt} + U\ \text{div}\ \underline{V} + \underline{p}\ :\ \underline{\nabla}\ \underline{V} + \text{div}\ \underline{q} = 0 \qquad (1.2.3)$$

and

$$U = \frac{3}{2}\ n\ kT$$

For an ionized gas, there are similar equations for each component, although now the interaction integral does not vanish, but leads to terms representing the transfer of energy and momentum between the two components. Alternatively, these equations may be combined and as in § 1, the mean velocity may be defined as:-

$$\rho \underline{V} = (\rho_+ + \rho_-)\underline{V} = \rho_+ \underline{V}_+ + \rho_- \underline{V}_-,$$

and $\underline{p}$, T etc. defined relative to V, whereupon:-

$$\frac{\partial \rho}{\partial t} + \text{div}\ .\ \rho \underline{V} = 0 \qquad (1.2.4)$$

$$\rho\ \frac{D\underline{V}}{Dt} + \underline{\nabla}\ .\ \underline{p} - Q(\underline{E} + \underline{V} \times \underline{B}) - \underline{j} \times \underline{B} = 0 \qquad (1.2.5)$$

and

$$\frac{DU}{Dt} + U \operatorname{div} \underline{V} + \underline{p} : \underline{\nabla} \underline{V} + \operatorname{div} \underline{q} - \underline{j} \cdot \underline{E} = 0 \qquad (1.2.6)$$

To illustrate important methods used in solving the B.E. we shall first consider some simple representations of a simple gas, and only gradually approach the complexities of the ionized gas in a magnetic field.

3. The Normal Solution: Hilbert's Procedure

To derive meaningful hydrodynamic equations it is useful to restrict attention first to those situations in which the rate of change of the distribution function is slow so that the collision frequency is much greater than any hydrodynamic frequency, i.e. if we introduce a macroscopic time scale T, length scale, L, and a characteristic velocity, $V = L\,T^{-1}$, then, if the external forces are small, so that $T^{-2} < 1$; the L.H.S. of the B.E. scales as T^{-1}. We can also define a collision time by $\tau^{-1} \doteq n\,\sigma_0\,V$, where σ_0 is a mean cross section, whereupon the condition, collision frequency is much greater than hydrodynamic frequency, becomes $\tau/T = \varepsilon \ll 1$, and the B.E. may be written:-

$$I(f,f) = \varepsilon \left\{ \frac{\partial}{\partial t} + \underline{v} \cdot \frac{\partial}{\partial \underline{x}} + \underline{A}_0 \cdot \frac{\partial}{\partial \underline{v}} \right\} f = \varepsilon\, D\, f \qquad (1.3.1)$$

It now makes sense to seek a solution expanded in powers of ε :-

$$f = f_0 + \varepsilon f_1 + \dots$$

which on being introduced into (1.3.1) reduces it to:-

$$I(f_0,\, f_0) = 0 \qquad (1.3.2)$$

$$I(f_0, f_1) = Df_0 \qquad (1.3.3)$$

The first equation here is satisfied by the locally Maxwellian distribution; i.e. by:-

$$f_0 = n(\underline{x},t) \left(\frac{m}{2\pi\, kT}\right)^{\frac{3}{2}} \exp\left\{ - \frac{m\,[\underline{v} - \underline{V}(\underline{x},t)\,]^2}{2\,kT(\underline{x},t)} \right\}$$

where n, T, V are undetermined functions of $\underline{x}$,t.

Hilbert observed that by writing $f_1 = f_0\,\phi$, (1.3.3) may be written:-

$$\int d^3v' f_0(\underline{v})\; f_0(\underline{v}')\; |\underline{v}-\underline{v}'|\; \sigma(\underline{v}-\underline{v}',\theta)[\phi(\overline{\underline{v}}) + \phi(\overline{\underline{v}}') - \phi(\underline{v}) - \phi(\underline{v}')\;] = Df_0$$

i.e.

$$\int K(\underline{v},\underline{v}')\; \phi(\underline{v}')\; d^3v' = Df_0$$

an integral equation in which conservation laws require that K should be symmetric in $\underline{v}$, $\underline{v}'$. This has the following interesting consequence:- that a solution can be obtained only if Df_o is orthogonal to the solution $h(\underline{v})$ of the homogeneous equation:-

$$\int K(\underline{v},\underline{v}')\ h(\underline{v}')\ d^3v' = 0$$

for:-

$$\int dv \int dv' h(\underline{v})\ K(\underline{v},\underline{v}')\ \phi(\underline{v}') = 0 = \int dv\ h(\underline{v})\ Df_o\ (\underline{v})$$

Since

$$\int K(v,v')\ \phi(v')\ dv'$$

gives the rate of change of ϕ produced by collision, the solutions to the homogeneous equation are the collision invariants, m, $m\underline{v}$, $\frac{1}{2}\ mv^2$, and the constraints on Df_o become the zero order hydrodynamic equations. Since for a Maxwellian distribution:-

$$P_{ij} = n\ kT\ \delta_{ij},\ \underline{q} = 0,\ U = \frac{3}{2}\ n\ kT\ ,$$

these become:-

$$\frac{\partial n}{\partial t} + \underline{V}\ .\ \underline{\nabla} n + n\ \underline{\nabla}\ .\ \underline{V} = 0 \qquad (1.3.4)$$

$$\frac{\partial \underline{V}}{\partial t} + \underline{V}\ .\ \underline{\nabla}\ \underline{V} + \frac{1}{nm}\ [\ \underline{\nabla}\ (nkT) - F\ ] = 0 \qquad (1.3.5)$$

$$\frac{\partial}{\partial t}\ (\frac{3}{2}\ n\ kT) + \underline{V}\ .\ \underline{\nabla}(\frac{3}{2}\ nkT) + \frac{5}{2}\ nkT\ div\ \underline{V} = 0 \qquad (1.3.6)$$

Furthermore, since f_o depends on x, and t through n, T and $\underline{V}$, we may write, with $\underline{c} = \underline{v} - \underline{V}$

$$Df_o = \{\ \frac{1}{n}\ (\frac{\partial}{\partial t} + \underline{V}\ .\ \underline{\nabla})\ n - (\frac{3}{2} - \frac{1}{2}\ \frac{mc^2}{kT})\ \frac{1}{T}\ (\frac{\partial}{\partial t} + \underline{V}\ .\ \underline{\nabla})T$$

$$+ \frac{m\underline{c}}{kT}\ .\ (\frac{\partial}{\partial t} + \underline{V}\ .\ \underline{\nabla})\underline{V} - \frac{m\underline{c}}{kT}\ .\ \underline{A}\ \}\ f_o$$

The time derivatives may be eliminated with aid of 3.4-6 and:-

$$Df_o = \{\ [\frac{m}{kT}\ \underline{c}\ .\ (\underline{c}\ .\ \underline{\nabla})\underline{V} - \frac{1}{3}\ \frac{mc^2}{kT} div\ \underline{V}\ ] - [\frac{5}{2} - \frac{1}{2}\ \frac{mc^2}{kT}]\frac{1}{T}(\underline{c}\ .\ \underline{\nabla})\ T\ \}\ f_o \qquad (1.3.7)$$

4. <u>Mean Free Time Theory for a Simple Gas</u>

A good many of the features of the B.E. may be retained by replacing the collision integral by a simple relaxation term, i.e. retaining only the tendency for a distribution function f to relax back to the

Maxwellian, and representing:-

$$I(f) \text{ by } \frac{1}{\tau}(f - f_o)$$

whereupon (1.3.3) becomes:-

$$\frac{1}{\tau}(f_o - f) = Df_o$$

or

$$f = f_o - \tau Df_o$$

and using (1.3.7)

$$f = f_o \{1 - \tau[\frac{m}{kT}\underline{c} \cdot (\underline{c} \cdot \underline{\nabla})\underline{V} - \tfrac{1}{3}\frac{mc^2}{kT}\text{div }\underline{V}] - (\frac{5}{2} - \tfrac{1}{2}\frac{mc^2}{kT})\frac{1}{T}\underline{c} \cdot \underline{\nabla}T\} \tag{1.4.1}$$

We need this to form the moments:-

$$p_{ij} = \int f\, mc_i c_j\, d^3v, \quad \underline{q} = \int f\, \tfrac{1}{2}mc^2\underline{c}\, d^3v$$

since $\langle \underline{c} \rangle$ and $\langle \frac{1}{2}c^2\underline{c} \rangle = 0$. Carrying out the integrals yields:-

$$p_{ij} = \frac{\tau}{2} n(kT)\, [\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} - \frac{2}{3}\text{div }\underline{V}\,\delta_{ij}] = -\frac{\tau}{2} p[\underline{\nabla}\ \underline{V}]$$

and

$$\underline{q} = -\frac{75}{32}\tau\frac{2kT}{m}k\underline{\nabla}T$$

PART II

Transport Processes in Fully Ionized Gases - Normal Solutions.

1. The Linearized Equations

When instead of a simple gas, an ionized gas in a magnetic field is considered several new complications arise, many of which may be clarified by a consideration of the simple m.f.t. theory. In the first place, instead of a simple gas, an ionized gas is a mixture of two gases, ions and electrons, which interdiffuse. As a result, even starting from a common Maxwell distribution the elimination of the time derivatives produces some new results, since neither component moves exactly as does the mixture. Next, for many plasmas, the gyro frequency $\omega = \frac{eB}{m}$ is comparable to or larger than the collision frequency, and the technique used for splitting up the terms in the B.E. becomes inappropriate. To handle the magnetic term it may itself be split into two parts, one involving the mean velocity $\frac{e}{m}\underline{V} \times \underline{B}$ which is small, and a second involving $\underline{c}(= \underline{v} - \underline{V})$ the "peculiar" velocity $\frac{e}{m}\underline{c} \times \underline{B}$, which is not necessarily small. The B.E. may then be written as:-

$$\underbrace{\left[\frac{\partial}{\partial t} + \underline{v} \cdot \frac{\partial}{\partial \underline{x}} + \frac{e}{m}(\underline{E} + \underline{V} \times \underline{B}) \cdot \frac{\partial}{\partial \underline{v}} \right] f}_{A} + \underbrace{\frac{e}{m}(\underline{c} \times \underline{B}) \cdot \frac{\partial f}{\partial \underline{v}}}_{B} - \underbrace{I(f)}_{C} = 0 \qquad (2.1.1)$$

A: B: C: : 1: ωT: T/τ. Different orderings are possible for these terms, e.g. ωT may be ~1 while T/τ is large, whereupon the ordering valid in the non magnetic problem becomes useful. ωT and T/τ may be of the same order, or finally, in a magnetic field, hydrodynamic behaviour is possible when $\omega T \gg 1 \gg T/\tau$. In any case, it is possible to start with a Maxwellian distribution:-

$$f^{o} = n(x)\left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left[-\tfrac{1}{2} m (\underline{v} - \underline{V})^2/kT \right]$$

where as before n, T, $\underline{V}$ are functions of position and time:- and where $\underline{V}$ is the velocity of the fluid as a whole. Now as in I, we may write the B.E. as:-

$$\left[I(f) - \frac{e}{m}\underline{c} \times \underline{B} \cdot \frac{\partial}{\partial \underline{v}} f \right]_i = \left[Df^{o} \right]_i \qquad (2.1.2)$$

and in forming the R.H.S. observe that space and time variation arises only through n, T, $\underline{V}$, and further, that the time derivatives may once more be eliminated by use of the zero order equations of motion. Now, however, certain complications arise, for in the equation for each component there appear terms such as $\underline{\nabla}\log n_i + \underline{\nabla}\log T$, which no longer is equivalent to $\underline{\nabla}\log p$; moreover $\frac{1}{n}\,{}_m\underline{F}_i$ is no longer eliminated by the equation of momentum conservation. Instead Df takes the form:-

$$[Df^o]_i = f_i^o \left\{\frac{m_i}{kT}\left[\underline{c}.\ (\underline{c}.\underline{\nabla})\underline{V} - \tfrac{1}{3} c^2 \mathrm{div}\ \underline{V}\right] - \left[\tfrac{5}{2} - \tfrac{1}{2}\frac{m_i c^2}{kT}\right]\underline{c}\cdot\underline{\nabla}\log T + \frac{n}{n_i}\,\underline{c}\cdot\underline{d}_i\right\} \qquad (2.1.3)$$

where:-

$$d_i = \underline{\nabla}\frac{n_i}{n} + \frac{1}{p}\left[\frac{n_i}{n} - \frac{n_i m_i}{\rho}\right]\underline{\nabla}\, p - \frac{1}{p}\left[n_i e_i - \frac{n_i m_i}{\rho} Q\right]\left[\underline{E} + \underline{V}\times\underline{B}\right] \qquad (2.1.4)$$

or more (anti!) symmetrically as:-

$$\underline{d}_1 = -\underline{d}_2 = \underline{\nabla}\frac{n_1}{n} + \frac{n_1 n_2 (m_2 - m_1)}{n\rho p}\underline{\nabla}\, p - \frac{n_1 m_2}{p\rho}(e_1 m_2 - e_2 m_1)\left[\underline{E} + \underline{V}\times\underline{B}\right] \qquad (2.1.5)$$

The problem of finding the normal solutions to the B.E. then reduces to that of solving the linear integro-differential equation:-

$$\left[I(f) - \frac{e}{m}\underline{c}\times\underline{B}\cdot\frac{\partial f}{\partial \underline{v}}\right]_i = P\ .\ \exp\left[-\frac{\frac{1}{2}m_i c^2}{kT}\right] \qquad (2.1.6)$$

2. Mean Free Time Model

As for the simple gas, the general form of the transport coefficients may be obtained by studying a simple model - one in which the collision integral is replaced by a simple relaxation term; $I(f) = \frac{1}{\tau}(f^o - f)$. Furthermore, we may introduce axis OX $||$ the magnetic field and represent the peculiar velocity c by:-

$$c_x = c_{||}\ ,\ c_y = c_\perp \cos\phi,\ c_z = c_\perp \sin\phi \qquad (2.2.1)$$

whereupon:-

$$\frac{e}{m}\underline{c}\times\underline{B}\cdot\frac{\partial f}{\partial\underline{v}} = -\omega\frac{\partial f}{\partial\phi} \qquad (2.2.2)$$

and the equation to be solved becomes:-

$$f - \omega\tau\frac{\partial f}{\partial\phi} = f^o\left[1 - \tau P\right] \qquad (2.2.3)$$

and on introducing the abbreviation:-

$$\underline{w} = \sqrt{\frac{m}{2kT}}\,\underline{c}$$

$$P = 2 \left[\frac{\partial V_i}{\partial x_j} - \tfrac{1}{3} \operatorname{div} . V \delta_{ij} \right] w_i w_j - \left[\left(\tfrac{5}{2} - w^2 \right) \tfrac{1}{T} \underline{\nabla} T - \tfrac{n}{n_i} \underline{d}_i \right] . \underline{c} \qquad (2.2.4)$$

In terms of ϕ, the differential equation contains terms ~ 1, $\cos\phi$, $\sin\phi$, $\cos^2\phi$, $\sin^2\phi$, $\sin\phi\cos\phi$, and the periodic contributions due to these terms are:-

1: $\frac{1}{1+\omega^2\tau^2} (\cos\phi - \omega\tau \sin\phi)$, $\frac{1}{1+\omega^2\tau^2} (\sin\phi + \omega\tau\cos\phi)$,

$$- \tfrac{1}{2} + \tfrac{1}{2} \frac{1}{1+4\omega^2\tau^2} \left[(\cos^2\phi - \sin^2\phi) - 4\,\omega\tau \cos\phi \sin\phi \right]$$

$$- \tfrac{1}{2} - \tfrac{1}{2} \frac{1}{1+4\omega^2\tau^2} \left[(\cos^2\phi - \sin^2\phi) - 4\,\omega\tau \cos\phi \sin\phi \right]$$

and

$$\frac{1}{1+4\omega^2\tau^2} \left[\cos\phi \sin\phi + \omega\tau (\cos^2\phi - \sin^2\phi) \right]$$

$$\begin{aligned} f = f_o \Big[1 - \tau \Big\{ & 2 \left[(\operatorname{div} \underline{V})_{\parallel} w_{\parallel}^2 + \tfrac{1}{2} (\operatorname{div} \underline{V})_{\perp} w_{\perp}^2 - \tfrac{1}{3} \operatorname{div} \underline{V} w^2 \right] \\ & - \left[\left(\tfrac{5}{2} - w^2 \right) \tfrac{1}{T} \nabla_{\parallel} . T - \tfrac{n}{n} d_{\parallel} \right] c_{\parallel} + \frac{1}{1+\omega^2\tau^2} \left[\frac{kT}{m} \left(\frac{\partial V_{\perp}}{\partial \bar{x}_{\parallel}} + \frac{\partial V_{\perp}}{\partial \bar{x}_{\perp}} \right) c_{\parallel} - \right. \\ & - \left(\tfrac{5}{2} - w^2 \right) \tfrac{1}{T} \underline{\nabla}_{\perp} T - \tfrac{n}{n_i} \underline{d}_{\perp} \quad (\underline{c}_{\perp} + \underline{c}_{\perp} \times \underline{b}\, \omega\tau) + \\ & + \frac{1}{2(1+4\omega^2\tau^2)} \left[(\underline{\nabla}_{\perp} \underline{V}_{\perp}) + \underline{b} \times (\underline{\nabla} \underline{V}) \times \underline{b} \right] : \underline{\underline{ww}} + \\ & + 4\,\omega\tau \left[(\underline{\nabla}_{\perp} \underline{V}_{\perp}) \times \underline{b} - \underline{b} \times (\underline{\nabla}_{\perp} \underline{V}_{\perp}) \right] : \underline{\underline{ww}} \Big\} \end{aligned} \qquad (2.2.5)$$

$$\begin{aligned} \underline{j} = n_1 e_1 \underline{c}_1 + n_2 e_2 \underline{c}_2 = & \; n e_1 \sqrt{\frac{kT}{m_1}} \tau_1 \left[\underline{d}_{\parallel} + \frac{\underline{d}_{\perp} + \underline{b} \times \underline{d}_{\perp}}{1 + \omega^2\tau^2} \right]_1 \\ & + n e_2 \sqrt{\frac{kT}{m_2}} \tau_2 \left[\underline{d}_{\parallel} + \frac{\underline{d}_{\perp} + \underline{b} \times \underline{d}_{\perp}}{1 + \omega^2\tau^2} \right]_2 \end{aligned} \qquad (2.2.6)$$

