CLM - P 183

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# ION CYCLOTRON INSTABILITIES IN SHORT MIRROR MACHINE PLASMAS

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# ION CYCLOTRON INSTABILITIES IN SHORT MIRROR MACHINE PLASMAS

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ABSTRACT

An appropriate finite geometry theoretical model of a magnetic mirror machine is examined for ion-cyclotron instabilities. The normal modes of the finite model are shown to be unstable even in the presence of large magnetic field gradients. The effect upon the ion cyclotron instability of particle mirroring is discussed. Electron Landau damping is shown to be stabilising for these modes, but low mirror ratio machines will require high electron temperatures to stabilise instabilities at frequencies near the high harmonics of $\omega_{ci}$.

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# C O N T E N T S

## 1. INTRODUCTION

The loss-cone of a mirror machine results in the setting up of a non-Maxwellian velocity distribution. In particular, the ion velocity distribution function has a hole at $v_{\perp} = 0$. This loss-cone type of ion distribution has been shown to be unstable to several classes of ion-cyclotron instabilities(1,2,3).

The theory of ion cyclotron instabilities in an infinite uniform plasma has been discussed by many authors(3,4,5) and recently the motion of the unstable wave packet has been determined by Beasley and Cordey(6), and Callen and McCune(7). These authors(6,7) found that at high densities the ion cyclotron instability in an infinite plasma became absolute. The wavelength of the absolute modes in a direction perpendicular to the magnetic field was of the order of an ion Larmor radius, however along the magnetic field the wavelength is highly extended and in many cases may be of the same order of magnitude as the inhomogeneity scale length of many mirror machine experiments. There is also experimental evidence in the PHOENIX II(8) mirror machine of ion cyclotron instabilities with long wavelength along the magnetic field. It is the non-uniformity of the plasma and magnetic field along the field lines that we shall be interested in here and <u>not</u> non-uniformities perpendicular to magnetic fields which have been discussed elsewhere(1,2).

The stability theory of an infinite homogeneous plasma has recently been extended by Rosenbluth, Berk and Pearlstein(9) to include small variations in plasma parameters along the magnetic field using the W.K.B. method. However when the wavelength of the unstable mode is of the same order of magnitude as the scale length of plasma inhomogeneity the wave packet concept and the W.K.B. method break down and we are only interested in the normal modes of the system.

In this paper we find the normal modes of an appropriate finite geometry model and discuss the effect of magnetic field inhomogeneity, plasma length and mirror frequency upon these modes.

The inhomogeneity of the magnetic field results in a variation of the ion cyclotron frequency along the length of the mirror and only a few of the particles will be resonant with the wave at any particular moment. In Section 3 the normal modes of our theoretical model are found and it is shown that strong magnetic field gradients do in fact reduce the growth rate of the unstable modes, but do not stabilise them completely.

In Section 4 we look at the effect of the mirroring of ions upon the frequency and growth rate of the instability. Ion mirroring only becomes important when the growth rate ($\gamma$) of the instability is less than the ion mirror frequency.

Finally in Section 5 we show that electron Landau damping can stabilise these modes and estimate the electron temperature required to stabilise instabilities at each harmonic of $\omega_{ci}$.

## 2. THEORETICAL MODEL AND EQUATIONS

From the theoretical(6,7) calculations of ion cyclotron instabilities in infinite homogeneous plasma it is clear that the most serious modes have a long wavelength in the direction of the magnetic field but a very short wavelength ($\sim a_i$) perpendicular to the magnetic field. This suggests that a more realistic theoretical model than the infinite geometry model would be a model which took into account the finite geometry of the plasma along the field lines. Thus we choose a slab model which is homogeneous and infinite in a plane perpendicular to the magnetic field (the $xy$ plane) but parameters may vary in the direction along the magnetic field (z-direction). In particular the variation of magnetic field $B_z$ and density $N$ in the z-direction should play an important part in determining the modes of the system. We take the plasma to be symmetrical about $z = 0$ and assume the plasma boundaries are at $z = \pm L/2$.

To mimic the oscillation of particles between the mirrors of a magnetic mirror machine we introduce a fictitious gravitational potential $\Phi(z)$ in our model. The ensuing analysis is considerably simplified by choosing the potential to be parabolic in the form $\Phi(z) = \frac{1}{2} m\omega_t^2 z^2$; then the frequency of oscillation of particles between the ends of the slab is $\omega_t$.

By taking the magnetic field in the form $\underline{B} = (0, 0, B_z(z))$ we have neglected all effects due to field curvature. This should be a reasonable approximation close to the central field lines of a magnetic mirror. We now find the equilibrium solution of the Vlasov equation for our model.

