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THEORY OF HIGH FREQUENCY ELECTROSTATIC WAVES IN A NON-UNIFORM PLASMA IN THE PRESENCE OF A MAGNETIC FIELD

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THEORY OF HIGH FREQUENCY ELECTROSTATIC WAVES IN A NON-UNIFORM PLASMA IN THE PRESENCE OF A MAGNETIC FIELD

by

C.N. LASHMORE-DAVIES

A B S T R A C T

High frequency ($\omega \gg \omega_{ci}, \omega_{pi}$) electrostatic waves which propagate in a non-uniform plasma in the presence of a uniform magnetic field are described. The analysis is carried out in a cylindrical geometry and both perfectly conducting and vacuum boundary conditions are included. Solutions of the dispersion relation corresponding to waves propagating perpendicularly (even for a uniform plasma column) and obliquely to the magnetic field are obtained. For the case of oblique propagation there are two branches to the dispersion relation for a given value of $|\ell|$ where ℓ is the azimuthal mode number. The relationship of these waves to the low frequency ($\omega \ll \omega_{ci}$) drift waves of a non-uniform plasma and to the lower branch of the modes discussed by Trivelpiece and Gould (1959) is considered. The attenuation of the waves due to collisions is computed. In the appendix the validity of the electrostatic approximation is examined.

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Berks.

December, 1968

LIST OF SYMBOLS USED

$\underline{E}$	- electric field
$\underline{H}$	- magnetic field
$\underline{B}$	- magnetic induction
$\underline{A}$	- vector potential
φ	- scalar potential
$\underline{J}$	- current density
$\underline{v}$	- average velocity of electrons
n	- number density
m	- mass of electron
$-e$	- charge of electron
κ	- Boltzmann's constant
T	- electron temperature
$v_T = \sqrt{\kappa T/m}$	- mean electron thermal velocity
ν	- collision frequency
ω	- wave frequency
ω_{ci}	- ion cyclotron frequency
ω_{pi}	- ion plasma frequency
ω_{ce}	- electron cyclotron frequency
ω_{pe}	- electron plasma frequency
ℓ	- azimuthal mode number
k	- axial wave vector
a	- plasma radius
ε_o	- dielectric constant of free space
μ_o	- magnetic permeability of free space
c	- velocity of light in free space
c_S	- ion sound velocity
c_A	- Alfvén velocity
β	- ratio of plasma pressure to magnetic pressure
r	- radial coordinate
θ	- azimuthal coordinate
z	- axial coordinate
t	- time

1. INTRODUCTION

The study of waves which can propagate in bounded or non-uniform plasmas in the presence of a magnetic field has attained great importance owing to their connection with the low frequency $(\omega \ll \omega_{ci})$ drift instabilities. These instabilities arise in a non-uniform plasma where there is a temperature gradient, a current flow, resistivity or finite ion Larmor radius (Mikhailovskii (1967); Kadomtsev (1965)). Under certain circumstances some of these low frequency waves are predominantly electrostatic and they can be described by an electrostatic potential φ where $\underline{E} = -\underline{\nabla}\varphi$. In this paper we shall consider the high frequency limit $\omega_{ci} \ll \omega \ll \omega_{ce}$, $\omega_{pi} \ll \omega$ of these electrostatic waves such that the ions can be considered as stationary. For short axial wavelengths these waves are just the electron plasma waves.

In this analysis the presence of a magnetic field is essential. We must also include the effect of a finite electron temperature since the low frequency drift waves have phase velocities which depend strongly on the temperature of the particles. We restrict our attention to frequencies well below the electron cyclotron frequency $\omega \ll \omega_{ce}$ and consider densities and magnetic fields such that $\omega_{pe}^2 \ll \omega_{ce}^2$. However, there is also a lower bound on the parameter ω_{pe}/ω_{ce} which is given in section 4.2. The frequency condition $\omega \ll \omega_{ce}$ is also the range in which helicon waves propagate. However, for helicon waves to be excited one requires a large enough concentration of electrons such that the first order currents are sufficient to perturb the zero order magnetic field. The limitation we have imposed on the parameter ω_{pe}/ω_{ce} excludes the helicon wave from the analysis. The precise conditions for the neglect of magnetic field perturbations (the validity of the electrostatic approximation) are given in the appendix.

The analysis given in this paper contains the lower branch of the modes discussed by Trivelpiece and Gould (1959) in the special case of a uniform plasma.