$$\begin{aligned} q = & - \frac{3}{2} \frac{k^2 T}{m_1} n_1 \tau_1 \left[\underline{\nabla}_{\parallel} T + \frac{1}{1+\omega^2\tau^2} \underline{\nabla}_{\perp} T + \underline{b} \times \underline{\nabla}_{\perp} T \right]_2 \\ & - \frac{3}{2} \frac{k^2 T}{m_2} n_2 \tau_2 \left[\underline{\nabla}_{\parallel} T + \frac{1}{1 + \omega^2\tau^2} \underline{\nabla}_{\perp} T + \underline{b} \times \underline{\nabla}_{\perp} T \right]_2 \\ & + \frac{5}{2} n (kT)^2 \left\{ \frac{\tau_1}{m_1} \left[\underline{d}_{\parallel} + \frac{\underline{d}_{\perp} + \underline{b} \times \underline{d}_{\perp}}{1 + \omega^2\tau^2} \right]_1 + \frac{\tau_2}{m_2} \left[\underline{d}_{\parallel} + \frac{\underline{d}_{\perp} + \underline{b} \times \underline{d}_{\perp}}{1 + \omega^2\tau^2} \right]_2 \right\} \end{aligned} \qquad (2.2.7)$$

$$p_{ij} = p_o - \left[\tau_1 \, p_1 + \tau_2 p_2 \right] \left[(\mathrm{div} \, . \, \underline{V})_{\|} - \tfrac{1}{3} \, \mathrm{div} \, . \, \underline{V} \right] \underline{b}\,\underline{b}$$

$$- [\tau_1 p_1 + \tau_2 p_2] \, [\tfrac{1}{2} \, (\mathrm{div} \, \underline{V})_{\perp} - \tfrac{1}{3} \, \mathrm{div} \, \underline{V}] \, [\underline{1} - \underline{b}\,\underline{b}\,]$$

$$- \{ p_1 \, \frac{1}{8} \, \frac{1}{1+4\omega^2\tau^2} [\underline{\nabla}_{\perp}\underline{V}_{\perp} + \underline{b} \times \underline{\nabla}\,\underline{V} \times \underline{b}\,] + 2\omega\tau [\underline{\nabla}_{\perp}\underline{V}_{\perp} \times \underline{b} - \underline{b} \times \underline{\nabla}_{\perp}\underline{V}_{\perp}] \mathrm{diag} \} 1{+}2$$

$$+ \frac{1}{4} \, \frac{\tau p}{1+\omega^2\tau^2} \, \{ (\frac{\partial \underline{V}_{\perp}}{\partial x_{\|}} + \frac{\partial \underline{V}_{\|}}{\partial x_{\perp}}) + \omega\tau \, \underline{b} \times (\frac{\partial \underline{V}_{\perp}}{\partial x_{\|}} + \frac{\partial \underline{V}_{\|}}{\partial x_{\perp}}) \, \} 1{+}2$$

$$+ \frac{1}{8} \, \frac{p}{1+4\omega^2\tau^2} \, [\underline{\nabla}_{\perp}\underline{V}_{\perp} + \underline{b} \times \underline{\nabla}\,\underline{V} \times \underline{b}] + 2\omega\tau [\underline{\nabla}_{\perp}\underline{V}_{\perp} \times \underline{b} - \underline{b} \times \underline{\nabla}_{\perp}\underline{V}_{\perp}] 1{+}2 \qquad (2.2.8)$$

From these expressions it is clear that, for two reasons:- (1) because the centre of mass of the total system does not coincide with the centre of either component, and (2) because trajectories are curved by the magnetic field, the transport processes are considerably complicated; the heat flux, for example depends on the vector d : which could be eliminated in favour of the current; or the pressure gradient and electric field. The current itself depends on temperature and density gradients as well as on the electric fields - moreover, currents either of heat or electricity, are not in the direction of applied forces. Hall currents flow; and conductivity is anisotropic. These complications persist when attempts are made to solve the integral equations posed by the Boltzmann equation. Note that this model has proved incapable of describing the explicit dependence of $\underline{j}$ on $\underline{\nabla}T$.

3. Methods Used to Obtain the Normal Solution

We will now consider the procedures that have been developed for solving the integral equation (2.1.2) which, on introducing $f_1 = f_o \Phi$ may be written as:-

$$I(f) = \int d^3v' \int d\Omega |\underline{v}-\underline{v}'| \, \sigma(|\underline{v}-\underline{v}'| \, , \theta) \, \{ f_o(\underline{v}) \, [1 + \Phi(\underline{v})] \, f_o(\underline{v}') \, [1+\Phi(\underline{v}')]$$

$$- f_o \, (\bar{\underline{v}}) \, [1 + \Phi(\bar{\underline{v}})] \, f_o \, (\bar{\underline{v}}) \, [\, 1 + \Phi(\bar{\underline{v}}') \,] \, \}$$

or

$$\int d^3v' \int d\,\Omega \,|\underline{v}-\underline{v}'|\; \sigma(\,|\underline{v}-\underline{v}'|,\, \theta)\; \{f_o(\underline{v})\; f_o(\underline{v}')\; [\Phi(\underline{v}) + \Phi(\underline{v}')\;]$$

$$- f_o(\bar{\underline{v}})\; f_o(\bar{\underline{v}}')\; [\;\Phi\,(\bar{\underline{v}}) + \Phi\,(\bar{\underline{v}}')\;]\;\} \qquad (2.3.1)$$

We must now consult the dynamics of a collision in order to discover the values of $\bar{\underline{v}}$, $\bar{\underline{v}}'$. On a collision, the centre of mass motion is constant; i.e.

$$\underline{V} = \frac{m_1\underline{v} + m_2\underline{v}'}{m_1 + m_2} = \frac{m_1\bar{\underline{v}} + m_2\bar{\underline{v}}'}{m_1 + m_2} \qquad (2.3.2)$$

and on removing this, the collision is described as the motion of a particle of the reduced mass:-

$$\frac{m_1\; m_2}{m_1 + m_2}\;,$$

moving with the relative velocity $\underline{g} = \underline{v}' - \underline{v}$. Conservation of energy on collision requires that only the direction of $\underline{g}$ be changed, e.g. that $\underline{g}'$ is:-

$$O\;\underline{g} = (\bar{\underline{v}}' - \bar{\underline{v}}) \qquad (2.3.3)$$

thus, the final velocities and the $\bar{\underline{v}}$ are given by:-

$$-\;\bar{\underline{v}}_1 = \underline{v}_{1\,f} = \underline{v}_1 + \frac{m_2}{m_1 + m_2}\,(\underline{g} - O\,.\,\underline{g})\;;\quad \underline{v}_{2f} = \underline{v}_2 - \frac{m_1}{m_1 + m_2}\,(\underline{g} - O\,.\,\underline{g})$$

$$(2.3.4)$$

From conservation of energy also:-

$$\tfrac{1}{2}\,m_1 v^2 + \tfrac{1}{2}\,m_2 v'^2 = \tfrac{1}{2}\,m_1 \bar{v}^2 + \tfrac{1}{2}\,m_2 \bar{v}'^2$$

hence:-

$$f_o(\bar{\underline{v}})\; f_o(\bar{\underline{v}}') \sim \exp\left[-\,\frac{\tfrac{1}{2}\,m_1\bar{v}^2 + \tfrac{1}{2}\,m_2\bar{v}'^2}{kT}\right] = f_o(\underline{v})\; f_o(\underline{v}')$$

and

$$I(f_o,\Phi) = \int d^3v' \int d\,\Omega|\underline{v}-\underline{v}'|\; \sigma(|\underline{v}-\underline{v}'|,\theta)\; f_o(\underline{v})\; f_o(\underline{v}')\;\;[\;\Phi\,(\underline{v}) +$$

$$+\;\Phi(\underline{v}') - \Phi(\bar{\underline{v}}) - \Phi(\bar{\underline{v}}')\;] \qquad (2.3.5)$$

Observe that $I(\underline{v}, \underline{v}')$ is unchanged by interchanging $m_1\;\underline{v}$ and $m_2 v'$ for since this changes the sign of g, (2.3.4) is unchanged, while all other terms are symmetric in these quantities, and the integral equation may be written:-

$$\int d^3v' \int d\Omega |v-v'| \ \sigma_{11}(|v-v'|,\Theta) \ f_1(v) \ f_1(v')[\Phi_1(v)+\Phi_1(v')-\Phi_1(\bar{v}') - \Phi_1(\bar{v})]$$

$$+ \int d^3v' \int d\Omega |v-v'| \ \sigma_{12}(|v-v'|,\Theta) \ f_1(v) \ f_2(v')[\Phi_1(v)+\Phi_2(v')-\Phi_1(\bar{v})-\Phi_2(\bar{v}')]$$

$$+\Omega \ f_o \frac{\partial\Phi}{\partial\phi}^1 = f_o^1 \ \{2 \ (\frac{\partial V_i}{\partial x_j} - \tfrac{1}{3} \operatorname{div} V \ \delta_{ij}) w_i w_j - \left[(\tfrac{5}{2} - w^2)\tfrac{1}{T} \nabla T - \tfrac{n}{n_i} \cdot d_i \right] \cdot c \ \}_1 \tag{2.3.6}$$

with a similar equation for Φ_2 .

One technique for solving these equations is a development of that due to Chapman and Cowling, and exploited by Landshoff and Marshall. In it, the function Φ, which must be proportional to a linear combination of the forces on the right is written as:-

$$\{V_{i,j}\} \ \{ \ [w_i w_j - \tfrac{1}{3} w^2] \ \Phi_1 \ (v^2) + [(b \times w)_i w_j] \ \Phi_2(v^2)$$

$$+ [(b \times w)_i \ (b \times w)_j - \tfrac{1}{3} \delta_{ij} \ (b \times w)^2] \ \Phi_3 \ \}$$

$$+ \{\Phi_4 \ (v^2)\underline{w} + \Phi_5(v^2) \ \underline{b} \times \underline{w} + \Phi_6(v^2) \ \underline{b} \ (\underline{b} \cdot \underline{w})\} \cdot (\underline{\nabla}\log T + \underline{d})$$

and the scalar quantities $\Phi_i(v')$ determined by a set of scalar equations.

Approximate solutions to these equations may be obtained by expanding in powers of certain polynomials:- the Sonine polynomials, which are orthogonal with $e^{-x^2} x^n$: i.e. such that:-

$$\int e^{-x^2} \ x^{2n+1} \ S_n^{m^1}(x) \ S_n^m \ (x) \ dx = \delta(m,m^1) \ \frac{\frac{1}{2}(n+m) \ !}{m!}$$

If the magnetic term is absent, a simple variational principal exists, which aids in obtaining approximate solutions. In a magnetic field, Marshall treated the differential term by using a complex representation of Φ , $\partial/\partial\phi$ then interchanged real and imaginary parts:- this, however, reduced the variational principle from a maximal to a simple stationary form. Simple rational expressions were obtained for the transport coefficient, the simplicity being enforced by the trial functions used.

An alternative method has been employed by Bernstein and Robertson. They first observe that if the limit $m/M \to 0$ is taken, conservation of energy permits a separation of the two equations for the perturbations Φ_- in the electron distribution and Φ_+ in the ion distribution; the equations becoming:-

$$f_-^o \left\{ 2\underline{\nabla}\,\underline{V} : (\underline{w}\,\underline{w} - \tfrac{1}{3} w^2 \underline{I}) + (w^2 - \tfrac{5}{2})\underline{c} \cdot \underline{\nabla} \log T - \underline{c} . \underline{d} - \frac{e}{m} \underline{B} \times \underline{c} \cdot \frac{\partial \Phi_-}{\partial \underline{c}} \right\}$$

$$= K_{--}\ \Phi_- + K_{+-} \Phi_-$$

and:-

$$f_+^o \left\{ 2\underline{\nabla}\,\underline{V} : (\underline{w}\,\underline{w} - \tfrac{1}{3} w^2 \underline{I}) + (w^2 - \tfrac{5}{2})\underline{c} \cdot \underline{\nabla} \log T - \frac{e_+}{m_+} \underline{B} \times \underline{c} \cdot \frac{\partial \Phi_+}{\partial \underline{c}} \right\} - I_{++}$$

$$= \left[I_{+-} - \frac{1}{nkT} \underline{j} \times \underline{B} \cdot \underline{c}\ f_+^o - \frac{2-}{kT} \underline{c} \cdot \underline{d}\ f_+^o \right] = 0 \qquad (2.3.7)$$

where from conservation of energy, the R.H.S. vanishes. Further, in their analysis, Bernstein and Robertson use a Fokker Planck representation of the interaction term which introduces some slight simplification. They now represent the solutions Φ as linear combinations of the forces as does Marshall, but instead of his simple complex representation, represent the angular dependence by a harmonic expansion, i.e.

$$\Phi^- = \Phi_{\ell \pm n} \Sigma\ T_{\ell \pm n} (\theta, \phi) = N_{\ell, n}\ P_\ell^n (\cos \theta) \begin{matrix} \sin n\phi \\ \cos n\phi \end{matrix}$$

where

$$N^2 = \frac{2\ell+1}{2\pi} \frac{(\ell - |n|)\ !}{(\ell + |n|)\ !} \frac{1}{1+\delta |n|}$$

is a normalizing factor.

If now the Φ's are substituted in (2.3.7) and the contributions to the various sources are separated, integro-differential equations are derived for the separate contributions to Φ - e.g. those arising from $\underline{d}$ become:-

$$f_-^o \left[d_z\ c\ T_{1,o} - I_- (\Phi_{1,o}, T_{1,o}, N_{1,o}) \right] = 0 \qquad (2.3.8)$$

$$f_-^o \left[d_y\ c\ T_{1,1} - I_- (\Phi_{1,1}, T_{1,1}, N_{1,1}) + \omega\, \Phi_{1,-1}\ N_{1,1}\ T_{1,1} \right] = 0$$

and

$$f_-^o \left[d_x\ c\ T_{1,-1} - I_- (\Phi_{1,-1}\ T_{1,-1}\ N_{1,-1}) - \omega\, \Phi_{1,1}\ N_{1,1}\ T_{1,-1} \right] = 0$$

ω being the gyro frequency.

Now since the integral operator I_- is invariant under rotation the spherical harmonics are eigen - functions thereof and;

$$I(\Phi\ T\ N) = K\ \Phi\ T\ N = \lambda\ .\ (c^2)\ \Phi\ T\ N\ ,$$

where λ is a function (unknown!) of c^2 ; and K is an integral operator acting on the functions $\Phi\ (c^2)$. If now we consider d $\|$ OY, the second two equations take the form:-

$$\frac{d_y c}{N_{1,1}} - K\,\Phi_{1,1} + \omega\,.\,\Phi_{1,-1} = 0$$

$$- K\,\Phi_{1,-1} - \omega\,\Phi_{1,1} = 0 \qquad (2.3.9)$$

By setting $\omega = 0$; and replacing $N_{1,1}$ by $N_{1,0}$ and dy by dz, the first equation in (2.3.8) may be reduced to (2.3.9). The problem of solving this equation is however still sufficiently formidable that recourse is had to a variational procedure - which is a suitable modification of that introduced by Hirschfelder et al. for normal gases, and used by Marshall.

4. The Variational Procedure

In the absence of a magnetic field, the equation to be solved takes on the form:-

$$\psi\ (c) = K\ \Phi\ (c) \qquad (2.4.1)$$

where ψ is a known function and K the integral operator representing the change in Φ produced by collisions. A variational principle may be derived by observing first that K is a symmetric operator, hence for any two functions of c, η and ξ ,

$$(\eta,\ K\ \xi) = \int d^3c\ \eta(c)\ K\ \xi(c) = (\xi,\ K\eta) \qquad (2.4.2)$$

Further

$$(\xi,\ K\ \xi) \leqslant 0.$$

To obtain approximate solutions to (2.4.1) consider:-

$$\lambda\ (\chi) = -(\chi,\psi)^2/(\chi,\ K\ \chi), \qquad (2.4.3)$$

If now, we insist that λ be stationary with respect to χ , then:-

$$\delta\lambda \quad -2\ \{[(\chi,K\chi)\ \ \psi - (\chi,\psi)\ K\chi]\ 2\ \frac{(\chi,\psi)}{(\chi,K\psi)^2}\ \ \delta\ \chi\ \} \qquad (2.4.4)$$

which vanishes if:-

$$\Psi = \frac{(\chi,\psi)}{(\chi,K\ \chi}\ \chi$$

satisfies (2.4.1). By carrying the variation out to 2nd order in $\delta\ \chi$, λ is shown to be a maximum at the solution.

In a magnetic field, we must consider a pair of equations:-

$$\psi(c) = \omega \frac{\partial}{\partial\phi} \Phi_2(c) + K\ (\Phi_1(c)) \tag{2.4.5}$$

$$- \omega \frac{\partial}{\partial\phi} \Phi_1(c) = K \Phi_2\ (c)$$

A partial integration shows:-

$$(\chi_1, \omega \frac{\partial}{\partial\phi} \chi_2) = - (\chi_2\ \omega \frac{\partial}{\partial\phi} \chi_1) \tag{2.4.6}$$

Now consider:-

$$\lambda_1 = - \frac{(\chi_1, \psi)^2}{[\ \chi_1, K\ \chi_1) + (\chi_1, \omega \frac{\partial}{\partial\phi} \chi_2)^2 / (\chi_2, k\ \chi_2)]} \tag{2.4.7}$$

If χ_1 is fixed, this is a minimum with respect to χ_2 when:-

$$- (\chi_2\ \omega \frac{\partial}{\partial\phi} \chi_1)^2 / (\chi_2, K\ \chi_2)$$

is a <u>maximum</u>, re χ_2 :

This, however, by comparison with (2.4.3) holds if:-

$$- \omega \frac{\partial}{\partial\phi} \chi_1 = K \frac{(\chi_1\ \omega \frac{\partial}{\partial\phi} \chi_2)}{(\ \chi_2,\ K\ \chi_2\ )} \chi_2$$

If, on the other hand, χ_2 is taken as fixed, λ_1 is stationary when:-

$$\frac{2(\chi_1, \psi)}{[(\chi_1, K\ \chi_1) + (\chi_1, \omega \frac{\partial}{\partial\phi} \chi_2)^2 / (\chi_2, K\ \chi_2)]^2} \left\{ [(\chi_1, K\ \chi_1) + (\chi_1,\ \omega \frac{\partial\chi_1}{\partial\phi})^2 / (\chi_2, K\ \chi_2)]\psi \right.$$

$$\left. - (\chi_1, \psi\ [K\ \chi_1\ + \omega \frac{\partial\chi_2}{\partial\phi}(\chi_1\ \omega \frac{\partial}{\partial\phi} \chi_2) / (\chi_2, K\ \chi_2)] \right\} = 0$$

i.e. when:-

$$\psi = \frac{(\chi_1, \psi)^2}{[(\chi_1, k\ \chi_1) + (\chi_1\ \omega \frac{\partial\chi_2}{\partial\phi})^2 / (\chi_2, K\ \chi_2)]} \left[K\ \chi_1 + \frac{(\chi_1, \omega \frac{\partial\chi_2}{\partial\phi})}{(\chi_2,\ K\ \chi_2)}\ \omega \frac{\partial\chi_2}{\partial\phi} \right] \tag{2.4.8}$$

This second stationary point may be shown to be a maximum, hence constants may be chosen so that χ_1 and χ_2 satisfy (2.4.5) if λ is a maximum re. χ_1 and a minimum re. χ_2. At this stage trial functions may be introduced (usually polynomials!) and constants selected to determine the extreme. Bernstein and Robinson evaluated this result by considering a Lorentz gas (in which I _ _ _ = 0), which may be solved exactly, and claim 10% accuracy for all results.