The Vlasov equation for ions and electrons in equilibrium is

$$m \underset{\sim}{v} \cdot \nabla \underset{\sim}{F} - \underset{\sim}{\nabla} \Phi \cdot \frac{\partial F}{\partial \underset{\sim}{v}} + e \underset{\sim}{v} \times \underline{B} \cdot \frac{\partial F}{\partial \underset{\sim}{v}} = 0 . \qquad \ldots (1)$$

For the theoretical model described in the last section $\underline{B} = (0, 0, B_z(z))$ and $\Phi = \frac{1}{2} m \omega_t^2 z^2$. Equation (1) may be written in the form

$$v_z \frac{\partial F}{\partial z} - \omega_c \frac{\partial F}{\partial \varphi} - \omega^2_t z \frac{\partial F}{\partial v_z} = 0$$

where

$$\varphi = \tan^{-1} v_y/v_x .$$

The characteristics of the above equation are $v_z^2 + \omega_t^2 z^2$ and $v_x^2 + v_y^2 = v_\perp^2$ , hence any function of the form

$$F(v_z^2 + \omega_t^2 z^2, v_\perp^2)$$

is a solution.

We now derive the linearised equations for the perturbed quantities. The perturbations are assumed to be electrostatic and have the form

$$\psi = \hat{\psi}(z)\, e^{i(k_\perp x + \omega t)}$$

$$f = F + \hat{f}(v_\perp, \varphi, v_z, z)\, e^{i(k_\perp x + \omega t)} .$$

The linearised Vlasov equation for the perturbed ion distribution function $\hat{f}_i$ is

$$\frac{\partial \hat{f}_i}{\partial \varphi} + \frac{\omega_{ti}^2}{\omega_{ci}} z \frac{\partial \hat{f}_i}{\partial v_z} - \frac{i(\omega + k_\perp v_\perp \cos\varphi)}{\omega_{ci}} \hat{f}_i - \frac{v_z}{\omega_{ci}} \frac{\partial \hat{f}_i}{\partial z} = \frac{e}{m_i \omega_{ci}} \left( i k_\perp \hat{\psi} \cos\varphi \frac{\partial F_i}{\partial v_\perp} + \frac{d\hat{\psi}}{dz} \frac{\partial F_i}{\partial v_z} \right) \quad \ldots (2)$$

The linearised Vlasov equation for the perturbed electron distribution function $\hat{f}_e$ may be considerably simplified by assuming $\frac{k_\perp v_\perp}{\omega} \ll 1$ (this inequality is usually satisfied since we are primarily interested in wavelengths of the order of the ion Larmor radius). The electron distribution $\hat{f}_e$ will then be independent of $\varphi$ and we may integrate the Vlasov equation for electrons with respect to $\varphi$ giving

$$\frac{\omega_{te}^2 z}{\omega} \frac{\partial \bar{f}_e}{\partial v_z} - i\bar{f}_e - \frac{v_z}{\omega} \frac{\partial \bar{f}_e}{\partial z} = - \frac{e}{\omega m_e} \frac{d\hat{\psi}}{dz} \frac{\partial \bar{F}_e}{\partial v_z} \quad \ldots (3)$$

where $\bar{f}_e = \int f_e d\varphi$ and $\bar{F}_e = \int F_e\, d\varphi$.

Equations (2) and (3) together with Poisson's equation which may be written in the form

$$k_\perp^2 \hat{\psi} - \frac{d^2\hat{\psi}}{dz^2} = 4\pi e \left\{ \int \hat{f}_i\, d^3v - \int \bar{f}_e\, v_\perp\, dv_\perp dv_z \right\} \quad \ldots (4)$$

are sufficient for the determination of $\hat{\psi}$ and $\hat{f}$.

The solution of equations (2) and (3) for the perturbed ion and electron distribution functions is the crucial problem here. The perturbed electron distribution $\hat{f}_e$ is obtained from equation (3) by expressing the dependent variables as series in $\omega_{te}/\omega$. The perturbed ion distribution may be found from equation (2) by expressing the dependent variables as asymptotic series in $a_i/L$, provided the growth rate $\gamma$ is greater than the ion mirror frequency $\omega_{ti}$. If $\gamma < \omega_{ti}$ then straightforward series expansions in $a_i/L$ of equation (4) are not asymptotically convergent (i.e. the terms increase in magnitude for all $a_i/L$). However we show in Section 4 that by expressing equation (2) as an integral equation one may obtained an asymptotic series representation for $\hat{f}_i$ for the special case of $\gamma < \omega_{ti}$. We now confine our attention to the more general case of $\gamma > \omega_{ti}$ and leave the discussion of the $\gamma < \omega_{ti}$ case to Section 4.

## 3. SOLUTION FOR GROWTH RATE GREATER THAN ION MIRROR FREQUENCY $\gamma > \omega_{ti}$

We begin by expressing $\hat{f}_i$ and $\hat{\psi}$ as asymptotic series in $a_i/L$ in the form,

$$\hat{f}_i = f_{io} + f_{i1}\left(\frac{a_i}{L}\right) + f_{i2}\left(\frac{a_i}{L}\right)^2 + \ldots$$

$$\hat{\psi} = \psi_o + \psi_1\left(\frac{a_i}{L}\right) + \psi_2\left(\frac{a_i}{L}\right)^2 + \ldots \; .$$

The series are substituted into equation (2) and terms of the same order in $\frac{a_i}{L}$ collected. Note that $\frac{\omega_{ti}}{\omega_{ci}}$ and terms of the form $\frac{v}{\omega_{ci}}\frac{\partial}{\partial z}$ are both of order $\frac{a_i}{L}$. The zero'th order terms of equation (2) give

$$\frac{\partial f_{io}}{\partial \varphi} - \frac{i(\omega + k_\perp v_\perp \cos\varphi)}{\omega_{ci}} f_{io} = -\frac{2ie}{m_i}\frac{k_\perp \psi_o}{\omega_{ci}}\frac{\partial F_i}{\partial v_\perp^2} v_\perp \cos\varphi \; .$$

The above equation may be solved by use of an integrating factor and the solution is

$$f_{oi} = \frac{2e}{m_i}\psi_o \frac{\partial F_i}{\partial v_\perp^2} - \frac{2e\omega}{m_i}\psi_o \frac{\partial F_i}{\partial v_\perp^2} \sum_{n,m} \frac{Jn(\frac{k_\perp v_\perp}{\omega_{ci}})\, Jm(\frac{k_\perp v_\perp}{\omega_{ci}}) e^{i(m-n)\varphi}}{\omega + n\omega_{ci}}$$

where we have made use of the fact that $f_{oi}$ must be periodic in $\varphi$ to eliminate the arbitrary constants.

For the electron distribution function we expand the dependent variables of equation (3) in powers of $\omega_{te}/\omega$. The zero, first and second order solutions are

$$\bar{f}_{eo} = 0 \quad , \quad \bar{f}_{e1} = -\frac{ie}{m_e}\frac{d\psi_o}{dz}\frac{\partial \bar{F}_e}{\partial v_z}$$

$$\bar{f}_{e2} = -\frac{e}{m_e\omega^2}\left\{ \omega_{te}^2\, z\, \frac{d\psi_o}{dz}\frac{\partial \bar{F}_e}{\partial v_z} - v_z \frac{d}{dz}\left(\frac{d_o}{dz}\frac{\partial \bar{F}_e}{\partial v_z}\right)\right\}.$$

Substituting the above expressions for $\bar{f}_e$ and $\hat{f}_i$ in Poisson's equation and retaining only lowest order non-zero terms in $a_i/L$ and $\omega_{te}/L$ we have

$$\frac{d}{dz}\left(\frac{hd\psi}{dz}\right) + \left\{\lambda + u\,(z,\,\omega)\right\}\psi = 0 \qquad \dots (5)$$

where

$$u(z,\omega) = 2\omega_{pi}\frac{\omega^2}{\omega_{peo}} \sum_{n=-\infty}^{\infty} \frac{n\omega_{ci} \iint J_n^2(k_\perp v_\perp/\omega_{ci})\, \frac{\partial \bar{F}_i}{\partial v_i^2} v_\perp dv_\perp dv_z}{\omega - n\,\omega_{ci}}$$

$h = N_e(z)/N_e(0) - \omega^2/\omega_{peo}^2$ , $\lambda = \omega^2 k_\perp^2/\omega_{peo}^2$, $\omega_{peo} = \omega_{pe}$ at $z = 0$,

$\bar{F}_i$ is the normalised distribution ($\iint \bar{F}_i\, v_\perp\, dv_\perp\, dv_z = 1$) and we have dropped the subscript on $\psi$.

For a large range of $z$, $u \ll 1$. However for the values of $z$ such that $\omega = \omega_{ci}(z)$ the wave is resonant with the local cyclotron frequency and $u$ is not small, so straightforward perturbation methods upon equation (5) will fail. However integrals of $u$ over the whole plasma should be small. This suggests that we should express equation (5) in integral form before we attempt a perturbation method of solution.

To put equation (5) in integral form we use the Green function $G_\lambda(z, z)$ which is the solution of the differential equation

$$\mathcal{L}\,\psi \equiv \frac{d}{dz}\left(\frac{h d\psi}{dz}\right) + \lambda\,\psi = \delta(z' - z) \qquad \ldots (6)$$

and satisfies the same boundary conditions as $\psi_0$ at $z = \pm L/2$ (the plasma boundary). The solution of equation (5) may then be written

$$\psi\,(z) = -\int_{-L/2}^{L/2} G_\lambda(z|z')\; u(z')\; \psi(z')\; dz' \; . \qquad \ldots (7)$$