The results of these rather elaborate calculations differ from the m.f. time theory in two ways; a value is given for τ, which depends on the particular cross section considered, and the simple rational functions:-

$$\frac{\tau}{1 + \omega^2\tau^2} \quad ; \quad \frac{\omega\tau^2}{1 + \omega^2\tau^2}$$

are replaced by more elaborate quantities of roughly the same form. In addition $\underline{j}$ contains a term $\sim \underline{\nabla}T$, the thermal diffusion effect which is missing from the m.f.t. theory.

Marshall's results are given in the accompanying table (Table 1); a comparison with the numerical results of Bernstein and Robinson is effected in the curves shown. (Figures 1-4)

5. Alternative Approaches to Solving the B.E.

In addition to the methods described above for obtaining the normal solution some other approaches have been used. One, due to Rosenbluth and Kaufmann, alters the ordering of the three terms in the B.E. The order selected is:-

$$B \gg A \gg C,$$

an order which is valid for rapid motions in strong fields. By selecting as a zero order distribution the Maxwellian, this is also valid for slow motions in strong fields. The distribution function is written as:-

$$f = f_o + f_1 + f_2$$

where:-

$$Df_o = \omega \frac{\partial}{\partial\phi} f_1$$

so that:-

$$f_1 = \frac{1}{\omega} f_o \{ (w^2 - \tfrac{5}{2}) \underline{c} \times \underline{b} . \underline{\nabla} \log T - \frac{m}{kT} (\underline{c} \times \underline{b}) . \underline{d}$$

$$- (\underline{\nabla}_\perp \underline{V}_\perp \times \underline{b} - \underline{b} \times \underline{\nabla}\, \underline{V}) : \underline{ww} \}$$

f_2 is now determined by:-

$$- \omega \frac{\partial}{\partial\phi} f_2 = I(f^o, f^1)$$

where $I = I(f^o, \Phi)$ and $\Phi = f_1/f_o$, and the problem is then reduced to that of calculating the integrals appearing in I. There is a happy agreement between transport coefficients calculated this way and the strong field limits of those calculated by Marshall. (As corrected by Haas and Vaughan Williams).

Yet another method is due to H. Grad. Here it is observed that what is desired from the B.E. is not the solution f, but the value of the moments $\underline{c}$, $\underline{cc}$. $\underline{cc}^2$ which have hydrodynamic significance, and further, that what is obtained is not the solution, f, but some rather crude approximation to the normal solution. This being so, it might be appropriate to consider not the equation for f, but the equations for the moments of f. The moment equations do not, of course, close unless forced to: but they may be forced to close by choosing a trial distribution function of the correct form. The first five moments, 1, $\underline{V}$, $\frac{1}{2}$ mc^2, yield hydrodynamic quantities of immediate interest, while the hydrodynamic equations involve a further eight, the remaining five components of $m\, c_i\, c_j$, and the three $\frac{1}{2}$ $mc^2\underline{c}$. Grad's method involves writing the distribution function as a linear combination of these moments: 1, $\underline{V}$, T, P_{ij} and q, chosen to be consistent. The moment equations then are forced to close after the first 13, and differential equations are obtained for the time dependence of the moments. This method has been applied to the ionized plasma by Kolodner, and by Lyley and Herdan; it is not noticably simpler than the normal solution procedure.

TABLE I

Transport Coefficients for a Fully Ionized Gas in a Magnetic Field

(Marshall, corrected by Haas and Vaughan-Williams)

Current density

$$\underset{\sim}{j} = \sigma_{11} \underset{\sim}{D}_{11} + \sigma_{\perp} \underset{\sim}{D}_{\perp} + \sigma_{H} \underset{\sim}{b} \times \underset{\sim}{D} + \phi_{11} \underset{\sim}{\nabla}_{11} T + \phi_{\perp} \underset{\sim}{\nabla}_{\perp} T + \phi_{H} \underset{\sim}{b} \times \underset{\sim}{\nabla} T$$

where:-

$$\underset{\sim}{V}_{11} = (\underset{\sim}{V} \cdot b), \quad \underset{\sim}{V} = \underset{\sim}{V}_{\perp} - \underset{\sim}{V}_{11}$$

for any vector $\underset{\sim}{V}$:

$$\underset{\sim}{B} = B \underset{\sim}{b}$$

and

$$\underset{\sim}{D} = \underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B} + \frac{1}{ne} \underset{\sim}{\nabla} p$$

Heat flux:-

$$\underset{\sim}{q} = - \lambda_{11} \underset{\sim}{\nabla}_{11} T - \lambda_{\perp} \underset{\sim}{\nabla}_{\perp} T - \lambda_{H} \underset{\sim}{b} \times \underset{\sim}{\nabla} T + \psi_{11} \underset{\sim}{j}_{11} + \psi_{\perp} \underset{\sim}{j}_{\perp} + \psi_{H} \underset{\sim}{b} \times \underset{\sim}{j}$$

or

$$\underset{\sim}{q} = - K_{11} \underset{\sim}{\nabla}_{11} T - K_{\perp} \underset{\sim}{\nabla}_{\perp} T - K_{H} \underset{\sim}{b} \times \underset{\sim}{\nabla} T - \beta_{11} \underset{\sim}{D}_{11} - \beta_{\perp} \underset{\sim}{D}_{\perp} - \beta_{H} \underset{\sim}{b} \times \underset{\sim}{D}$$

Stress tensor is most intelligible in component form. If OX $\parallel$ $\underset{\sim}{b}$ and:-

$$S_{ij} = \tfrac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \tfrac{1}{3} \delta_{ij} \operatorname{div} v\right)$$

$$p_{xx} = p - 2\mu S_{xx}$$

$$p_{yy} = p - \frac{2\mu}{1+r_+^2} \left\{ S_{yy} + \tfrac{1}{2} r_+^2 (S_{yy} + S_{zz}) + r_+ S_{yz} \right\}$$

$$p_{zz} = p - \frac{2\mu}{1+r_+^2} \left\{ S_{zz} + \tfrac{1}{2} r_+^2 (S_{yy} + S_{zz}) - r_+ S_{yz} \right\}$$

$$p_{xy} = p_{yx} = - \frac{2\mu}{1+\frac{1}{4}r_+^2} \left\{ S_{xy} + \tfrac{1}{2} r_+ S_{xz} \right\}$$

$$p_{xz} = p_{zx} = - \frac{2\mu}{1+\frac{1}{4}r_+^2} \left\{ S_{xz} - \tfrac{1}{2} r_+ S_{xy} \right\}$$

$$p_{yz} = p_{zy} = \frac{2\mu}{1+\frac{1}{4}r_+^2} \left\{ S_{yx} - \tfrac{1}{2} r_+ (S_{yy} - S_{zz}) \right\}$$

To define the coefficients in these expressions we introduce the following collision frequencies.

$$\nu_- = \tfrac{2}{3} \sqrt{\frac{2\pi}{m_-}} \; n\, e^4 \log \Lambda/(kT)^2 \quad \text{with } \Lambda = 2(kT)^{3/2}/e^3(\pi n)^{\frac{1}{2}}$$

$$\nu_+ = \frac{2\sqrt{2}}{5} \sqrt{m_-/m_+} \cdot \nu_-$$

the gyro frequencies

$$\Omega_- = (e\,B/m)_- \qquad \Omega_+ = (e\,B/m)_+ \quad ,$$

and the ratios:-

$$r = \Omega_-/\nu_- \qquad r_+ = \Omega_+/\nu_+ \; ,$$

the mass ratio:-

$$M_- = m_- / (m_- + m_+) \quad ,$$

the unmodified transport coefficients,

$$\sigma_o = ne^2/m_-\nu_- \qquad \lambda_o = \sigma_o\, k^2T/e^2 \; , \; \phi_o = \sigma_o/e \; , \; \psi_o = kT/e$$

and the r dependent factors,

$$\Delta_1^{-1} = r^4 + 6.28\, r^2 + 0.93 \qquad \Delta_2^{-1} = r^4 + 16.20\, r^2 + 44.3$$

$$\Delta_3^{-1} = r^2 M_- + 9M_- + 3.39\sqrt{M_-} + 0.32 \qquad \Delta_4^{-1} = r^2 + 3.48, \quad \Delta_5^{-1} = r^2 + 12.72$$

The transport coefficients for a magnetized plasma then become:-

$$\sigma = \sigma_o . 1.93/2 \qquad \phi = 0.777\, \phi_o$$

$$\sigma_\perp = \sigma_o \tfrac{1}{2}(r^2 + 1.8)\; \Delta_1 \qquad \phi_\perp = -\,\phi_o\, 0.75(r^2 - 0.966)\; \Delta_1$$

$$\sigma = \sigma_o \tfrac{1}{2}\, r(r^2 + 4.4)\; \Delta_1 \qquad \phi = -\,\phi_o\, 2.15\, r\, \Delta_1$$

$$\lambda = 1.02\; \lambda_o \qquad \psi = 3.30\; \psi_o$$

$$\lambda_\perp = 1.25\; \lambda_o[(3M_- + 0.56\sqrt{M_1})\; \Delta_3 + (5.43r^2 + 36.1)\; \Delta_2 - 3.56\; \Delta_5]$$

$$\psi_\perp = \psi_o\, [2.5r^2 + 11.5]\; \Delta_4$$

$$\lambda = 1.25\; \lambda_o\, [rM_-\Delta_3 - 9.28r\; \Delta_2 - r\; \Delta_5] \qquad \psi = \psi_o\, 1.5r\; \Delta_4$$

$$\mu = \tfrac{1}{4}\, n\, kT/\nu_+$$

The second form of the thermal conductivity may be written in terms of the first, thus:-

$$K = \lambda \; - \phi\; \psi \qquad \beta = -\,\sigma\; \psi$$

$$K_\perp = \lambda_\perp + \phi\; \psi \; - \phi_\perp \psi_\perp \qquad \beta_\perp = \sigma\; \psi \; - \sigma_\perp\, \psi_\perp$$

$$K = \lambda \; - \phi_\perp \psi \; - \phi\; \psi_\perp \qquad \beta = -\,\sigma\; \psi_\perp - \sigma_\perp\, \psi$$

In the limit of strong magnetic fields $r, r_+ \rightarrow \infty$

$$\sigma = 0.8\ \sigma_o \qquad \phi = 0.68\ \phi_o$$

$$\sigma_\perp = 0.5\ \sigma_o\ r^{-2} \qquad \phi_\perp = 0.75\ \phi_o\ r^{-2}$$

$$\sigma = r\ \sigma_\perp \qquad \phi = 2.13\ \phi_o\ r^{-3}$$

$$\lambda = \lambda_o \qquad \psi = 3.3\ \psi_o$$

$$\lambda_\perp = 0.7 \sqrt{\left(\frac{m_+}{m_-}\right)}\ \lambda_o r^{-2} \qquad \psi_\perp = 2.5\ \psi_o$$

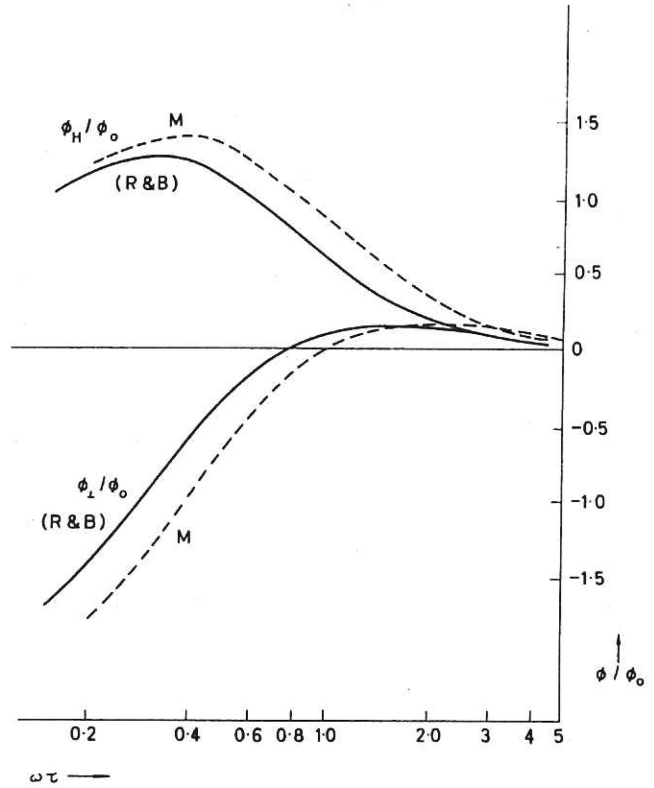
$$\lambda = -\ 0.5\ \lambda_o\ (0.4 + r^2 m_-/m_+)\ r^{-1} \qquad \psi = 1.5\ \psi_o\ r^{-1}$$

$$p_{xx} = p_o - 2\ \mu\ S_{xx}$$

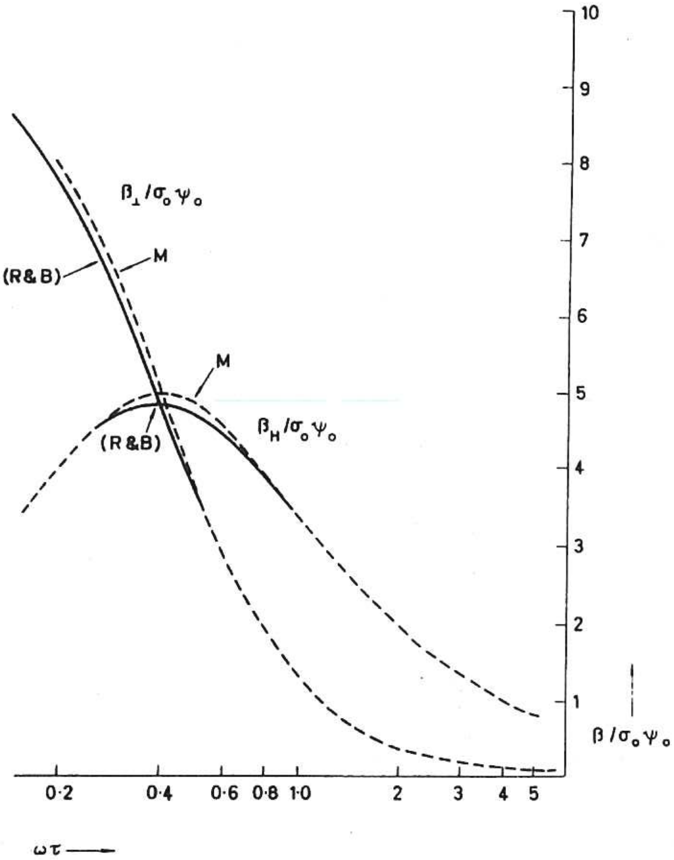
$$p_{yy} = p_o - \mu(S_{yy} + S_{zz}) = p_{zz}$$

$$p_{xy} = p_{yx} = -\ \frac{p_o}{\Omega_+}\ S_{xy} = -\ p_{xz} = -\ p_{zx}$$

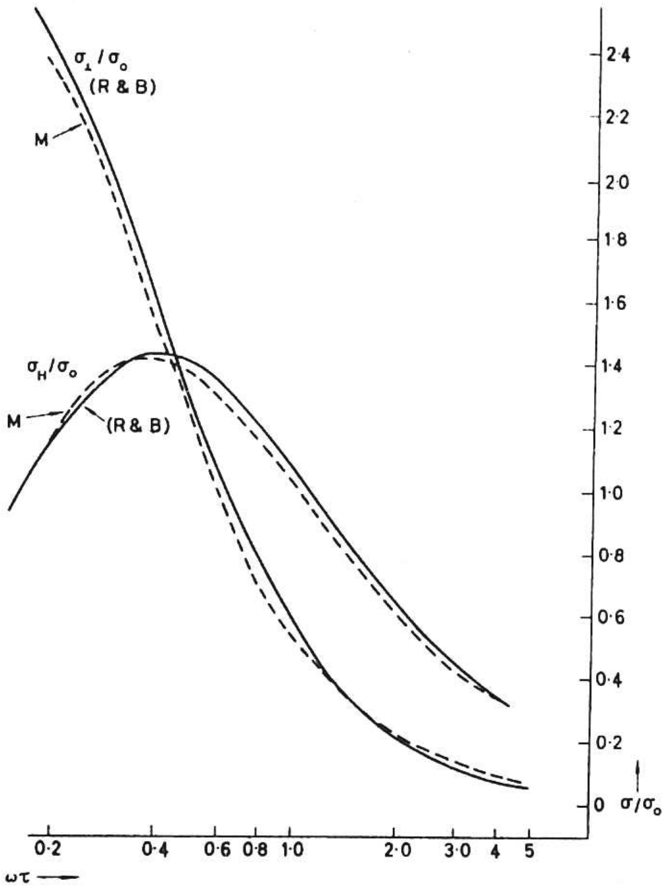
$$p_{yz} = -\ p_{zy} = \frac{p_o}{4\Omega_+}\ (S_{yy} - S_{zz})$$



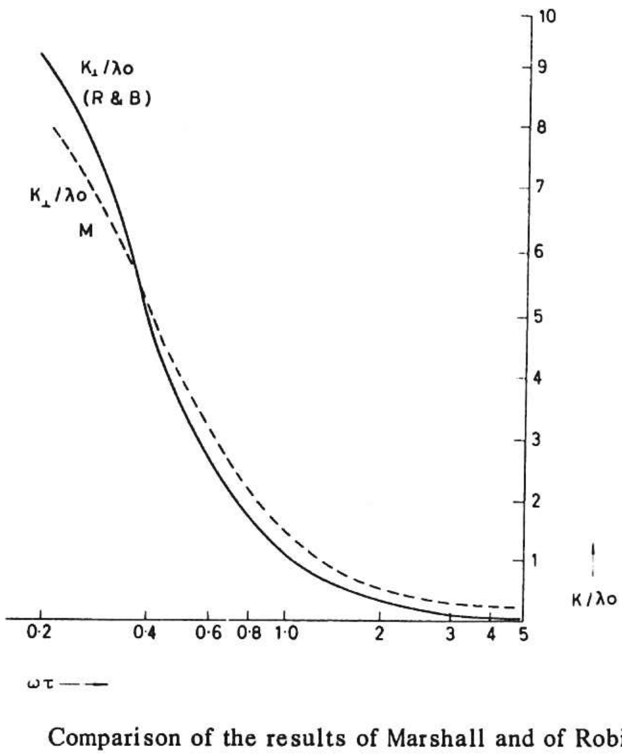
CLM-P 18 Fig. 1 Comparison of the results of Marshall and of Robinson and Bernstein. Thermal diffusion $\phi_\perp, \phi_H$



CLM-P 18 Fig. 2 Comparison of the results of Marshall and of Robinson and Bernstein. Thermal diffusion $\beta_\perp$ & β_H



CLM-P 18 Fig. 3 Comparison of the results of Marshall and of Robinson and Bernstein. Electrical conductivity $\sigma_\perp, \sigma_H$



CLM-P 18 Fig. 4 Comparison of the results of Marshall and of Robinson and Bernstein. Thermal conductivity K

PART III

KINETIC THEORY OF PLASMA

The Transport Equation

1. Introduction

Having shown how the Boltzmann equation leads to the appearance of transport coefficients and to phenomena associated with "real" fluids, we turn to the prior question, that of determining the correct form of the transport equation. Our procedure is first to show how the Boltzmann equation can be expanded for small angle scattering, which dominates the collision process. Having done this we are free to discuss the effect of long-range corelations on the small angle scattering and to develop forms for the transport equation which are valid in this region. We may then compare the expanded B.E. with the long range equation of transport, and from this justify our use of the B.E. in those situations for which the meaning of the transport coefficients is unambiguous.