Now if the boundary conditions are homogeneous we may express $G_\lambda$ in terms of the eigenfunctions $\chi_m$ of

$$\mathcal{L}\,\psi = 0 \qquad \ldots (8)$$

(equation (6) with the r.h.s. set equal to zero). The form of $G_\lambda$ is

$$G_\lambda = \Sigma\; \frac{\chi_m(z)\; \chi_m^*(z')}{\lambda - \lambda_m} \; .$$

Substituting the above expression into equation (7) we have

$$\psi(z) = -\frac{\chi_m(z)}{\lambda - \lambda_m}\int_{-L/2}^{L/2} \chi_m^*(z')u(z')\psi(z')dz - \sum_{p \neq m} \frac{\int \chi_p^*(z')u(z')\psi(z')dz'}{\lambda - \lambda_p}\,\chi_p(z)$$
$$\ldots (9)$$

Equation (9) is exact and we now show how one may use a perturbation technique to solve it. First of all if we fix the normalisation of $\psi$ such that $\psi(z) \to \chi_m(z)$ as $u(z) \to 0$, then

$$\lambda = \lambda_m - \int_{-L/2}^{L/2} \chi_m^*(z')\ u(z')\ \psi(z')\ dz' \qquad \ldots (10)$$

and equation (9) becomes

$$\psi(z) = \chi_m(z) - \sum_{p\neq m} \int_{-L/2}^{L/2} \frac{\chi_p^*(z')u(z')\psi(z')\ dz'\chi_p(z)}{\lambda - \lambda_p} \qquad \ldots (11)$$

Equations (10) and (11) are now in convenient form for our perturbation procedure to evaluate the eigenvalue and eigenfunctions for functions $u(z)$ with singularities. The perturbation technique we use consists of substituting successively better approximations for $\psi(z)$ into equations (10) and (11) to obtain the new approximation for the eigenvalue and eigenfunction. To start the iterative procedure we put $\psi(z) = \chi_m(z)$ and $\lambda = \lambda_m$ in equations (10) and (11). The $\ell^{th}$ iterative is

$$\psi_m^{(\ell)} = \chi_m(x) - \sum_{p\neq m}\left[\frac{u_{pm}}{\lambda^{(\ell-1)} - \lambda_p}\right]\chi_p + \sum_{pq\neq m} \frac{u_{pq}\ u_{qm}\ \chi_p}{[(\lambda^{\ell-2} - \lambda_p)(\lambda^{\ell-2} - \lambda_q)]} + \ldots$$

and eigenvalue is

$$\lambda^{(\ell)} = \lambda_m - u_{mm} + \sum_{p\neq m} \frac{u_{mp}\ u_{pm}}{\lambda^{(\ell-2)} - \lambda_p} + \ldots$$

where

$$u_{pn} = \int_{-L/2}^{L/2} \chi_p^*\ u\chi_n\ dz' \ .$$

One can show that the series are convergent (see Morse and Feshbach[(10)]) and that the difference between $\psi$ and $\psi_m^{(\ell)}$ is of order of $(\gamma/\omega_{ci})^{\ell+1}$. The first iterate for $\lambda$ gives

$$\lambda \equiv \frac{\omega^2 k_\perp^2}{\omega_{peo}^2} = \lambda_m + \frac{2\omega^2}{\omega_{peo}^2} \int_{-L/2}^{L/2} \omega_{pi}^2 \sum \frac{n\omega_{ci} D_n \chi_m^2 \, dz}{\omega - n\omega_{ci}(z)} \qquad \ldots (12)$$

where

$$D_n = - \iint \frac{\partial \bar{F}}{\partial v_\perp^2} J_n^2(k_\perp v_\perp/\omega_{ci}) \, v_\perp dv_\perp dv_\parallel \quad .$$

Equation (12) is the dispersion equation for $\omega$ as a function of $k_\perp$ for the $m^{th}$ mode. The functions $\lambda_m$ and eigenvalues $\lambda_m$ of equation (8) may be determined either numerically in the case of a complicated electron density profile $h(z)$ or analytically for simple $h(z)$.

To simplify equation (12) so that we may solve for $\omega$ analytically we assume that the magnetic field variation is parabolic and hence take

$$\omega_{ci} = \omega_{co} (1 + \beta z^2) \; .$$

Putting $\omega = \omega_o - i\Upsilon$ in equation (12), with $\omega_o = \omega_{peo}\lambda_m^{1/2}/k_\perp$ and neglecting terms of order $\Upsilon^2$ we have

$$\Upsilon = i \int_{-L/2}^{L/2} \frac{\chi_m^2 \, \omega_{pi}^2}{k_\perp^2} \; \frac{n\omega_{ci} D_n \, dz}{\omega_o - i\Upsilon - n\omega_{co}(1 + \beta z^2)} \qquad \ldots (13)$$

where we have dropped the summation over $n$ and concentrated on instabilities close to a particular harmonic, this approximation is valid provided $|\omega - n\omega_{ci}| \ll |\omega - (n \pm 1)\, \omega_{ci}|$ .