2. An Expansion of the B.E.

Consider the Boltzmann collision integral for an ionized gas of like particles (for the moment). Using:-

$$\sigma = \sigma_R = \left(\frac{e}{2m}\right)^2 \left(g \sin \frac{\theta}{2}\right)^{-4}$$

where $\underline{g} = \underline{v} - \underline{v}'$ and further, using the result (2.3.4) to write:-

$$\bar{\underline{v}} = \bar{\underline{v}} - \tfrac{1}{2}\,\Delta\,\underline{g}$$

$$\bar{\underline{v}}' = \bar{\underline{v}}' + \tfrac{1}{2}\,\Delta\,\underline{g}$$

where $\Delta \underline{g}$ = change in $\underline{g}$ on collision, i.e. $\bar{\underline{g}} - \underline{g} = \Delta g$, enables this to be written:-

$$I = \left(\frac{e^2}{m}\right)^2 \int d^3v' \int d\phi\, d\theta \sin\theta \frac{g}{g^4 \sin^4(\theta/2)} [f(\underline{v}' + \tfrac{1}{2}\Delta\underline{g})\, f(\underline{v} - \tfrac{1}{2}\Delta g) - f(\underline{v}')f(\underline{v})] \quad (3.1.1)$$

If $\underline{m}$, $\underline{n}$ are unit vectors orthogonal to $\underline{g} = g\,\hat{\underline{g}}$, we may write:-

$$\Delta\underline{g} = 2\,g \sin\left(\frac{\theta}{2}\right)\left\{-\,\hat{\underline{g}} \sin\frac{\theta}{2} \sin\phi + \underline{m}\cos\frac{\theta}{2}\cos\phi + \underline{n}\cos\left(\frac{\theta}{2}\right)\sin\phi\right\} \quad (3.1.2)$$

Now we may expand the quantity within the square brackets thus:-

$$f(\underline{v}' + \tfrac{1}{2}\Delta\underline{g})\, f(\underline{v} - \tfrac{1}{2}\Delta g) - f(\underline{v}')\, f(\underline{v})$$

$$= \left[f(\underline{v}) \frac{\partial f(\underline{v}')}{\partial \underline{v}'} - f(\underline{v}') \frac{\partial f(\underline{v})}{\partial \underline{v}} \right] . \tfrac{1}{2}\Delta\,\underline{g} + \tfrac{1}{2}\left[f \frac{\partial^2 f(\underline{v}')}{\partial \underline{v}'\partial \underline{v}'} + f(\underline{v}') \frac{\partial^2 f(\underline{v})}{\partial \underline{v}\partial \underline{v}} - 2\frac{\partial f(\underline{v}')}{\partial \underline{v}'} \frac{\partial f(\underline{v})}{\partial \underline{v}} \right] \tfrac{1}{4}\Delta\,\underline{g}\;\Delta\,\underline{g} + 0(\Delta\underline{g})^3 \quad (3.1.3)$$

If we are interested only in scattering at small $\Delta\underline{g}$, i.e. small angle scattering, this Taylor expansion may be substituted in the collision integral (3.1.1) and if the series is cut off at the second term, the expansion (3.1.2) used for $\Delta\underline{g}$ and the integral over ϕ performed, there results:-

$$I=8\,\pi(\tfrac{e^2}{m})^2 \int d^3v' \int^{\pi/2} d(\tfrac{\theta}{2}) \cos(\tfrac{\theta}{2}) \Big\{ - \frac{\underline{g}}{g^3 \sin(\frac{\theta}{2})} . \left[f(\underline{v})\frac{\partial f(\underline{v}')}{\partial \underline{v}} - f(\underline{v}')\frac{\partial f(\underline{v})}{\partial \underline{v}} \right]$$

$$+ \tfrac{1}{2}\left[f(\underline{v}) \frac{\partial^2 f(\underline{v}')}{\partial \underline{v}'\partial \underline{v}'} + f(\underline{v}') \frac{\partial^2 f(\underline{v})}{\partial \underline{v}\partial \underline{v}} - 2\frac{\partial f(\underline{v}')}{\partial \underline{v}} \frac{\partial f(\underline{v})}{\partial \underline{v}} \right]$$

$$. \frac{1}{g}\left[\frac{\underline{gg}}{g^2} \sin\frac{\theta}{2} + \tfrac{1}{2}(\underline{mm} + \underline{nn})\left(\frac{1}{\sin\frac{\theta}{2}} - \sin\frac{\theta}{2}\right) \right] \Big\} \quad (3.1.4)$$

Now:-

$\int_0^{\pi/2} d(\frac{\theta}{2}) \cos(\frac{\theta}{2}) \sin(\frac{\theta}{2}) = 1$, while $\int_0^{\pi/2} d(\frac{\theta}{2}) \cos(\frac{\theta}{2})/\sin(\frac{\theta}{2}) = \log \sin(\frac{\theta}{2})\Big|_0^{\pi/2}$ diverges; however, the divergence arises from the lower limit $\theta/2 = 0$, and this represents just those long range effects where co-operative phenomena may be important. The scattering angle at long ranges may be related to the impact parameter b, indeed, using the impulse approximation:-

$$\frac{v_\perp}{v_o} = \sin\theta = \frac{1}{v_o}\int F_\perp\, dt = \frac{1}{v_o^2}\int F_\perp\, dx$$

$$= \frac{e^2}{m_r v_o^2} \int_{-\infty}^{\infty} \frac{b}{(x^2+b^2)^{3/2}}\, dx = \frac{1}{b}\, \frac{e^2}{m_r v_o{}^2} \int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^{3/2}} = \frac{2e^2}{b m_r v_o^2}$$

$$\therefore \sin(\tfrac{\theta}{2}) \simeq \frac{\theta}{2} \simeq \tfrac{1}{2}\sin\theta \simeq \frac{e^2}{b m_r v_o^2}$$

To approximate the maximum impact parameter for which collective effects are unimportant, we may observe, in the most primitive fashion that the plasma acts as a dielectric medium, for which, roughly,

$$\varepsilon = 1 - \frac{\omega_o^2}{\omega^2}\,,$$

where:-

$$\omega_o^2 = 4\,\pi\, n\, e^2/m$$

is the plasma frequency. If also, we Fourier analyze the potential due to a particle passing the test particle at a distance b, i.e.

$$\phi\ (\omega) = \int_{-\infty}^{-o} \frac{e^{i\omega t}}{[b^2+v^2t^2]}\ dt = \frac{1}{v}\, K_o\ (\frac{\omega b}{v})\ ,$$

this is exponentially small for values of $\omega \gg v/b$. However, frequencies $< \omega_o$ are not transmitted, but exponentially damped in the plasma, hence if $v/b < \omega_o$ the interaction is screened. This gives:-

$$b_{max} \simeq v/\omega_o$$

and

$$- \log\ (\sin\frac{\theta}{2})_{min} = \log\Lambda \simeq \log\left[\frac{e^2}{b_{max} m v^2}\right] = \log(\frac{e^2\omega_o}{mv^3}) \simeq \log\left[\frac{\pi n e^6}{(kT)^2}\right]^{\frac{1}{2}}$$

$$\simeq \log\left\{2\sqrt{\pi}\left[\frac{n^{\frac{1}{3}}e^2}{kT}\right]^{3/2}\right\} \simeq \log\left\{2\sqrt{\pi}\,(\frac{V}{T})^{3/2}\right\} \qquad (3.1.5)$$

where V and T are the average interparticle potential and the average kinetic energy. The parameter Λ or better (V/T) is usually small and plays an important role in our theory.

(3.1.4) may be reduced to:-

$$I = 4\,\pi(\frac{e^2}{m})^2 \log\Lambda \int d^3v' \ - \frac{\underline{g}}{g^3} \cdot \left[f(\underline{v})\ \frac{\partial f(\underline{v}')}{\partial \underline{v}'} - f(\underline{v}')\ \frac{\partial f(\underline{v})}{\partial \underline{v}} \right]$$

$$+ \frac{1}{2}\left[f(\underline{v})\ \frac{\partial^2 f(\underline{v}')}{\partial \underline{v}'\partial \underline{v}'} + f(\underline{v}')\ \frac{\partial^2 f(\underline{v})}{\partial \underline{v}\partial \underline{v}} - 2\ \frac{\partial f(\underline{v}')}{\partial\ \underline{v}'}\ \frac{\partial f(\underline{v})}{\partial \underline{v}} \right] : \frac{1}{2}\ \frac{\underline{\underline{mm}} + \underline{\underline{nn}}}{g}$$

$\underline{\underline{mm}} + \underline{\underline{nn}}$ is the unit tensor normal to $\hat{\underline{g}}$, i.e.

$$\underline{\underline{mm}} + \underline{\underline{nn}} = (\underline{\underline{1}} - \hat{\underline{g}}\hat{\underline{g}})$$

If:-

$$\underline{\underline{w}} = \frac{1}{g}\ (\underline{\underline{1}} - \hat{\underline{g}}\hat{\underline{g}}) = \frac{\underline{\underline{mm}} + \underline{\underline{nn}}}{g}\ ;\quad \underline{\underline{w}} = \frac{\partial}{\partial \underline{g}}\ \frac{\partial}{\partial \underline{g}}\ |g| \qquad (3.1.6)$$

and

$$\frac{\partial}{\partial \underline{g}} \cdot \underline{\underline{w}} = -\ 2\ \frac{\underline{g}}{g^3}$$

I may be written:-

$$I = 8\pi\left(\frac{e^2}{m}\right)\log\Lambda\int d^3v'\ \frac{1}{2}\frac{\partial\cdot \underline{\underline{w}}}{\partial \underline{g}}\cdot\left[f\frac{\partial f(\underline{v}')}{\partial \underline{v}'} - f(\underline{v}')\frac{\partial f(\underline{v})}{\partial}\right]$$

$$+\frac{1}{2}\underline{\underline{w}} : \frac{\partial}{\partial \underline{v}}\left[f(\underline{v}')\frac{\partial f(\underline{v})}{\partial \underline{v}} - f(\underline{v})\frac{\partial f(\underline{v}')}{\partial \underline{v}'}\right]$$

$$+\frac{1}{4}\underline{\underline{w}} : \left[f(\underline{v})\frac{\partial^2 f(\underline{v}')}{\partial \underline{v}'\partial \underline{v}'} - \frac{\partial f(\underline{v}')}{\partial \underline{v}'}\frac{\partial f(\underline{v})}{\partial \underline{v}}\right]$$

the last term here may be integrated by parts, and if we use :-

$$\frac{\partial \underline{\underline{w}}}{\partial \underline{g}} = -\frac{\partial \underline{\underline{w}}}{\partial \underline{v}'};$$

there results:-

$$I = 2\pi\left(\frac{e^2}{m}\right)^2\log\Lambda\ \frac{\partial}{\partial v_j}\int d^3v'\left[f(v')\frac{\partial f(v)}{\partial v_i} - f(v)\frac{\partial f(v')}{\partial v_i}\right]w_{ij} \tag{3.1.7}$$

(c.f. Landau). This may also be written as:-

$$I = -2\pi\left(\frac{e^2}{m}\right)^2\log\Lambda\ \frac{\partial}{\partial v_i}\left[\int d^3v'\left\{\frac{\partial f(v')}{\partial v_j}w_{ij} - f(v')\frac{\partial w_{ij}}{\partial v_j}\right\} f(v)\right]$$

$$+ 2\pi\left(\frac{e^2}{m}\right)^2\log\Lambda\ \frac{\partial^2}{\partial v_i\partial v_j}\left[\int d^3v'\left\{f(v')\,w_{ij}\right\} f(v)\right] \tag{3.1.8}$$

which has the form:-

$$I(f) = \frac{\partial}{\partial \underline{v}}\cdot(\underline{D}f) + \frac{\partial^2}{\partial \underline{v}\partial \underline{v}} : (\underline{D}f) \tag{3.1.9}$$

of the Fokker-Planck equation.

3. The Fokker-Planck Equation

The form of I, (3.1.9) may be approached by a somewhat different route and arises from a study of the rate of change of a Markovian probability distribution. Suppose that $P(x,t)$ represents the distribution function for a quantity x at time t: and suppose moreover, that during the interval $t \to t + \delta t$, x changes to x' with a probability $w(x, x' - x)$. Then, $P(x, t + \delta t)$ may be expressed in terms of $P(x', t)$ as:-

$$P(x, t+\delta t) = \int dx'\ w(x', x - x')\ P(x', t)$$

$$= \int d\xi\ \ w(x-\xi, \xi)\ P(x-\xi, t)$$

If w is a rapidly decreasing function of ξ, i.e. if P develops by small steps, then we may use a Taylor expansion on the R.H.S., so that:-

$$P(x, t+\delta t) = \int d\xi\, w(x,\xi)\, P(x,t) - \frac{\partial}{\partial x}\int d\xi\, \xi\, w(x,\xi)\, P(x,t)$$

$$+ \tfrac{1}{2}\frac{\partial}{\partial x}\frac{\partial}{\partial x}\int d\xi\, \xi\, \xi\, w\, P(x,t)$$

Now, the first integral here = 1, since $\int w(x,\xi)\, d^3\xi = 1$, the second = $\langle\xi\rangle$, the third $\langle\xi\xi\rangle$, thus:-

$$\frac{\partial P}{\partial t} = \frac{P(x,t+\delta t) - P(x,t)}{\delta t} = -\frac{\partial}{\partial x}[D_1 P(x,t)] + \tfrac{1}{2}\frac{\partial^2}{\partial x \partial x}[D_2 P(x,t)] + \ldots$$

where

$$D_1 = \frac{1}{\tau}\langle\xi\rangle, \quad D_2 = \frac{1}{\tau}\langle\xi\,\xi\rangle, \quad \tau = \delta t$$

For I(f,f) we write:-

$$I(f,f) = -\frac{\partial}{\partial \underline{v}}\cdot \underline{D}_1 f(v) + \tfrac{1}{2}\frac{\partial^2}{\partial \underline{v}\partial \underline{v}} : \underline{D}_2 f(v) \tag{3.2.1}$$

where

$$\underline{D}_1 = \frac{1}{\tau}\langle\underline{\Delta}v\rangle, \quad \underline{D}_2 = \frac{1}{\tau}\langle\underline{\Delta}v\ \underline{\Delta}v\rangle$$

For an unmagnetized plasma the quantities $\underline{D}_1$, $\underline{D}_2$ can be expressed in terms of the fluctuating microfield within the plasma, i.e. the change in velocity in a time τ of a particle initially at $\underline{x}$ having velocity $\underline{v}$ is:-

$$\Delta\underline{v} = \frac{e}{m}\int_{t-\tau}^{t} \underline{E}(x', t')\, dt'$$

where

$$\underline{x}' = \underline{x}_o + \int \underline{v}(x',t')\, dt'$$

and

$$\underline{v}(x,t) = \underline{v}_o + \frac{e}{m}\int \underline{E}(x',t')\, dt'$$

If $(e/m)\underline{E}$ small, we may expand $\underline{\Delta}v$ thus:-

$$\Delta v_i = \frac{e}{m}\int_{t-\tau}^{t} dt' \left\{ E_i(x_o + v_o t', t') + \frac{\partial E_i}{\partial x_j}\int_{t-\tau}^{t'} dt'' \int_{t-\tau}^{t''} \frac{e}{m} E_j(x_o + v_o t''', t''')\, dt''' \right\} \tag{3.2.2}$$

while to the same order in ${}^{eE}/m$;

$$\underline{\Delta}v\ \underline{\Delta}v = \frac{e^2}{m^2}\int_{t-\tau}^{t} dt'\, \underline{E}(x_o + v_o t', t') \int_{t-\tau}^{t} dt''\, E(x_o + v_o t'', t'')$$

$$= \frac{e^2}{m^2}\int_{t-\tau}^{t} dt' \int_0^{\tau} ds\, \underline{E}(x_o + v_o t', t')\, \underline{E}[x_o + v_o(t'-s), t-s] \tag{3.2.3}$$

This equation may be derived in an alternative way which requires, however, a return to the Liouville equation.

$$\frac{\partial F}{\partial t} + \underline{v}_i \cdot \frac{\partial F}{\partial \underline{x}_i} + \frac{e}{m} \underline{E}_i \cdot \frac{\partial}{\partial \underline{v}_i} F = 0 \qquad (3.2.4)$$

Now, the Liouville function F can be written :-

$$F_1(x_1) \Psi (x_2 \ldots x_N; x_1)$$

and by integrating over $x_2 \ldots x_N$, an equation for $F_1(x_1)$ may be deduced. Since:-

$$\underline{E}_i = - \frac{\partial}{\partial \underline{x}_i} \Phi_i = \frac{\partial}{\partial \underline{x}_i} \Sigma_j \frac{e}{|\underline{x}_i - x_j|}$$

depends upon the $\underline{x}_j$, however, the integration cannot be carried out explicitly, but involves a term of the form:-

$$- \frac{\partial}{\partial \underline{v}} \cdot \Sigma_j \frac{e^2}{m} \frac{\partial}{\partial \underline{x}_1} \Phi (\underline{x}_1, \underline{x}_j) \Psi (\underline{x}_2 \ldots \underline{x}_N; \underline{x}_1) F_1(\underline{x}_1) dx_2 \ldots dx_N,$$

$$\text{i.e. } \frac{e^2}{m} \frac{\partial}{\partial v} \cdot \underline{E} F$$

For any <u>given complexion</u>:-

$$F = \Pi_i \delta[\underline{x}_1 - \underline{X}_i (t)] \delta[\underline{v}_i - \underline{V}_i (t)],$$

is a rapidly varying function, and we prefer to work with a smoothly varying quantity, f_1, which represents the probability of a particle being at $\underline{x}_1$, $\underline{v}_1$ given some initial probability distribution :-

$$p_o(x_1 \ldots x_N, v_1 \ldots v_N)$$

for the entire distribution.

Then, F has the form:-

$$p_o (\underline{x}_1^o \ldots \underline{x}_N^o, \underline{v}_1^o \ldots \underline{v}_N^o) \Pi_1 \delta[\underline{x} - \underline{X}_i(t)] \delta[\underline{v} - \underline{V}_i(t)]$$

where

$$\underline{X}(t) = \underline{x}_1^o + \int \underline{V}_i (t') dt'$$

$$\underline{V}(t) = \underline{v}(o) + \frac{e}{m} \int \underline{E}(\underline{X}(t'), t') dt'$$

From this we may form F_1, again by integrating over $dx_2 \ldots dx_N$ and noting that if p_o is smooth, we expect F_1 to be a slowly varying function. In fact we expect F_1 to satisfy an equation of the Boltzmann type, and to change significantly in times of order τ_e, the collision period. This means that instead of considering the equation of motion for F_1 we obtain an adequate description by considering the motion of the coarse-grained distribution:-

$$f = \frac{1}{\tau}\int_{t-\tau}^{t} F_1(t')dt'$$

where $\tau \ll \tau_o$. This satisfies:-

$$\frac{\partial f}{\partial t} + \underline{v}\cdot\frac{\partial f}{\partial \underline{x}} + \frac{e}{m}\frac{\partial}{\partial \underline{v}}\cdot\frac{1}{\tau}\int_{t-\tau}^{t}\underline{E}(t', x(t'))\ .\ F_1(t')dt' = 0$$

i.e.

$$\frac{\partial f}{\partial t} + \underline{v}\cdot\frac{\partial f}{\partial \underline{x}} + \frac{e}{m}\frac{\partial}{\partial \underline{v}}\cdot\frac{1}{\tau}\ d^3\underline{x}\ d^3\underline{v}\int_{t-\tau}^{t'} dt'\underline{E}(t',\underline{x})\ \delta\ [\underline{x} - \underline{X}(t')]\,\delta\,[\underline{v} - \underline{V}(t')]F_1(t'-\tau) \tag{3.2.5}$$

where the last term is explicitly:-

$$-\frac{e^2}{m}\frac{\partial}{\partial \underline{v}}\cdot\frac{1}{\tau}\int_{t-\tau}^{t} dt'\ d^3x_2\ \dots\ d^3_N\ d^3v_2\dots\ d^3v_N\ \sum_j \frac{\partial\Phi}{\partial \underline{x}_i}(\underline{x}_i,\underline{x}_j)$$

$$\psi(\underline{x}_2\dots x_N,\ \underline{v}_2\dots\underline{v}_N, x_1)\ F_1(\underline{x}_1, v_1, t-\tau)\ \delta\ [\underline{x}_1 - \underline{X}_1(t')]\ \delta\ [\underline{v}_1 - \underline{V}_1(t')] \tag{3.2.6}$$

Now the electric field may contain some mean part $\underline{E}_o$, but will certainly contain a rapidly fluctuating part $\underline{E}_f$. $\underline{E}_o$ is easily handles, contributing a term :-

$$\frac{e}{m}\underline{E}_o\cdot\frac{\partial f}{\partial \underline{v}},$$

and to treat $\underline{E}_f$ we may observe that for most plasmas, where the ratio of kinetic to potential energy is small, the field dependent quantities $\underline{X}$ and $\underline{V}$ may be calculated in perturbation theory: thus:-

$$\underline{V}(t) = \underline{v}_o + \frac{e}{m}\int^{t}\underline{E}\ (\underline{x}_o + \underline{v}_o t',\ t')dt'$$

$$X(t) = \underline{x}_o + \underline{v}_o t + \frac{e}{m}\int_0^{t} dt'\int_0^{t'} dt''E(\underline{x}_o + \underline{v}_o t'',\ t'')$$

and the δ functions in (3.2.5) may similarly be expanded, thus:-

$$\delta(\underline{x} - \underline{X}(t'))\delta(\underline{v} - \underline{V}(t'')) = \delta(\underline{x} - \underline{x}_o - \underline{v}_o t')\ \delta(\underline{v} - \underline{v}_o)$$

$$- \delta(\underline{x} - \underline{x}_o - \underline{v}_o t')\frac{\partial}{\partial \underline{v}}\delta(\underline{v} - \underline{v}_o t', t')\ .\ \frac{e}{m}\int^{t'}\underline{E}\ (x_o + v_o t'', t'')dt''$$

$$- \delta(\underline{v} - \underline{v}_o)\frac{\partial}{\partial \underline{x}}\cdot\delta(\underline{x} - \underline{x}_o - v_o t')\ .\ \frac{e}{m}\int^{t'} dt''\int^{t''} dt'''\ \underline{E}(x_o + v_o t''', t''')$$

These may now be handled, as usual, by a partial integration, and use made of the fact that F_1 is slowly varying, so that $F(t-\tau, v) = f(t,v)$ whereupon (3.2.5) becomes

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{x}} + \frac{e}{m} \underline{E}_o \cdot \frac{\partial f}{\partial \underline{v}} - \frac{e^2}{m^2} \frac{\partial}{\partial v_j} \left\{ \frac{1}{\tau} \int_{t-\tau}^{t} dt' \frac{\partial}{\partial x_j} E_i (\underline{x}_o + \underline{v}_o t', t') . \right.$$

$$\int_{t-\tau}^{t'} dt'' \int_{t-\tau}^{t''} E_j(\underline{x}_o + \underline{v}_o t''', t''') f - \frac{e^2}{m^2} \frac{\partial}{\partial v_i} \frac{\partial}{\partial v_j} \left\{ \frac{1}{\tau} \int_{t-\tau}^{t} dt' E_i (\underline{x}_o + \underline{v}_o t', t') \right.$$

$$\left. \int_{t-\tau}^{t'} dt'' \, E_j (\underline{x}_o + \underline{v}_o \, t'', t'') \, f(v) \right\} \qquad (3.2.7)$$

an equation of the Fokker Planck form with the coefficient given in (3.2.2) and (3.2.3). These expanded forms may be written exactly as those in (3.2.3), i.e.

$$\left\langle \underline{E} \int_{t-\tau}^{t} \underline{E} \, dt' \right\rangle = \int d^3x_2 \ldots d^3x_N, \, d^3v_2 \ldots d^3n_N \sum_j \sum_k \frac{\partial \Phi}{\partial \underline{x}_1} (x_1, x_j) \int_{t-\tau}^{t} dt' \, \frac{\partial \Phi}{\partial \underline{x}_1}(x_1, x_k)$$

$$\psi(X_2 \ldots X_N, \, V_2 \ldots V_N, x_1)$$

i.e. as mean values $\langle E_i \, E_j \rangle$

Now, the correlation functions:-

$$\left\langle \Phi(t) \int_{t-\tau}^{t} \Phi(t') dt' \right\rangle = \left\langle \int_{0}^{\tau} \Phi(t) \, \Phi(t-s) ds \right\rangle \qquad (3.2.8)$$

may be simplified somewhat by assuming a property of $\langle \underline{E}(t) \, \underline{E}(t-s) \rangle$ which we will be able to demonstrate, namely, that the fields are strongly co-related for small times, but that the product $\underline{E} \cdot \underline{E}$ becomes small and fluctuates about zero for long times; indeed, in times $\sim 1/\omega_o$ the correlation is already small. This suggests that if the upper limit of integration is taken as $\tau \gg \omega_o^{-1}$, which implies $\tau_c \gg \omega_o^{-1}$, a condition which is usually satisfied: then the upper limit in (3.2.8) may be extended to infinity and the required quantities become:-

$$\left\langle \int_{0}^{\infty} \Phi(t) \, \Phi(t-s) ds \right\rangle$$

and if the field is derivable from a potential :-

$$\left\langle E_i\ E_j \right\rangle = \left\langle \frac{\partial\Phi}{\partial x_i}\ \frac{\partial\Phi}{\partial x_j} \right\rangle.$$

The correlation function may be expressed in terms of the spectrum of Φ for if:-

$$\Phi\ (\underline{x},t) = \int d^3k\ d\omega\ e^{i\underline{k}\ \cdot\ \underline{x}}\ e^{i\omega t}\ \Phi(\underline{k},\omega)$$

then:-

$$\left\langle \int_0^\infty \Phi\ (\underline{x},t)\ \Phi\ (\underline{x}-\underline{v}s,t-s)ds \right\rangle = R\left\langle \int d^3k\ d\ \omega \int d^3k'\ d\omega' \int_0^\infty ds \right.$$

$$\left. e^{i\underline{k}\ \cdot\ \underline{x}}\ e^{i\omega t}\ e^{i\ \underline{k}'\cdot\ (\underline{x}-\underline{v}s)}\ e^{i\omega'(t-s)}\ \Phi(\underline{k},\omega)\ \Phi(\underline{k}',\omega') \right\rangle$$

$$= R\left\langle \int ds \int d^3k\ d\omega\ e^{-i(\underline{k}\ \cdot\ \underline{v}\ +\omega)s} \int d^3k'\ d\omega'\ e^{i(\underline{k}+\underline{k}')\ \cdot\ \underline{x}}\ e^{i(\omega\ +\ \omega')t} \right.$$

$$\left. \Phi(\underline{k},\omega)\ \Phi(\underline{k}',\omega') \right\rangle$$

The inner integral however, is the energy spectrum.

$$\left\langle \Phi(\underline{k},\omega)\ \Phi(\underline{k},\ \omega) \right\rangle$$

hence, the correlation function:-

$$\left\langle \Phi(\underline{x},t)\ \Phi(\underline{x}-\underline{v}s,t-s) \right\rangle = R\int d^3k\ d\omega\ \Phi(\underline{k},\omega)\ \Phi(\underline{k},\omega)\ e^{-i(\underline{k}.\underline{v}-\omega)s}$$

$$= \frac{1}{(2\pi)^4}\left\langle \int d^3k\ d\omega\ \Phi(\underline{k},\omega)\ \Phi(\underline{k},\omega)\ \cos\ (\underline{k}\ \cdot\ \underline{v}+\ \omega)s \right\rangle$$

is the cosine transform of the energy spectrum, (the Wiener - Kinchin theorem). It follows that:-

$$\left\langle \int_0^\infty ds\ \Phi(\underline{x},t)\ \Phi(\underline{x}-\underline{v}s,t-s) \right\rangle = \left\langle \int d^3k\ d\omega\ \Phi(k,\omega)\ \Phi\ (\underline{k},\omega)\delta(\omega+\underline{k}\ \cdot\ \underline{v}) \right\rangle \tag{3.2.9}$$

4. Calculation of the Spectrum

To calculate the spectrum we can again assume that the electric field within the plasma is weak: and the interactions are small. At the same time, the effect of the field on the distribution function must be retained in any calculation of the field, which otherwise diverges. Our first object then must be to calculate the response of the plasma to a field in that approximation in which the particle interaction is neglected. This, however, requires a solution to the Vlasov equation:-

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{x}} + \frac{e}{m} \underline{E} \cdot \frac{\partial f}{\partial \underline{v}} = 0 \qquad (3.3.1)$$

and since the fields are assumed small, we may use a perturbation solution of this about some distribution f_o, assumed known,- whereupon, on Fourier transforming:-

$$f = f_o + f', \quad f'(\omega,\underline{k}) = -\frac{e}{m} \frac{\underline{E}(\omega,\underline{k}) \cdot \frac{\partial}{\partial \underline{v}} f_o}{i(\omega + \underline{k} \cdot \underline{v})} \qquad (3.3.2)$$

or if $\underline{E}$ is derivable from a potential:-

$$\underline{E} = -i\underline{k}\Phi$$

$$f' = \frac{e}{m} \frac{\underline{k}, \frac{\partial f_o}{\partial \underline{v}}}{(\omega + \underline{k}, \underline{v})} \Phi$$

The charge induced by a potential Φ, then becomes if f_o^-, f_o^+

are the unperturbed distributions of electrons and ions:-

$$q_{ind}(\omega,\underline{k}) = \int d^3v \frac{1}{(\omega + \underline{k}, \underline{v})} \underline{k} \cdot \frac{\partial}{\partial \underline{v}} \left[\frac{e^2}{m_-} f_o^- + \frac{e_+^2}{m_+} f_o^+ \right] \cdot \Phi(\omega,\underline{k})$$

$$= -k^2 K(\omega,\underline{k}) \Phi(\omega,\underline{k}) \qquad (3.3.3)$$

If now, a test charge e_1 is introduced into a plasma, the charge produced is:-

$$q^*(\omega,k) = e_1 \int e^{-i(\omega t + \underline{k} \cdot \underline{x})} \delta(\underline{x} - \underline{v} \cdot t) d^3x\, dt$$

$$= 2\pi e_1 \delta(\omega + \underline{k} \cdot \underline{v}) \qquad (3.3.4)$$

and the induced potential may be written, from:-

$$k^2\Phi = 4\pi q = 4\pi (q_{ind} + q^*) = -4\pi k^2 K \Phi + 8\pi^2 e_1 \delta(\omega + \underline{k} \cdot \underline{v}) \qquad (3.3.5)$$

whence:-

$$\Phi = 8\pi^2 e_1 \frac{\delta(\omega + \underline{k}, \underline{v})}{k^2(1 + 4\pi K)} = 8\pi^2 \frac{e_1 \delta(\omega + \underline{k}, \underline{v})}{k^2 \varepsilon(\omega,\underline{k})}$$

introducing the dielectric coefficient:-

$$\varepsilon(\underline{k},\omega) = 1 + 4\pi K(\underline{k},\omega)$$

$$= 1 + \frac{\omega_o^{1}}{k^2} \int \frac{dV_{11}}{(V_{11} - V_p)} \frac{\partial}{\partial v_{11}} \left\{ g_- + \frac{\omega_+^2}{\omega_o^2} g_+ \right\} \qquad (3.3.6)$$

where

$$-v_p = \omega/k,\ \omega_o{}^2 = 4\pi m_- e^2/m_- ,\ \omega_+^2 = 4\pi m_+ e_+^2/m_+$$

$$g_\pm = \frac{1}{n}\int d^2v_\perp\ f_o^\pm(\underline{v}) ,\ \underline{v}_\perp = \underline{v} - v_{\parallel}\,\hat{\underline{k}},\ v_{\parallel} = \hat{\underline{k}}.\ \underline{v} \qquad (3.3.7)$$

To handle the singular integrals in (3.3.6) requires some care; however several different arguments, e.g. solving an initial value problem, considering a perturbation which is adiabatically switched on, or considering a model of residual collision process as in (2.4.1), all lead to the conclusion that, with $P\int$ = Cauchy principle value,

$$\int dt\ \frac{g(t)}{t-x} = P\int dt\ \frac{g(t)}{t-x} + i\pi\ g(x) = \int_L dt\ \frac{g(t)}{t-x} \qquad (3.3.8)$$

hence ε is complex. In normal systems an imaginary part to ε represents the loss due to collisions; here the loss process is Landau damping.

The field at a charge e_i introduced by its own presence is:-

$$E(\underline{v}t,t) = R_e\int\frac{d^3k\ d\omega}{(2\pi)^4}\ 8\pi^2\ e_1\ i\underline{k}\ e^{\ i(\underline{k}\cdot\underline{v}t + \omega t)}\ \frac{\delta(\omega+\underline{k}\cdot\underline{v})}{k^2\varepsilon(\omega,\underline{k})} \qquad (3.3.9)$$

$$= \frac{1}{2\pi^2}\ e_1\int\underline{k}\ \text{Im}\ \frac{\{k^2\varepsilon(\omega,\underline{k})\}}{|k^2\varepsilon(\omega,\underline{k})|^2}\ \delta(\omega+\underline{k}\cdot\underline{v})\ d^3k\ d\omega \qquad (3.3.10)$$

The fluctuating field in the plasma may be calculated by noting that each charge in the plasma itself produces a polarization, like that of a test charge; hence, if particles are distributed at points $X_i(t)$

$$q(\underline{x},t) = \Sigma\ e_i\ \delta(\underline{x} - \underline{X}_i(t))$$

$$q(\omega,k) = 2\pi\sum_i e_i\int dt\ e^{-i\ \underline{k}.X_i\ (t)\ -\ i\omega t}$$

$$\Phi(\omega,k) = 8\pi^2\sum_i e_i\int dt\ \frac{e^{-i[\underline{k}.X_i(t)\ +\omega t\ ]}}{k^2\varepsilon(\omega,\underline{k})} \qquad (3.3.11)$$

The motion of each particle is approximately constant for times of order τ_c provided the mean field is small, i.e.

$$n^{1/3}\ e^2 \ll kT,$$

and provided no close collision occurs; hence if:-

$$t' = t + s,\ \underline{x}(t') = \underline{x}(t) + \underline{v}s$$

and

$$\Phi(\omega,\underline{k})\Phi(\omega',\underline{k}') = \frac{(8\pi^2)^2}{k^2\varepsilon(\omega,\underline{k})} k'^2\varepsilon(\omega'\underline{k}')\sum_{i,j}\int dt\int ds\ e^{i(\underline{k}.\underline{X}_i(t)+\omega t)} e^{i[\underline{k}'(\underline{X}_j(t)+\underline{V}s]}$$

$$\times\ e^{i\,\omega'(t+s)}$$

Further, because the correlation between any pair of particles is small the random phase approximation may be used to reduce the double sum to a single one:-

$$< \sum_i \exp\,[i(\underline{k}' + \underline{k})\ .\ \underline{x}_i(t)] >$$

and the mean value of this (only quantity involving $\underline{x}_i$)

$$= \frac{1}{V}\int d^3 x_i\ e^{i(\underline{k}' + \underline{k})\ .\ x_i} = (2\pi)^3\ \delta(\underline{k} + \underline{k}')$$