Equation (13) can be solved analytically in the limiting cases of weak and strong magnetic field gradients.

(a) <u>Solution for Weak Magnetic Field Gradient $\Upsilon \gg |\omega_o - n\omega_{ci}|$</u>

In this case we may neglect all the terms of the denominator of equation (13) except the $i\Upsilon$ term. Hence

$$\Upsilon = \left[ - \int_{-L/2}^{L/2} \frac{\chi_m^2 \, \omega_{pi}^2 \, \omega_{ci} \, D_n}{k_\perp^2} \, dz \right]^{\frac{1}{2}} .$$

This is a very similar expression to that of the growth rate in a uniform plasma except now $\omega_{pi}$, $D_n$ etc. are integrated over the square of the potential distribution.

(b) Solutions for Strong Magnetic Field Gradients $\Upsilon \ll |\omega_o - \omega_{ci}|$

In equation (13) we now neglect $i\Upsilon$ compared with $\omega_o - n\omega_{ci}$ and we find that

$$\Upsilon = i\omega_o \int_{-L/2}^{L/2} \frac{\omega_{pi}^2}{k_\perp^2} \chi_m^2(z) \frac{n\omega_{ci} \, D_n \, dz}{\omega_o - n\omega_{co} \, (1 + \beta z^2)} .$$

The integrand of the above equation has poles at

$$z = \pm z_p \quad , \quad z_p = \left\{ \frac{\omega_o - n\omega_{co}}{\beta \, n\omega_{co}} \right\}^{\frac{1}{2}} .$$

Using the Landau prescription to deal with these poles gives

$$\Upsilon = - \frac{\pi n}{\beta} \left\{ \frac{\omega_{pi}^2 \, \chi_m^2 \, \omega_{ci} \, D_n}{k_\perp^2 \, z} \right\}_{z = z_p} . \qquad \ldots (14)$$

For instability we must have $D_n < 0$ at $z = z_p$ which is equivalent to $\frac{\partial F_i}{\partial v_\perp} > 0$ for a range of $v_\perp$ i.e. $F_i(v_\perp^2, v_\parallel)$ is of the loss cone type. One can see that as we increase the gradient of the magnetic field through the parameter $\beta$ the growth rate decreases, however it should be pointed out that one cannot keep increasing $\beta$ indefinitely since the equilibrium motion of the plasma will become non-adiabatic for large $\beta$.

As an example of the use of the above theoretical calculations we now present the results for the PHOENIX II mirror machine parameters. The $\chi_m$ and $\lambda_m$ were determined numerically from equation (8), using a finite difference scheme described by Richtmeyer[11]. The first eigenfunction $\chi_1(z)$ is shown in Fig.1(a) for the density profile of Fig.1(b). The boundary conditions applied at A and B are $\underline{D}$ and $\psi$ continuous. The sharp boundaries at the ends of the plasma give rise to good wave reflection at these points. More shallow boundaries may be treated using the above analysis but the solution of equation (8) is more complex. However the sharp boundary approximation is thought to be the most appropriate one for PHOENIX II. In Fig.2 we show the growth rate as a function of density for this mode, obtained by use of equation (14) (since the strong field gradient approximation is appropriate for PHOENIX parameters).

To summarise this section we have shown how one may obtain the eigenvalues and eigenfunctions of our theoretical model for the case when $\gamma > \omega_{ti}$. The growth rate of the instability decreases with increasing field gradient but does not go to zero.

## 4. SOLUTION FOR GROWTH RATE LESS THAN ION MIRROR FREQUENCY $\gamma < \omega_{ti}$

In Section 2 it was assumed that the wave amplitude e folded in a time which was short compared to the period oscillation of the ions between the mirrors i.e. $\gamma > \omega_{ti}$. However at low densities near the instability threshold $\gamma$ will be less than $\omega_{ti}$. In this section we obtain the growth rates and frequencies of these slow growing instabilities for our theoretical model. To simplify the problem we ignore the variation of the cyclotron frequency along the plasma since one would expect the variation in $\omega_{ci}$ to have qualitatively the same effect upon the slow growing instabilities as on the fast growing instabilities of Section 3.

Our starting point for the analysis will be equations (2)-(4).

If we follow the same procedure as that described in Section 3 to solve equation (2) by expanding $f_i$ in a series on $a_i/L$, we find that the first order term is of the form $a/\{L(\omega - \omega_{ci})^2\}$. Thus for $\gamma < \omega_{ti} = a\omega_{ci}/L$, the series does not asymptotically approach $f_i$. However by writing equation (2) as an integral equation one can obtain an asymptotic expansion in $a_i/L$ for $f_i$.