∴ further:-

$$\left\langle \int e^{-i(\omega + \omega')t}\ dt \right\rangle = 2\ \pi\ \delta\ (\omega + \omega')$$

$$\int ds\ e^{i(\omega' + \underline{k}\cdot\underline{v})s} = 2\pi\ \delta(\omega + \underline{k}\cdot\underline{v})$$

while

$$\sum_i \psi\ (\underline{v}_i) = \int d^3v\ f(\underline{v})\ \psi(\underline{v})$$

Therefore:-

$$\left\langle\Phi(\underline{k},\omega)\ \Phi(\underline{k}',\omega')\right\rangle = (2\pi)^5(8\pi)^2\ \frac{\delta(\underline{k}+\underline{k}')\ \delta(\omega+\omega')}{|k^2\varepsilon(\omega,\underline{k})|^2}\ e^2\int d^3v\ \delta(\omega+\underline{k}\cdot\underline{v})\ f(v)$$

The power spectrum is then:-

$$\frac{1}{(2\pi)^4}\int d^3k'd\omega'\ \left\langle\Phi(k,\omega)\,\Phi(\underline{k}'\omega')\right\rangle = \left|\frac{32\pi^3}{k^2\varepsilon(\underline{k},\omega)}\right|^2\int d^3v\ \delta(\omega+\underline{k}\cdot\underline{v})\ \sum_{\pm} e_\pm^2\ \ f_\pm\ (\underline{v}) \qquad (3.3.12)$$

and the diffusion coefficient; using (3.2.7) and (3.2.9) becomes:-

$$D_{ij} = 32\pi^3\ \frac{e^2}{m^2}\int d^3v'\int\frac{d^3kd\omega k_i\ k_j\delta(\omega+\underline{k}.\underline{v})}{(2\pi)^4|k^2\varepsilon(\underline{k}\cdot\omega)|^2}\ \delta(\omega+\underline{k}\cdot\underline{v}')\ \Sigma\ e_\pm^2\ f_\pm\ (\underline{v}') \qquad (3.3.13)$$

The friction involves a term of the form:-

$$-\frac{e^2}{m^2}\left\langle\frac{\partial}{\partial x_j}\ E_i\int dt'\int^{t'} dt''\ E_j(t'')\right\rangle$$

in using the property that the quantity under the first integral sign ε = function of $t - s$, we may write this as:-

$$- \frac{e^2}{m^2} \left\langle \int ds E_j(x,t)\, s \frac{\partial}{\partial x_j} E_i(\underline{x}-\underline{v}s, t-s) \right\rangle - \frac{e^2}{m^2} \frac{\partial}{\partial v_j} \left\langle \int ds E_j(\underline{x},t) E_i(\underline{x}-\underline{v}s, t-s) \right\rangle = \frac{\partial}{\partial v_j} D_{ij}.$$

5. The Dominant Approximation

The integrals required to evaluate the D_{ij} take the form:-

$$\int d^3g \int d^3k \; d\omega \; \delta\underline{k} \quad \frac{\delta(\omega + \underline{k}\;\underline{v})}{\left|k^2 \varepsilon\;(\underline{k}\;\omega)\right|^2} \; \delta\;(\underline{k}\;\underline{g})\;\underline{k} \; \frac{\partial}{\partial \underline{v}} \; f(\underline{v} + \underline{g}) \qquad \text{from(3.3.10)}$$

$$\int d^3g \int d^3k \; d\omega \; \delta\underline{k} \quad \underline{k} \; \frac{\delta(\omega + \underline{k}\;\underline{v})}{\left|k^2 \varepsilon\;(\underline{k},\omega\;\right|^2} \; \delta(k\;g)\; f(\underline{v} + \underline{g}) \qquad \text{from(3.3.13)}$$

where $\underline{g} = \underline{v}' - \underline{v}$.

The integral over $d\omega$ is trivial, and on splitting $d^3k = k^2 dk d\Omega$ these become:

$$\int d^3g \int d\Omega \; \delta\;(\hat{\underline{k}},g)\; \hat{\underline{k}}\;\hat{\underline{k}}\; dk \; \frac{k^3}{\left|k^2 \varepsilon(\underline{k}\;,\;\underline{k}\cdot\underline{v})\right|^2} \qquad (3.4.1)$$

Now

$$k^2\varepsilon(k,\omega) = k^2 + \omega_o^2 \int_L \frac{dv_{11}}{(v_{11} - v_\rho)} \cdot \frac{\partial}{\partial v_{11}} \{g_- + \frac{\omega^2}{\omega_o^2} g_+\}$$

$$= k^2 + k_D^2 \left[X\left(\frac{\omega}{kv_\theta}\right) + iY\left(\frac{\omega}{kv_\theta}\right)\right] \qquad (3.4.2)$$

$$\int \frac{dk\;k^3}{(k^2 + k_o^2 X)^2 + k_o^4 Y^2} = \log\left(\frac{k_{max}}{k_o}\right) + \tfrac{1}{4} \log\left[\frac{[1 + (\frac{k_o^2}{k_m^2})X]^2 + Y^2}{X^2 + Y^2}\right] - \frac{X}{4}\{\tan^{-1}\left|\frac{X}{Y}\right| - \frac{\pi}{2}\} \qquad (3.4.3)$$

In this integral k_{max} is an upper limit which is forced upon us by the existence of close encounters, for which the field strength becomes large, and the linearisation in $\underline{E}$ underlying this treatment breaks down. Since:

$$\left(\frac{n^{\frac{1}{3}}}{k_D}\right)^2 = n^{2/3} \frac{m v_\theta^2}{4\pi n e^2} = \frac{1}{2\pi} \frac{\frac{1}{2} m v_\theta^2}{n^{1/3} e^2} = \frac{1}{2\pi} \frac{T}{V}$$

hence if $k_{max} \geqslant n^{\frac{1}{3}}$ the large value of the ration T/V ensures that $(k_{max/k})$ is large. In the dominant approximation only this term is retained; and the integrals became:

$$\tfrac{1}{4} \int d^3g \int d\;\Omega\;\underline{k}\;\delta(\hat{\underline{k}}\cdot g)\hat{\underline{k}} \cdot \frac{\partial f}{\partial \underline{v}}\;(\underline{v} + \underline{g})\;\tfrac{1}{2}\;\log\Lambda \qquad (3.4.4)$$

$$\tfrac{1}{4} \int d^3g \int d\;\Omega\;\hat{\underline{k}}\;\hat{\underline{k}}\;\delta\;(\hat{\underline{k}}\cdot\underline{g})\;f(\underline{v} + \underline{g})\;\tfrac{1}{2}\;\log\Lambda \qquad (3.4.5)$$

Note that (3.4.5) $= \frac{\partial}{\partial \underline{v}} \cdot$ (3.4.4), $\therefore \frac{e_1}{m_1} \underline{E} = - \frac{\partial}{\partial \underline{v}} \cdot \underline{D}$ while in (3.4.4) the angular integration merely selects those parts of $\hat{\underline{k}}\;\hat{\underline{k}}$ orthogonal to $\underline{g}$, i.e.

$$\int d\;\Omega\;\hat{\underline{k}}\;\hat{\underline{k}}\;\delta(\hat{\underline{k}}.\underline{g}) = \frac{\pi}{g}\;(1 - \hat{\underline{g}}\;\hat{\underline{g}}) = \underline{w} \qquad \text{c.f. (3.1.6)}$$

hence

$$D_{ij} = 2\ \pi(\frac{e^2}{m})^2\ \log \Lambda \int d^3v' w_{ij}\ f\ (\underline{v}')$$

Now, using the relations displayed between the integrals, the F.P.E. may be written:

$$-\frac{\partial D_{ij}}{\partial v_i}\frac{\partial f}{\partial v_j} - \frac{\partial}{\partial v_i}(\frac{\partial D_{ij}}{\partial v_j})\ f + \frac{\partial^2}{\partial v_i\ \partial v_j}(D_{ij}\ f)$$

$$= -2\frac{\partial}{\partial v_i} D_{ij}\frac{\partial f}{\partial v_j} - \frac{\partial^2 D_{ij}}{\partial v_i\ \partial v_j} f + \frac{\partial^2 D_{ij}}{\partial v_i\ \partial v_j} f + \frac{\partial D_{ij}}{\partial v_i}\frac{\partial f}{\partial v_j} + D_{ij}\frac{\partial^2 f}{\partial v_i\ \partial v_j}$$

$$= D_{ij}\frac{\partial f}{\partial v_i\ \partial v_j} - \frac{\partial D_{ij}}{\partial v_i}\frac{\partial f}{\partial v_i}$$

$$= 2\ \pi(\frac{e^2}{m})^2\ \log\Lambda\ \frac{\partial}{\partial v_j}\int d^3v\ w_{ij}\ [\frac{\partial f(\underline{v}')}{\partial v_i'}\ f(\underline{v}) - f(\underline{v}')\ \frac{\partial f(v)}{\partial v_i}] \qquad (3.4.6)$$

This however is Landau's form (3.1.8), which thus represents the dominant approximation to a kinetic equation which includes both the static correlation effects which produce screening with the dynamic effects that represent the production of plasma oscillations. It is of particular interest to observe that our approach to this has required that f_o be substantially uniform over time $> \omega_o^{-1}$; thus, if we wish to discuss the attenuation of radio frequency oscillations propagating through the plasma, the usual Boltzmann treatment is inadequate; instead correlation effects must be determined for the distribution function perturbed by the incident r-f field.

PART IV

The Kinetic Equation in a Magnetic Field

1. Fokker Planck Equation in a Magnetic Field

As in the absence of a magnetic field, it is possible to use the Liouville equation to describe the dynamics of a complete system, and again one may integrate this to obtain the Boltzmann equation

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{x}} + \frac{e}{m}\left[\underline{E} + \frac{\underline{v}}{c} \times \underline{B}\right] + \frac{ne}{m}\frac{\partial}{\partial \underline{v}} \cdot \int \underline{F}_{12}\ f(\underline{v}_1, \underline{x}_1, \underline{v}_2, \underline{x}_2) d^3x_2\ d^3v_2 \qquad (4.1.1)$$

where now the interaction forces $\underline{F}_{12}$ include, in general, terms of the form:

$$\frac{\partial}{\partial \underline{r}_1}\ \frac{e_2}{|\underline{r}_1 - \underline{r}_2|} \text{ and } \frac{\underline{v}_1}{c} \times \frac{\partial}{\partial \underline{r}_1} \times \frac{e_2\ \underline{v}_2/c}{|\underline{r}_1 - \underline{r}_2|}$$

the first representing electric and the second magnetic interactions. Note that the ratio of the 2nd to the 1st is v^2/c^2, hence the 2nd is negligible except for relativistic systems, for which the Coulomb representation of the field is inadequate; thus retardation effects becoming significant wherever the (local) magnetic interaction matters. For temperatures $T > 500$ keV the entire calculation should be made relativistic - a complication I propose to avoid.

If the magnetic interaction is omitted 4.1.1 may be coarse grained as before, and the last term written:

$$\frac{\partial}{\partial \underline{v}}\frac{1}{\tau}\iint_{t-\tau}^{t} dt'\ \frac{e}{m}\underline{E}\ (\underline{x}', t')\ \delta[\underline{x}' - X(t')]\ \delta[\underline{v}' - \underline{V}(t')]\ d^3x' d^3v' f(\underline{v}) \qquad (4.1.2)$$

and again the δ functions may be expanded to first order in $\underline{E}$, (4.1.2) becoming:-

$$\frac{e}{m}\frac{\partial}{\partial \underline{v}} \cdot \frac{1}{\tau} \int dt' \int d^3x' d^3v' \cdot \delta[\underline{x}' - \underline{X}_0(t')]\ \delta[\underline{v}' - \underline{V}_0(t)] \cdot \{\underline{E}(x', t')$$
$$+ \frac{\partial}{\partial \underline{x}} \cdot \Delta\ \underline{x}\ \underline{E}(x', t') + \frac{\partial}{\partial \underline{v}} \cdot \Delta\ \underline{v}\ \underline{E}(x' t')\} \qquad (4.1.3)$$

The complications introduced by the magnetic field are two-fold. In the first place $\underline{X}_0(t)$, $\underline{V}_0(t)$ are no longer linear, but represent helical motions; i.e., if $V(0) = (V_{\parallel}, V_{\perp} \cos\phi, V_{\perp} \sin\phi)$

$$V(t) = [V_{\parallel}, V_{\perp} \cos(\Omega t + \phi), V_{\perp} \sin(\Omega t + \phi)] \qquad (4.1.4)$$

and if $X(O) = (z_o, x_o, y_o)$

$$X_o(t) = [z_o + V_{\parallel} t,\ x_o + \frac{V_\perp}{\Omega}(\sin(\Omega t + \phi) - \sin\phi),\ y_o - \frac{V_\perp}{\Omega}(\cos(\Omega t + \phi) - \cos\phi)] \quad (4.1.5)$$

Furthermore, the quantities $\Delta\, \underline{v}$, $\Delta\, \underline{x}$ must be determine by:

$$\dot{\underline{v}} + \Omega\, \hat{\underline{b}} \times \underline{v} = \frac{e}{m}\, \underline{E}(x,t)$$

i.e.

$$\Delta v_{\parallel} = \frac{e}{m} \int_o^t E_{\parallel}\ dt'$$

$$\Delta\, \underline{v}_\perp = \mathrm{Re}\, \exp i\Omega t \cdot \int_o^t \frac{e}{m}(\underline{E}_\perp + i\, \underline{b} \times \underline{E}) \exp - i\Omega t'\ dt'$$

$$= \frac{e}{m} \int_o^t \underline{E}_\perp \cos\Omega(t - t') + \underline{b} \times \underline{E} \sin\Omega(t - t') \cdot dt' \qquad (4.1.6)$$

and

$$\Delta x_{\parallel} = \frac{e}{m} \int_o^t dt' \int_o^{t'} dt'' E_{\parallel}$$

$$\Delta\, \underline{X}_\perp = \frac{c}{B} \int_o^t dt'\ \underline{E}_\perp(t') \sin\Omega(t - t') + \underline{b} \times \underline{E}(t')[\cos\Omega(t - t') - 1] \quad (4.1.7)$$

Now, if the correlation time $\tau \ll \Omega^{-1}$, the term introduced by the field $\sim e^{i\Omega t} \approx 1$, and these expressions reduce to those in the absence of a magnetic field. Since the correlation time $\sim \tau \sim \omega_o^{-1}$ this will be true whenever $\omega_p \gg \Omega$ i.e. $\lambda_D \ll r_L$, (as might be expected!) - i.e. whenever

$$\frac{4\pi n e^2}{m} \quad \frac{e^2 B^2}{m^2 c^2} = \frac{4\pi n m c^2}{B^2} \gg 1.$$

For a field of 5 k.g. this holds for electron densities > 5.10^{10} and ion density $n > 2.10^8$, thus for most problems in which kinetic theory results are useful, the Landau form of the F.P.E. forms an adequate description. For very diffuse plasma, however, modifications are required. (4.1.3) may be written:-

$$\frac{\partial}{\partial \underline{v}} \cdot \left\langle \frac{e}{m}\, \underline{E}\ f \right\rangle + \left(\frac{e}{m}\right)^2 \frac{\partial}{\partial \underline{v}} \cdot \left\langle \frac{\partial}{\partial x_{\parallel}}\, \underline{E} \int dt' \int_o^{t'} dt''\ E_{\parallel}\ (t'')\ f \right\rangle$$

$$+ \frac{e}{m} \frac{\partial}{\partial \underline{v}} \cdot \left\langle \frac{\partial}{\partial x_\perp} \cdot \underline{E}\, \frac{c}{B} \int dt'\ \underline{E}(t') \sin\Omega(t - t') + \underline{b} \times \underline{E}(t')[\cos\Omega(t - t') - 1] f \right\rangle$$

$$+ \left(\frac{e}{m}\right)^2 \frac{\partial}{\partial \underline{v}} \frac{\partial}{\partial v_{\parallel}} \left\langle \underline{E} \int E_{\parallel} dt \right\rangle f + \left(\frac{e}{m}\right)^2 \frac{\partial}{\partial v} \frac{\partial}{\partial_\perp} \left\langle \underline{E} \int \underline{E}_\perp \cos\Omega(t - t') + \underline{b} \times \underline{E}\ \sin\Omega(t' - t) f \right\rangle \quad (4.1.8)$$

now the term

$$\left\langle \frac{\partial}{\partial x_{\parallel}}\, \underline{E} \int dt' \int^{t'} dt''\ E_{\parallel}\ (t'') \right\rangle f$$

may be handled by using a Fourier representation of $\underline{E}$, i.e:

$$\frac{\partial}{\partial x_{\parallel}} e^{i\underline{k} \cdot \underline{x}}\ \underline{E}(\underline{k}, v) \int dt' \int dt''\ E_{\parallel}\ e^{i\underline{k}' \cdot \underline{x}}\ e^{ik_{\parallel} v_{\parallel}(t - t')} e^{i\underline{k}_\perp \cdot \Delta x_\perp}$$

On integrating by parts

$$\frac{\partial}{\partial \underline{v}} \cdot \frac{\partial}{\partial x_{\parallel}} \underline{E} \int dt' \int dt'' \; E_{\parallel} = \frac{\partial}{\partial \underline{v}} \cdot \frac{\partial}{\partial x_{\parallel}} \underline{E} \int dt' (t - t') E_{\parallel} = \frac{\partial}{\partial \underline{v}} \frac{\partial}{\partial v_{\parallel}} \underline{E} \int E_{\parallel} \, dt$$

and (4.1.8) becomes

$$\frac{e}{m} \frac{\partial}{\partial \underline{v}} \cdot \langle \underline{E} \rangle f + \frac{e}{m} \frac{c}{B} \frac{\partial}{\partial \underline{v}} \cdot \left\langle \frac{\partial}{\partial x_{\perp}} \cdot \underline{E} \int dt' \underline{E}_{\perp} \sin\Omega(t - t') + \underline{b} \times \underline{E} (\cos\Omega(t - t') - 1) \right\rangle f$$

$$+ 2 \left(\frac{e}{m}\right)^2 \frac{\partial}{\partial \underline{v}} \frac{\partial}{\partial v_{\parallel}} \cdot \left\langle \underline{E} \int dt' E_{\parallel} \right\rangle f$$

$$+ \left(\frac{e}{m}\right)^2 \frac{\partial}{\partial \underline{v}} \cdot \frac{\partial}{\partial \underline{v}_{\perp}} \cdot \left\langle \underline{E} \int dt' \underline{E}_{\perp} \cos\Omega(t - t') + \underline{b} \times \underline{E} \sin\Omega(t - t') \right\rangle f \qquad (4.1.9)$$