We begin by using the transformation

$$z = J/\omega_{ti} \sin\theta$$

$$v_z = J \cos\theta$$

to replace $z$ and $v_\parallel$ by $J$ and $\theta$. Equation (2) becomes

$$\frac{\partial \hat{f}_i}{\partial \varphi} - \frac{\omega_{ti}}{\omega_{ci}} \frac{\partial \hat{f}_i}{\partial \theta} - \frac{i(\omega + k_\perp v_\perp \cos\varphi)}{\omega_{ci}} \hat{f}_i = - \frac{e}{m_i \omega_{ci}} \left\{ ik_\perp \cos\varphi \frac{\partial F_i}{\partial v_\perp} \hat{\psi} + \frac{d\hat{\psi}}{dz} \frac{\partial F_i}{\partial v_z} \right\} \quad \ldots (15)$$

The transformation

$$\eta = \tfrac{1}{2} \left\{ \varphi - \frac{\omega_{ci}}{\omega_{ti}} \theta \right\}$$

$$\sigma = \tfrac{1}{2} \left\{ \varphi + \frac{\omega_{ci}}{\omega_{ti}} \theta \right\}$$

then reduces equation (15) to a form convenient for integration

$$\frac{\partial \hat{f}_i}{\partial \eta} - i \left\{ \frac{\omega}{\omega_{ci}} + \frac{k_\perp v_\perp}{\omega_{ci}} \cos(\eta + \sigma) \right\} \hat{f}_i = - \frac{e}{m_i \omega_{ci}} \left\{ ik_\perp \cos(\eta + \sigma) \frac{\partial F_i}{\partial v_\perp} \hat{\psi} + \frac{d\hat{\psi}}{dz} \frac{\partial F_i}{\partial v_z} \right\} \quad \ldots (16)$$

The solution of equation (16) is

$$\hat{f}_i = - \frac{e}{m_i \omega_{ci} I} \int \left\{ i\, k_\perp \cos(\eta + \sigma) \frac{\partial F_i}{\partial v_\perp} \hat{\psi} + \frac{d\hat{\psi}}{dz} \frac{\partial F_i}{\partial v_z} \right\} d\eta \quad \ldots (17)$$

where the integrating factor

$$I = \exp\left[ - i \left\{ \omega\eta + k_\perp v_\perp \sin(\eta + \sigma) \right\} \Big/ \omega_{ci} \right]$$

and the periodicity of $\hat{f}_i$ in $\theta$ and $\varphi$ is used to eliminate the constant of interaction.

We may now express $\hat{f}_i$ as an asymptotic series in $a_i/L$ using the above integral equation form for $\hat{f}_i$. The zero'th order terms give

$$\hat{f}_{io} = - \frac{i\, e\, k_\perp}{m_i\, \omega_{ci}\, I} \frac{\partial F}{\partial v_\perp} \int \psi \cos(\eta + \sigma) I \, d\eta \, .$$

Substituting the above expression for $\hat{f}_{io}$ into Poisson's equation, we may write the eigenvalue problem in the same form as equation (5) of Section 3, but $u$ is now an operator

$$u(\psi) = - \frac{i\omega^2}{\omega_{peo}^2} \, \omega_{pi}^2 \int d^3 v \, \frac{\partial \bar{F}}{\partial v_\perp} \int \psi \cos(\eta + \sigma) I \, d\eta \quad \ldots (18)$$

Using the same perturbation technique to evaluate the eigenvalues and eigenfunctions as described in Section 3, the dispersion equation (corresponding to equation (12) of Section 3) is

$$\lambda \equiv \frac{\omega^2 k_\perp}{\omega_{peo}^2} = \lambda_m - \int \chi_m^* \, u(\chi_m) \, dz \, . \qquad \ldots (19)$$

The general method of evaluating the integral on the r.h.s. of equation (18) is to express the $\chi_m$ as a Fourier series and then transform to $J$, $\theta$ coordinates to evaluate the integrals. To simplify the analysis we choose a particularly simple form for $\chi_m$

$$\chi_m = \sum_{q=1}^{\infty} a_q \sin q z \pi / L \, .$$

Substituting the above form for $\chi_m$ into equation (19) and integrating over $\theta$ and $\varphi$ gives the dispersion equation

$$\frac{\omega^2 k_\perp^2}{\omega_{peo}^2} = \lambda_m - \frac{2L\omega_{pi}^2\omega^2}{\omega_{peo}^2} \int \frac{\partial \bar{F}}{\partial v_\perp^2} \sum_{n=-\infty}^{\infty} n\,\omega_{ci} \sum_{\substack{p=-\infty \\ p\,odd}}^{\infty} \frac{\{\Sigma a_q J_p(\alpha_q)\}^2}{\omega_{ti}(\omega - n\omega_{ci} - p\omega_{ti})} J_n^2\left(\frac{k_\perp v_\perp}{\omega_{ci}}\right) v_\perp dv_\perp JdJ \quad \ldots (20)$$

where $\alpha_q = \frac{q\pi J}{\omega_{ti}L}$ and $\bar{F}$ is normalised so that $\int \bar{F}\, d^3v dz = 1$. The reason why only odd values of $p$ occur in summation over $p$ is due to the antisymmetric form of $\chi_m$; if we had taken $\chi_m$ symmetric about $z = 0$ then only even values of $p$ would have appeared in equation (20).

In general the dispersion equation (20) will have to be solved numerically. However equation (20) is very similar in form to equation (12) of Section 3 (except that the resonances in the ion term occur at harmonics of $\omega_{ti}$ at each harmonic of $\omega_{ci}$) and using the results of Section 3 we can make several deductions about instability at frequencies $\omega \sim n\omega_{ci} + p\omega_{ti}$. First of all a necessary condition for instability is $\frac{\partial F_i}{\partial v_\perp} > 0$ for some $v_\perp$, that is the ion distribution must be of loss-cone type. As we go to higher harmonics of $\omega_{ti}$ (increasing $|p|$) the numerator of the ion term in equation (20) decreases, hence the growth rate of the instability will decrease with increasing p. Variation in $\omega_{ti}$ over $J$ and $v_\perp$ will be stabilising like the variation in $\omega_{ci}$ in Section 3, the amount of stabilisation depending on the detailed variation of $\omega_{ti}$ as a function of $J$ and $v_\perp$.

## 5. ELECTRON LANDAU DAMPING

In Sections 3 and 4 in calculating the perturbed electron distribution $\hat{f}_e$ it has been tacitly assumed that $\mathrm{Im}\omega > \omega_{te}$. We now drop this assumption and discuss qualitatively the amount of electron Landau damping required to stabilise the instability discussed in Section 3.

To determine the effect of electron Landau damping quantitatively one requires a detailed knowledge of the shape of the potential well to determine the equilibrium electron motion. With a knowledge of the potential and assuming real $\omega > \omega_{te}$ one may then reproduce a similar calculation to that of Section 4 and determine the imaginary part of the electron term. This is a long and tedious process and the result would be somewhat dependent upon the shape of the potential well and the ensuing electron distribution. To obtain qualitatively an estimate of the imaginary part of the electron term (Landau damping contribution) we may proceed as follows.

Assuming that the electrons are held in by a parabolic magnetic potential well then the electron mirror frequency $\omega_{te}$ will be proportional to $v_{\perp}$. An electron distribution of the form $e^{-(v^2+\omega_0^2 z^2)/\alpha_e^2}$ will then give rise to an additional imaginary term of the form

$$- 2i\pi^{\frac{1}{2}} \frac{\omega\,\omega_{pe}^2}{\omega_{te}\,\alpha_e} \; e^{-\omega^2/\omega_{te}^2} \quad \psi$$

to be added to the l.h.s. of equation (5). The dispersion equation (12) then becomes

$$\frac{\omega^2}{\omega_{po}^2} k_{\perp}^2 = \lambda_m + 2i\pi^{\frac{1}{2}} \frac{\omega^2}{\omega_{te}\,\alpha_e^2} \; e^{-\omega^2/\omega_{te}^2} + \frac{2\omega^2}{\omega_{peo}^2} \int_{-L/2}^{L/2} \omega_{pi}^2 \sum \frac{n\omega_{ci} D_n \chi_m^2}{\omega - n\omega_{ci}} \, dz \qquad \ldots (21)$$

The plasma is marginally stable where $\mathrm{Im}\omega = 0$. Using the results of Section 3 for the evaluation of the integral in equation (21) when $\mathrm{Im}\omega = 0$, we find that for stability at a frequency close to the $n^{th}$

harmonic

$$\omega_{ci}\omega_{pe}\,\frac{\omega^2 e^{-n\omega_{ci}^2/\omega_{te}^2}}{\omega_{te}\alpha^2 e} + \frac{\pi^{\frac{1}{2}} n}{\beta}\left[\frac{\omega_{pi}^2\,\chi_m^2\,\omega_{ci}\,D_n}{\alpha_\perp^2\, z}\right]_{z=z_p} > 0\ .$$

This inequality may be written approximately in the form

$$-\frac{n\omega_{ci}}{\omega_{te}}\; e^{-n^2\omega_{ci}^2/\omega_{te}^2} > \frac{T_e}{T_i(R-1)}$$

where $R$ is the mirror ratio.

Thus one can see from the above expression that one requires large values of $\omega_{te}$ (hence hot electrons) to stabilise the high harmonics of $\omega_{ci}$ in mirrors with low mirror ratio (small R). In mirrors with a large mirror ratio a fairly low electron temperature should stabilise several harmonics of $\omega_{ci}$. However it must be emphasised again that the results of this Section are only qualitative and the quantitative effects of electron Landau damping will be very much dependent upon the shape of the potential well and ensuing electron distribution function.