As before E may be expressed in terms of a potential Φ, and Φ may be Fourier analysed, whereupon the required correlation functions become:

$$\left\langle \underline{E} \int \underline{E}(t-s, x(t-s)) ds \right\rangle = \int \frac{d^3k \, d\omega}{(2\pi)^4} \underline{k} \, \underline{k} \langle \Phi(\underline{k},\omega) \Phi \; (\underline{k},\omega) \rangle \cdot \int ds \; e^{-i\underline{k} \cdot \underline{\Delta} x(s)} \cdot e^{-i\omega s}$$

$$= \int \frac{d^3k \, d\omega}{(2\pi)^4} \underline{k} \; \underline{k} \; P(\underline{k},\omega) \; R(\underline{k},\omega)$$

and

$$\left\langle \underline{E} \, e^{i\Omega t} \; \underline{E}(t-s) \; e^{-i\Omega(t-s)} \right\rangle = \int \frac{d^3k \, d\omega}{(2\pi)^4} \underline{k} \, \underline{k} \; P(\underline{k},\omega) \; R(\underline{k},\omega + \Omega)$$

The term

$$\frac{\partial}{\partial \underline{v}} \cdot \frac{\partial}{\partial \underline{x}_{\perp}} : \left\langle \underline{E} \cdot \int [\underline{E}_{\perp} \times \underline{b} (1 - \cos\Omega s) - \underline{E}_{\perp} \sin\Omega s \,] \, ds \right.$$

may be written

$$- \frac{\partial}{\partial \underline{v}} \left\langle \frac{\partial}{\partial \underline{x}_{\perp}} : \int \frac{d^3k \, d\omega}{(2\pi)^4} \frac{d^3k' d\omega'}{(2\pi)^4} \Phi(\underline{k}) \Phi(\underline{k}') \; e^{i(\underline{k} + \underline{k}') \cdot \underline{x}} \; e^{i(\omega + \omega')t} \right.$$

$$\left. \int ds \; \underline{k} \; \underline{k}' \times \underline{b} \; \exp i(\underline{k} \cdot \Delta x + \omega s) \; (1 - \cos\Omega s) - \underline{k} \; \underline{k}_{\perp}' \; \exp i(\underline{k} \cdot \Delta \underline{x} + \omega s) \; \sin\Omega s \right\rangle$$

since

$$\left\langle e^{i(\underline{k} + \underline{k}') \cdot \underline{x}} \right\rangle = (2\pi)^3 \delta(\underline{k} + \underline{k} \;)$$

this vanishes and (4.1.9) becomes:

$$\frac{e}{m} \frac{\partial}{\partial \underline{v}} \cdot \underline{E} \; f + 2 \left(\frac{e}{m}\right)^2 \frac{\partial}{\partial \underline{v}} \frac{\partial}{\partial \underline{v}_{\parallel}} \cdot \int \frac{d^3k \, d\omega}{(2\pi)^4} P(\underline{k},\omega) \; R(\underline{k},\omega) \; \underline{k} \; k_{\parallel} \; f$$

$$+ \left(\frac{e}{m}\right)^2 \frac{\partial}{\partial \underline{v}} \frac{\partial}{\partial \underline{v}_{\perp}} : \int \frac{d^3k \, d\omega}{(2\pi)^4} P(\underline{k},\omega) [\tfrac{1}{2}(\underline{k} + i \; \underline{b} \times \underline{k}) \underline{k} \; R(\underline{k}, \omega + \Omega)$$

$$+ \tfrac{1}{2}(\underline{k} + i \; \underline{b} \times \underline{k}) \underline{k} \; R(\underline{k}, \omega - \Omega) \,] \; f \qquad (4.1.10)$$

The problem now is reduced to that of evaluating R and P and of course, carrying out the required integrations. Now:-

$$R(\omega,\underline{k}) = \int ds \exp i(\underline{k},\Delta\underline{x} + \omega s)$$

$$= \int ds \exp i\{(k_{\parallel} v_{\parallel} + \omega) s + \frac{k_x v_\perp}{\Omega} [\sin(\Omega s + \phi) - \sin\phi]$$

$$+ \frac{k_y v_\perp}{\Omega} [\cos(\Omega s + \phi) - \cos\phi]\} \qquad (4.1.11)$$

Write $k_x = k_\perp \cos\psi, \qquad k_y = k_\perp \sin\psi,$ then

$$R = \int ds \exp\{i(k_{\parallel} v_{\parallel} + \omega) s + i \frac{k_\perp v_\perp}{\Omega} [\sin(\Omega s + \phi - \psi) - \sin(\phi - \psi)]\}$$

$$= \sum_n \exp - i \frac{k_\perp v_\perp}{\Omega} \sin(\phi - \psi) \cdot \exp in(\phi - \psi) \cdot \int ds\ Jn(\frac{k_\perp v_\perp}{\Omega}) e^{i(k_{\parallel} v_{\parallel} + \omega + n\Omega)s}$$

$$= \sum_{n,m} Jn(\frac{k_\perp v_\perp}{\Omega}) J_m(\frac{k_\perp v_\perp}{\Omega} \cdot e^{i(n-m)(\phi-\psi)} \cdot [2\pi\delta(\omega + n\Omega + k_{\parallel} v_{\parallel}) - \frac{i}{k_{\parallel} v_{\parallel} + \omega + n\Omega}] \qquad (4.1.12)$$

We will evaluate the power spectrum exactly as in the non-magnetic case, except that now our test particles must be allowed to move along helical orbits instead of straight lines; and the dielectric coefficient becomes complicated by the presence of the magnetic field.

2. The Dielectric Coefficient of a Magnetized Plasma

We will require the field induced in a plasma by the presence of charge, q, and as a preliminary, we may ask what charge is induced in a magnetized plasma by a potential Φ. This can be discovered from the relevant Vlasov equation:

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{x}} + \frac{e}{m} (\frac{\underline{v}}{c} \times \underline{B}) \cdot \frac{\partial f}{\partial \underline{v}} - \frac{e}{m} \underline{\nabla} \Phi \frac{\partial fo}{\partial \underline{v}} = 0$$

This has as its solution:

$$f' = \frac{e}{m} \int dt' \ \Phi \ i \ \underline{k} \cdot \frac{\partial fo}{\partial \underline{v}} \qquad (4.2.1)$$

i.e.

$$f' = i\frac{e}{m} \Phi(k,\omega) e^{i(\underline{k}\cdot\underline{x} + \omega t)} \cdot \int_0^\infty ds \ e^{i\underline{k}\cdot\Delta x(s) + i\omega s} \{k_{\parallel} \frac{\partial f}{\partial v_{\parallel}} + k_x \frac{\partial f}{\partial v_x} + k_y \frac{\partial f}{\partial v_y}\}$$

To carry out the integrals in (4.2.1), we will generally need to know the phase dependence of f, for:

$$\underline{k} \cdot \frac{\partial}{\partial \underline{v}} f = k \frac{\partial f}{\partial v_{\parallel}} + \frac{\underline{k} \cdot \underline{v}_\perp}{v_\perp} \frac{\partial f}{\partial v_\perp} + \underline{b} \cdot \frac{\underline{k} \times \underline{v}_\perp}{v_\perp^2} \frac{\partial f}{\partial \phi}$$

however, it is usually true that the distribution function f may be described as only weakly dependent upon the phase, and in the dielectric coefficient we will ignore this phase dependence; whereupon, using the notation of (4.1.1):-

$$f \doteq i\frac{e}{m}\Phi(\underline{k},\omega)e^{i(\underline{k}\cdot\underline{x}+\omega t}\int_0^\infty \exp i(\underline{k}\cdot\underline{\Delta x}+\omega s)\cdot[k \quad \frac{\partial f}{\partial v_{\parallel}} + K_\perp v_\perp \cos(\Omega t+\phi-\psi)\frac{\partial f}{\partial v_\perp}]$$

$$= i\frac{e}{m}\Phi(\underline{k},\omega)e^{i\underline{k}\cdot x+\omega t}\cdot R(\underline{k},\omega)k_{\parallel}\frac{\partial f}{\partial v_{\parallel}} + \tfrac{1}{2}[e^{i(\phi-\psi)}R(\omega+\Omega,k)+e^{-i(\phi-\psi)}R(\omega-\Omega,k)]k_\perp\frac{\partial f}{\partial v_\perp} \tag{4.2.2}$$

The induced charge may be written:

$$q = i\frac{e^2}{m}\Phi\, e^{i(\underline{k}\cdot\underline{x}+\omega t)}\cdot\int d^3v\, R(\underline{k},\omega)k_{\parallel}\frac{\partial f}{\partial v_{\parallel}} + \tfrac{1}{2}[e^{i(\phi-\psi)}R(\omega+\Omega,\underline{k})$$

$$+ e^{-i(\phi-\psi)}R(\omega-\Omega,\underline{k})]k_\perp\frac{\partial f}{\partial v_\perp}$$

Carrying out the integral over the phases yields

$$\int d\phi\, R(k,\omega)k_{\parallel}\frac{\partial f}{\partial v_{\parallel}} = \sum_n J_n^2[2\pi\delta(\omega+n\Omega+k_{\parallel}v_{\parallel}) - \frac{i}{k_{\parallel}v_{\parallel}+\omega+n\Omega}]$$

$$= \Sigma\, J_n^2\left(\frac{k_\perp v_\perp}{\Omega}\right) 2\pi\delta_+\ (\omega+n\Omega+k_{\parallel}v_{\parallel})$$

$$\int d\phi\, R(\underline{k},\omega+\Omega)e^{i(\phi-\psi)} = 2\pi\Sigma\, J_n\left(\frac{k_\perp v_\perp}{\Omega}\right)J_{n+1}\left(\frac{k_\perp v_\perp}{\Omega}\right)\cdot\delta_+(\omega+(n+1)\Omega+k_{\parallel}v_{\parallel})$$

$$= 2\pi\sum_n J_nJ_{n-1}\ \delta_+\ (\omega+n\Omega+k_{\parallel}v_{\parallel})$$

$$\int d\phi\, R(\underline{k},\omega-\Omega)e^{-i(\phi-\psi)} = 2\pi\sum_n J_n\ J_{n+1}\ \delta_+(\omega+n\Omega+k_{\parallel}v_{\parallel})$$

hence on using

$$J_{n+1}(x) + J_{n-1}(x) = \frac{2n}{x}J_n$$

$$q = \frac{i2\pi e^2}{m}\Phi\, e^{i(\underline{k}\cdot\underline{x}+\omega t)}\cdot\int dv_{\parallel}dv_\perp v_\perp\cdot\sum_n J_n^2\left(\frac{k_\perp v_\perp}{\Omega}\right)\cdot\delta_+(\omega+n\Omega+k_{\parallel}v_{\parallel})\cdot[k_{\parallel}\frac{\partial f}{\partial v_{\parallel}} + \frac{n\Omega}{v_\perp}\frac{\partial f}{\partial v_\perp}]$$

The integral here is singular and must be treated by the Landau procedure, so that the dielectric coefficient becomes complex; although now, each term has distinct zeros, and a large number of resonances appear. The potential due to a test charge $q^*(\underline{k},\omega)$ is then given by:-

$$\Phi = \frac{4\pi\ q^*(\underline{k},\omega)}{k^2 - 2\pi i\omega_o^2\int dv_{\parallel}dv_\perp v_\perp\sum_n J_n^2\left(\frac{k_\perp v_\perp}{e}\right)\delta_+(\omega+n\Omega+k_{\parallel}v_{\parallel})[k_{\parallel}\frac{\partial f}{\partial v_{\parallel}} + \frac{n\Omega}{v_\perp}\frac{\partial f}{\partial v_\perp}]} \tag{4.2.3}$$

3. Field of a Test Particle

We may now calculate the field due to a particle of charge e moving through the plasma with a velocity:

$$\underline{v} = [v_{\parallel},\quad v_\perp\cos(\Omega t+\phi),\ v_\perp\sin(\Omega t+\phi)]$$

and position:

$$\underline{x} = \{z_o + v_{\parallel}t,\ x_o + \frac{v_\perp}{\Omega}[\sin(\Omega t+\phi) - \sin\phi],\ y_o - \frac{v_\perp}{\Omega}[\cos(\Omega t+\phi) - \cos\phi]\}$$

so that:

$$q^* = e\,\delta[x - x_o(t)]$$

$$= e\, e^{-i\underline{k}\cdot\underline{x}_o}\int_{-\infty}^{\infty}\exp -i\{k_{\shortparallel}v_{\shortparallel}t + \omega t + \frac{k_\perp v_\perp}{\Omega}[\sin(\Omega t + \phi - \psi) - \sin(\phi - \psi)]$$

$$= e\, e^{-i\underline{k}\cdot\underline{x}_o}\Sigma\, J_n\left(\frac{k_\perp v_\perp}{\Omega}\right)e^{-in(\phi-\psi)}\,e^{i\frac{k_\perp, v_\perp}{\Omega}\sin(\phi-\psi)}\cdot 2\pi\delta(\omega + n\Omega + k_{\shortparallel}v_{\shortparallel}) \qquad (4.3.1)$$

$\therefore$ the potential induced by a test particle is:

$$\Phi = \frac{8\pi^2 e\, e^{-i\underline{k}\cdot x_o}\sum_{n,m} J_nJ_m\left(\frac{k_\perp v_\perp}{\Omega}\right)e^{i(m-n)(\phi-\psi)}\delta(\omega + n\Omega + k_{\shortparallel}v_{\shortparallel})}{k^2 - 2\pi i\omega_o^2\int dv_{\shortparallel}dv_\perp v_\perp J_n^2\left(\frac{k_\perp v_\perp}{\Omega}\right)\delta + (\omega + n\Omega + k_{\shortparallel}v_{\shortparallel})[k_{\shortparallel}\frac{\partial f}{\partial v_{\shortparallel}}\ (\frac{n\Omega}{v_\perp}\frac{\partial f}{\partial v_\perp}]} \qquad (4.3.2)$$

The self-field on the particle requires a knowledge of $i\underline{k}\Phi(x_o)$

$$\underline{E} = \frac{1}{(2\pi)^4}\int d^3k d\omega\ i\underline{k}\Phi(\underline{x}_o) = \Sigma\int\frac{(-i)mJ_n\left(\frac{k_\perp v_\perp}{\Omega}\right)e^{in(\phi-\psi)}\,e^{i\frac{k_\perp v_\perp}{\Omega}\sin(\phi-\psi)}\ i\underline{k}d^3kd\omega}{[k^2 - \omega_o^2\,K(n\Omega - k_{\shortparallel}v_{\shortparallel}, \underline{k})]} \qquad (4.3.3)$$

If into this we place $\underline{k} = k_{\shortparallel}, k_\perp\cos\psi, k_\perp\sin\psi$, and note $\underline{v} = (v_{\shortparallel}, v_\perp\cos\phi, v_\perp\sin\phi)$ we may integrate over ψ, obtaining:

$$\underline{E}\cdot\hat{\underline{b}} = \mathrm{Re}\ \Sigma\int\frac{i\,k_{\shortparallel}\,J_n^2\left(\frac{k_\perp v_\perp}{\Omega}\right)}{k^2 - \omega_o^2\,K(n\Omega - k_{\shortparallel}v_{\shortparallel}, \underline{k})}$$

$$\underline{E}\cdot\hat{\underline{v}}_\perp = \mathrm{Re}\ \Sigma\int\frac{i\,k_\perp[J_n(J_{n+1} + J_{n-1})]}{k^2 - \omega_o^2\,K}$$

$$\underline{E}\cdot\underline{b}\times\hat{\underline{v}}_\perp = -\mathrm{Re}\ \Sigma\int\frac{1}{2}\,\frac{k_\perp\,J_n(J_{n+1} - J_{n-1})}{k^2 - \omega_o^2\,K}$$

Of particular interest here is the last term, which involves the Hermitian part of $k^2\varepsilon$, and is a force normal to the particle trajectory. For an isotropic distribution function f, $\frac{\partial}{\partial v_\perp} \parallel v_\perp$, and this does not contribute to the F.P.E.

4. The Spectrum

The power spectrum may be evaluated from (4.3.2), thus:-

$$\int \frac{d^3k\, d\omega}{(2\pi)^4} P = \int \frac{d^3k\, d\omega}{(2\pi)^4} \cdot \int \frac{d^3k' d\omega'}{(2\pi)^4} \cdot \Phi(\underline{k},\omega)\ \Phi\ (\underline{k}',\omega')$$

$$= \int \frac{d^3k\, d\omega}{(2\pi)^4} \cdot \int \frac{d^3k' d\omega'}{(2\pi)^4} \cdot (8\pi^2)^2 e^2 \sum_{i,j} e^{i\underline{k}\cdot\underline{x}_i} e^{i\underline{k}'\cdot\underline{x}_j} \sum_{n,m,s,t,} \frac{J_n J_m(\frac{k_\perp v_\perp{}^i}{\Omega}) J_s J_t(\frac{k_\perp{}' v_\perp{}^j}{\Omega})}{k^2\varepsilon(\omega,k)k'^2\varepsilon(\omega'k')},$$

$$\cdot\ \delta(\omega + n\Omega + k_{\parallel}v_{\parallel}^i)\delta(\omega' + t\Omega + k_{\parallel}'v_{\parallel}^j)\ \exp\ i(m-n)(\phi^i - \psi) + i(s-t)(\phi^j - \psi')\phi^i \qquad (4.4.1)$$

and on using the random phase approximation on $\sum_{i,j}$, and carrying out the (trivial) integrations over the k', ω' and ω and recalling that ϕ, and ψ are the phases of $\underline{v}_\perp$ and $\underline{k}_\perp$

$$P(\omega,\underline{k}) = 32\pi^3 e^2 \int d^3v\ \frac{f(\underline{v}) \sum_{n,m,s,t} J_n J_m J_s J_t\ \exp\ i[(m-n)(\phi-\psi)+(s-t)(\phi-\psi)]}{k^2\varepsilon(\underline{k}, -n\Omega + k_{\parallel}v_{\parallel})k^2\varepsilon(-\underline{k}, k_{\parallel}v_{\parallel} - s\Omega)} \qquad (4.4.2)$$

The diffusion coefficients now take the form:

$$\int \frac{d^3k}{(2\pi)^3} \int d^3\omega' \cdot T(\underline{k},\underline{k}) \cdot \Sigma\ \frac{J_q J_r J_s J_t(\frac{k_\perp v_\perp{}'}{\Omega})\ f(\underline{v}')}{k^2\varepsilon[\underline{k}, -(r\Omega + k_{\parallel}v_{\parallel})] \cdot k^2\varepsilon[-\underline{k}, k_{\parallel}v_{\parallel} - s\Omega]}$$

$$\cdot\ \Sigma\ J_n J_m(\frac{k_\perp v_\perp}{\Omega}) 2\pi i\delta + [(n-r)\Omega + k_{\parallel}\ (v_{\parallel} - v_{\parallel}')]$$

$$\cdot\ \exp\ i(n-m)(\phi-\psi) \cdot\ \exp\ i[(p-q)(\phi'-\psi) + (s-t)(\phi'+\psi)] \qquad (4.4.3)$$

where the $T(\underline{k},\underline{k})$ are tensors constructed from $\underline{k}$ and $\underline{b}$.