## 6. CONCLUSION

It has been shown that ion cyclotron instabilities with a highly extended wavelength along the magnetic field exist in magnetic mirror machines even when the magnetic field has a strong gradient. The growth rates of the unstable modes are found analytically for the cases of both strong and weak magnetic field inhomogeneity.

For the slow growing instabilities ($\gamma < \omega_{ti}$), the growth rate of the unstable modes peaks at harmonics of $\omega_{ti}$ ($\omega \sim n\omega_{ci} + p\omega_{ti}$), but the growth rate decreases as $|p|$ increases.

Finally we show that electron Landau damping is stabilising and estimate quantitatively the temperature of the electrons required to quench instabilities at each harmonic of $\omega_{ci}$. Although it should be easy to quench instabilities at the first few harmonics of $\omega_{ci}$ by heating electrons, it will be quite difficult to stabilise the high harmonics of $\omega_{ci}$ especially in mirror machines with a low mirror ratio.

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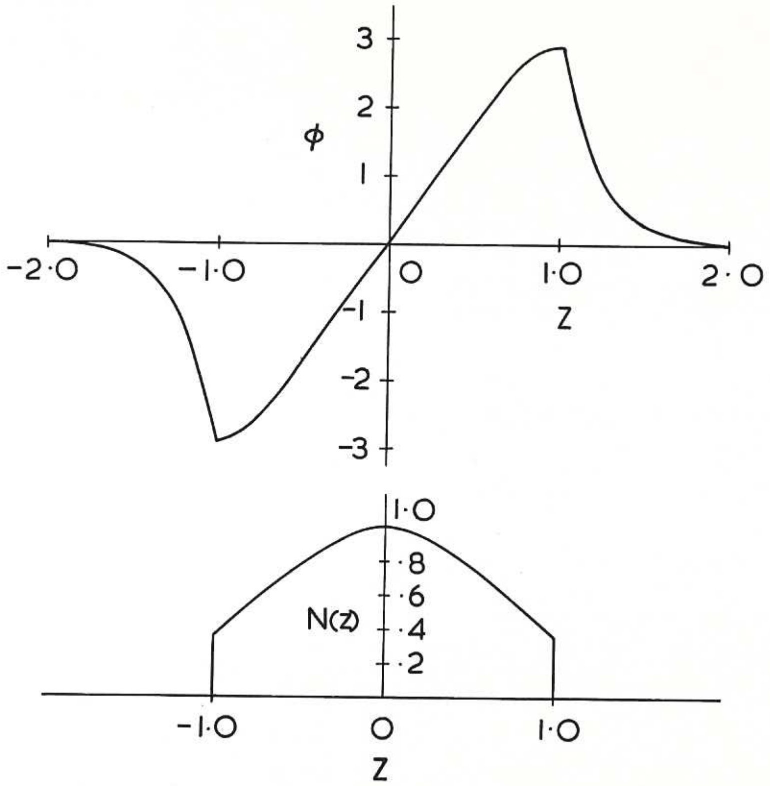


Fig. 1 (CLM-P 183)

The upper figure is the first eigenfunction for the density profile of the bottom figure

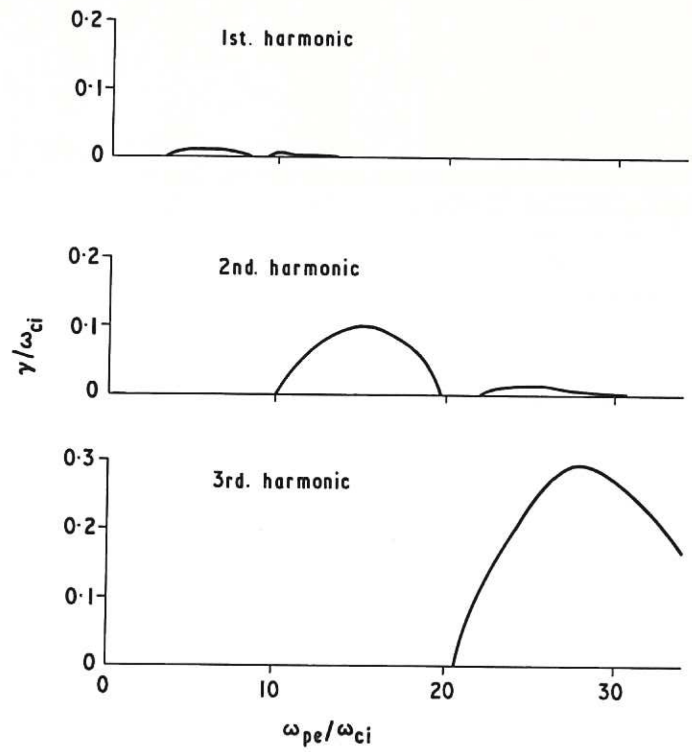


Fig. 2 (CLM-P 183)

Growth rate as a function of density for the first three harmonics of $\omega_{ci}$