Some simplification is obtained if we consider the average of the diffusion coefficients over all phase, whereupon they reduce to

$$\frac{\partial}{\partial\underline{v}}\frac{\partial}{\partial\underline{v}} : \underline{D}\ f = \frac{\partial^2}{\partial v_{\parallel}^2}\int k_{\parallel}^2\ P\ \Sigma\ J_n^2\ (\frac{k_\perp v_\perp}{\Omega})\ \delta_+(o)$$

$$+ \frac{\partial}{\partial v_{\parallel}}\ (\frac{\partial}{\partial v_{\parallel}} + \frac{1}{v_\perp})\int k_\perp k_{\parallel}\ P\ \tfrac{1}{2}\ \Sigma\ J_n\ (J_{n+1} + J_{n-1})\delta_+(o)$$

$$+ \tfrac{1}{2}(\frac{\partial}{\partial v_\perp} + \frac{1}{v_\perp})\ (\tfrac{1}{2}\ \frac{\partial}{\partial v_\perp} + \frac{1}{v_\perp})\int k_\perp^2\ P\ \Sigma\ J_n^2[\delta_+(-1) + \delta_+(1)]$$

$$+ \frac{\partial}{\partial v_\perp}\int k_\perp^2\ P\ \Sigma\ J_n\ (J_{n+2} + J_{n-2})[\delta_+(-2) + \delta_+(1)]$$

$$+ \tfrac{1}{2}(\frac{\partial}{\partial v_\perp} + \frac{1}{v_\perp})(\tfrac{1}{2}\ \frac{\partial}{\partial v_\perp} + \frac{1}{v_\perp})\int k_\perp^2 P\ \Sigma\ J_n[J_{n-2} + J_{n+2}][\delta_+(-1) - \delta_+(1)] \qquad (4.4.4)$$

with the following abbreviations:

$$\int = \int\int\int\int dv_{\parallel}' \; dv_{\perp}' \, v_{\perp}' \, dk_{\parallel} \, dk_{\perp} \, k_{\perp}$$

$$P = 16\pi^2 \frac{e^4}{m^2} \cdot \sum_{ps} \frac{J_p^2\left(\frac{k_\perp v_\perp'}{\Omega}\right) J_s^2\left(\frac{k_\perp v_\perp'}{\Omega}\right)}{k^2\varepsilon(\underline{k}, -p\Omega - k_{\parallel}v_{\parallel}')k^2\varepsilon(-\underline{k}, k_{\parallel}v_{\parallel}' - s\Omega)}$$

$$k^2\varepsilon = k^2 - \omega_o^2 \int dv_{\parallel} \, dv_\perp v_\perp \sum_n J_n^2 2\pi i \; \delta_+(\omega + n\Omega + k_{\parallel}v_{\parallel})\left[k_{\parallel} \frac{\partial \hat{f}}{\partial v_{\parallel}} + \frac{n\Omega}{v_\perp} \frac{\partial \hat{f}}{\partial v_\perp}\right]$$

$$\delta_+(t) = \delta_+[k_{\parallel}(v_{\parallel} - v_{\parallel}') + (n - p + t)\Omega]$$

The functions involved in this expression have been explored to some extent by N. Rostoker, who concludes that at worst the screening parameter is altered, a poor reward for such effort. On the other hand, the analysis presented here is incomplete; for in obtaining (4.4.4) we have assumed f_o to be phase independent, whereas, in general one must deal with contributions depending on directions across the B. field. A possible procedure here is to Fourier expand f in ϕ, but the reader will be spared this. A more serious defect may be the omission of electromagnetic as opposed to electrostatic interactions, which was rather glibly effected on the first page; for what co-operative effects may be important here is far from clear, and the relative inefficiency of the plasma as a current screen may overweigh the $(v)^2$ in the inter-particle force. It is certainly true that in many instabilities, e.g. the mirror type, the inductive fields are more important than those produced by charge separation, and similar phenomena may play a role in particle interactions. This presentation must, therefore, be considered as a sketch of the process of particle interaction in a magnetic field rather than as a complete account.

PART V

Correlation Functions and Scattering of Radiation from a Plasma

1. The Correlation Functions in a Plasma

Since our procedure has given a value for the potential produced by a particle in a plasma, namely (in the absence of a magnetic field)

$$\Phi(\underline{k},\omega) = 8\pi^2 e_i \frac{\delta(\omega + \underline{k}\cdot\underline{v})}{k^2\ \varepsilon\ (\underline{k},\ \omega)} e^{-i\ \underline{k}\cdot\underline{x}_i} \tag{5.1.1}$$

we may now use the Vlasov equation to calculate the disturbance that this produces in the distribution function, i.e.

$$f' = 8\pi^2\ e_1 \frac{\delta(\omega + \underline{k}\cdot\underline{v})}{k^2\ \varepsilon\ (\underline{k},\ \omega)}\ e^{-i\ \underline{k}\cdot\underline{x}_i}\ \frac{e_2}{m_2} \frac{\underline{k}\cdot \frac{\partial f_o}{\partial v}}{(\omega + \underline{k}\cdot\underline{v})} \tag{5.1.2}$$

Now the probability of finding a particle at $(\underline{x}_1,\ v_1,\ t_1)$ and a particle at $(\underline{x}_2,\ \underline{v}_2,\ t_2)$ is clearly:

$$P(1,2) = f(v_1,x_1,t_1)\ f(v_2,x_2,\ 0) + f(x_2.v_2,\ 0)\ f(1,2,x_1,v_1.t_1)$$

However, the last term is just (5.1.2); i.e. f' the change in f(1) produced by a particle at 2. hence:-

$$f(\underline{x}_1,\underline{v}_1,t_1,2) = \frac{e_1 e_2}{m_1}\ \frac{1}{2\pi^2}\int d^3k\ d\omega\ \frac{\delta(\omega + \underline{k}\cdot\underline{v}_2)\, e^{ik\cdot(\underline{x}_1 - \underline{x}_2)}}{[k^2\varepsilon(\underline{k},\omega)](\omega + \underline{k}\cdot\underline{v}_1)}\cdot e^{i\omega t}\underline{k}\cdot\frac{\partial f_o}{\partial \underline{v}_1}(1) \tag{5.1.3}$$

Having the (space dependent) perturbation induced in the distribution function by the potential of a charged particle, it is possible to calculate spatial distribution of electrons and ions in a plasma, in this approximation. If the zero order position of electrons and ions are x_i and X_i, then the plasma potential is:

$$\Phi = 8\pi^2 \sum_i e_- \frac{\delta(\ \omega + \underline{k}\cdot\underline{v}_i)\ e^{-i\underline{k}\cdot\underline{x}_i} + e_+\delta(\omega + \underline{k}\cdot\underline{v}_i)\, e^{-ik\cdot\underline{X}_i}}{k^2\ \varepsilon\ (\underline{k},\ \omega)} \tag{5.1.4}$$

and

$$n_-(\underline{k},w) = \sum_i e^{-i\underline{k}\cdot\underline{x}_i}\ \delta(\omega + \underline{k}\cdot\underline{v}_i) + \frac{e^2}{m_-}\int d^3v\ \frac{\delta(\omega + \underline{k}\cdot\underline{v}_i)\, e^{-i\underline{k}\cdot\underline{x}_i}}{\underline{k}^2\ \varepsilon(\underline{k},\ \omega)(\omega + \underline{k}\cdot\underline{v})}\underline{k}\cdot\frac{\partial f_o}{\partial\underline{v}} \tag{5.1.5}$$

From this, the charge density is:-

$$q_1 = e_- n_- (\underline{k},\omega) + e_+ n_+ (\underline{k},\omega)$$

$$= 2\pi \Sigma \{e_- e^{-i\underline{k}\cdot\underline{x}_i} \delta(\omega + \underline{k}\cdot\underline{v}_i) + e_+ e^{-i\underline{k}\cdot\underline{X}_i} \delta(\omega + \underline{k}\cdot\underline{V}_i)\}$$

$$\cdot \left\{1 + \frac{\left[\omega_o^2 \int \frac{d^3v}{(\omega + \underline{k}\cdot\underline{v})} \underline{k}\cdot \frac{\partial f_-}{\partial \underline{v}} + \omega_+^2 \int \frac{d^3v}{(\omega + \underline{k}\cdot\underline{v})} \underline{k}\cdot\frac{\partial f_+}{\partial \underline{v}}\right]}{k^2 \varepsilon(\underline{k},\omega)}\right\} \qquad (5.1.6)$$

But

$$k^2 \varepsilon(\underline{k},\omega) = k^2 - \omega_o^2 \left[\int \frac{d^3v}{(\omega + \underline{k}\cdot\underline{v})} k \cdot \frac{\partial f_-}{\partial \underline{v}} + \frac{\omega_+^2}{\omega_o^2} \int \frac{d^3v}{(\omega + \underline{k}\cdot\underline{v})} \underline{k}\cdot\frac{\partial f_+}{\partial \underline{v}}\right]$$

hence

$$4\pi \quad q_1 = 8\pi^2 \Sigma \frac{[e_- \delta(\omega + \underline{k}\cdot\underline{v}_i) e^{-i\underline{k}\cdot\underline{x}_i} + e_+ \delta(\omega + \underline{k}\cdot\underline{V}_i) e^{-i\underline{k}\cdot\underline{X}_i}]}{\varepsilon(\underline{k},\omega)}$$

and the calculation is self consistent.

2. Scattering of Radiation from a Plasma

It is of interest to note an observable phenomena which depends on the details of the electron correlation function; this is the scattering of radiation by a plasma. To treat this, we consider a plasma in which the distribution function may be written $f_o(v) + f_1(x,v,t)$ and consider the effect on this of an electric field $E(x,t)$. After Fourier transforming we obtain for the induced currents:-

$$\underline{j}_{ind}(\omega,k) = -\frac{e^2}{m} \int d^3v \frac{[\underline{E}(\omega,k)\cdot\frac{\partial f_o}{\partial \underline{v}} \underline{v} + \Sigma \underline{E}(\Omega,K)\cdot\frac{\partial f}{\partial \underline{v}} (\omega - \Omega, K - k)\underline{v}]}{i(\omega + \underline{k}\cdot\underline{v})} \qquad (5.2.1)$$

If the phase velocity is high $\omega/k \gg v_\Theta$; then

$$\underline{j}_{ind}(\omega,k) = \frac{1}{4\pi}\frac{\omega_o^2}{i\omega} \{\underline{E}(\omega,k) + \frac{1}{\omega_o^2} \Sigma \Delta \omega_o^2 (\omega - \Omega, k - K) \underline{E}(\Omega,K)\}$$

Maxwell's equations for this field become:

$$\nabla^2 \underline{E} - \frac{1}{c^2}\frac{\partial^2 \underline{E}}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial \underline{j}}{\partial t} + \text{grad}(\text{div } E)$$

or, on Fourier transforming, and using the equation of continuity:-

$$\left(\frac{\omega^2}{c^2} - k^2\right) \underline{E}(\omega,\underline{k}) = -\frac{4\pi i\omega}{c^2} \underline{j} - \underline{k}(\underline{k}\cdot\underline{E})$$

or

$$\left(\frac{\omega^2 + \omega_o^2}{c^2} - k^2\right) \underline{E}(\omega,\underline{k}) = \left[\frac{\omega^2}{c^2} - \underline{k}\,\underline{k}\,\cdot\right] \Sigma \frac{\Delta\omega_o^2}{\omega^2} (\omega - \Omega, \underline{K} - \underline{k}) \underline{E}(\Omega,\underline{K}) \qquad (5.2.2)$$

Now, if there is incident on the plasma a field of frequency Ω and wave number $\underline{K}$ ($\underline{K}^2 = \Omega^2/c^2$), a scattered wave will be produced given by:-

$$\underline{E}(\omega,\underline{k}) = \frac{[\omega^2 - c^2\underline{k}\ \ \underline{k}.]\ \Delta\omega_o^2(\omega - \Omega, k - K)\ \underline{E}\ (\Omega,\underline{K})}{\left[\frac{\omega^2 + \omega_o^2}{c^2} - k^2\right]} \qquad (5.2.3)$$

For scattered and incident waves well above the plasma frequency, this yields to an expression for the intensity at large distances in the form:-

$$\frac{d^2 I_s\ (\omega'\underline{k}')}{d\ \omega\ d\ \Omega} = I_o(\Omega,\underline{K})\ 4\pi\ \left(\frac{e^2}{mc^2}\right)^2 \left|\Delta\ n(\omega,\underline{k})\right|^2\ [1 - (\sin\theta\ \cos\phi)^2] \qquad (5.2.4)$$

where θ, ϕ are the scattering angles, with respect to incident direction and polarization and the last factor $= \frac{1}{2}(1 + \cos^2\theta)$ for an unpolarised incident beam. If the only polarization of the electron density is that produced by random fluctuations Δn is given by (5.1.5) i.e:

$$\Delta n(\omega,\underline{k}) = \sum_i (1 + G_-)\ \frac{e^{-i\underline{k}\cdot\underline{x}_i}\ \delta(\omega + \underline{k}\cdot\underline{v}_i) + G_+ \frac{e_+}{e_-}\delta(\omega + \underline{k}\cdot\underline{V}_i)\, e^{-i\underline{k}\cdot\underline{X}_i}}{1 - G_- - G_+} \qquad (5.2.5)$$

where

$$G_{\pm} = 4\pi\ \left(\frac{e^2}{m}\right)_{\pm}\ \frac{1}{k^2}\int \frac{\underline{k}\cdot\frac{\partial f_o}{\partial v}}{(\omega + \underline{k}\cdot\underline{v})}\ d^3v \sim \frac{k_D^2}{k^2} \qquad (5.2.6)$$

Now, to form $(\Delta n)^2$, the random phase approximation may be invoked whereupon

$$\left|\Sigma\ e^{-i\underline{k}\cdot\underline{x}_i}\ \delta(\omega + \underline{k}\cdot\underline{v}_i)\right|^2 = \int d^3v\ f(v)\ \delta(\omega + \underline{k}\cdot\underline{v})$$

and

$$|\Delta n_-(\omega,k)|^2 = N\left\{\left|\frac{1 - G_+}{1 - G_- - G_+}\right|^2 \int d^3v\ f_-(\underline{v})\ \delta(\omega + \underline{k}\cdot\underline{v})\right.$$

$$\left. + \left|\frac{G_-}{1 - G_- - G_+}\right|^2 \int d^3v\ f_+\ (\underline{v})\ \delta(\omega + \underline{k}\cdot\underline{v})\right\} \qquad (5.2.7)$$

The scattered radiation is then given by 5.2.4 and 5.2.7. If $(k_D/k)^2 \gg 1$ several interesting features appear. Since f_+ is much narrower than f_-, the second term dominates near $\Delta\omega = 0$; but since G_- increases with ω, the scattered wave first increases, then decreases, with a half width determined by $f_+(\omega/k)$, however, at large frequency shifts the first term dominates. When $\Delta\omega = \omega_o$ the dielectric coefficient $(1 - G_- - G_+)$ becomes small over a narrow region and a sharp narrow peak corresponding to the emission of plasma oscillation appears.

If $T_- \gg T_+$ ion sound waves may appear, and to a sharp peak appears at $(\Delta\omega/\Delta k) \simeq (KT_-/m_+)$ from the centre. If $k_D \ll k$ the G's are small, and only the first terms persist.

The interest in this process lies in the possibility of exploring the correlation function directly; the difficulty lies in the small value of the Thomson cross section $8\pi(e^2/mc^2)^2 \simeq 10^{-25}$ cm^2 which determines the scale of the phenomena. If because of an instability $\Delta n(\omega,\underline{k})$ becomes very large for some narrow range of $\omega,\underline{k}$ a much more spectacular effect would be expected.

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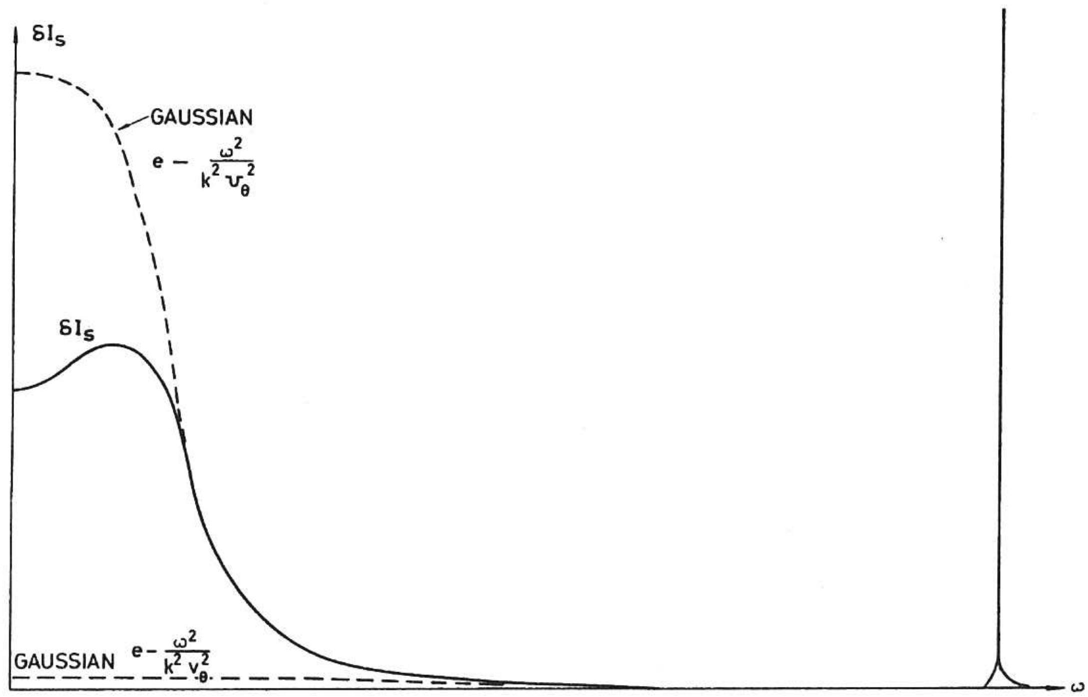

CLM-P 18 Fig. 5 Scattering of radiation from a plasma formof δI_s vs ω