In section 2 of this paper we derive the dispersion equation for the electrostatic potential φ for an infinitely long cylindrical plasma in which the plasma density depends only on r. In section 3 we consider the boundary conditions which together with the dispersion equation determine the dispersion relation for the wave. Solutions of the dispersion relation which correspond to waves propagating across the magnetic field and, obliquely to it, are given in section 4. In section 5 the attenuation is considered and in section 6 the conclusions of the work are given.

2. THE DISPERSION EQUATION

As previously mentioned, we consider a frequency range $\omega \gg \omega_{ci}$, $\omega \gg \omega_{pi}$ such that the motion of the ions can be neglected. The ions simply provide a background of positive charge. We also restrict the analysis to frequencies such that $\omega \ll \omega_{ce}$ and take $\omega_{pe}^2 \ll \omega_{ce}^2$. For this range the motion of the electrons perpendicular to the magnetic field is adequately described by the electron fluid equation. We also use the electron fluid equation to describe the motion parallel to the magnetic field, a procedure which is justified by the final result.

The equations needed for a complete description of the wave are the following:

$$\frac{d\underline{v}}{dt} + \frac{\kappa T}{nm} \underline{\nabla} n + \nu \underline{v} = - \frac{e}{m} \underline{E} - \frac{e}{m} \underline{v} \times \underline{B}, \qquad \ldots (1)$$

$$-e \frac{\partial n}{\partial t} + \underline{\nabla} \cdot \underline{J} = 0, \qquad \ldots (2)$$

$$\underline{\nabla} \cdot \underline{E} = - \frac{e}{\varepsilon_0} (n - n_0), \qquad \ldots (3)$$

where equations (1), (2) and (3) are respectively, the electron fluid equation, the continuity equation and Poisson's equation, in MKS units. We assume that the electric field is derivable from a scalar potential φ,

$$\underline{E} = - \underline{\nabla}\varphi \; . \qquad \ldots (4)$$

The conditions under which this is justified are considered in the appendix.

We choose a system of co-ordinates in which $\underline{B}_o$ is parallel to the axis and the plasma has cylindrical symmetry. Perturbations of quantities about their equilibrium values are assumed to have the following form

$$F = f(r) \exp i(kz + \ell\theta - \omega t) \; . \qquad \ldots (5)$$

We may write the relevant variables of the problem as a sum of a constant part, suffix o and a perturbation, suffix 1 :

$$\underline{v} = \underline{v}_o + \underline{v}_1 \; ,$$

where

$$\underline{v}_o = \frac{-\kappa T}{eB_o} \frac{1}{n_o} \frac{dn_o}{dr} \hat{\iota}_\theta$$

(the diamagnetic drift velocity) and $\hat{\iota}_\theta$ is the unit vector in the θ-direction,

$$n = n_o + n_1$$

$$\underline{E} = \underline{E}_1$$

$$\underline{B} = \underline{B}_o$$

We have assumed that the plasma is neutral in equilibrium and therefore there are no zero-order electric fields. The absence of magnetic field perturbations is a result of equation (4).

In order to derive the dispersion equation we obtain the components of $\underline{v}$ perpendicular and parallel to the magnetic field from equation (1) (neglecting terms of order ν/ω_{ce} and ω/ω_{ce}), substitute them into equation (2) using the definition of $\underline{J}(= - ne\,\underline{v})$ and then use equations (3) and (4) giving*,

$$\nabla^2\varphi + \frac{\left(1+\frac{i\nu}{\omega}\right)}{\left(1+\frac{i\nu}{\omega}-\frac{k^2 v_T^2}{\omega^2}\right)} \frac{e\ell}{\omega\varepsilon_0 B_0}\frac{1}{r}\frac{dn_0}{dr}\varphi + \frac{1}{\left(1+\frac{i\nu}{\omega}-\frac{k^2 v_T^2}{\omega^2}\right)}\frac{k^2 n_0 e^2}{\varepsilon_0 m\,\omega^2}\varphi = 0 \qquad \ldots (6)$$

An equation similar to this has been solved by Barrett, Franklin and Jones (1967). However, they only considered the case $\ell = 0$ when the term in $\frac{dn_0}{dr}$ vanishes. Since we are primarily concerned with the high frequency limit of the low frequency drift waves we are not interested in propagation purely along $\underset{\sim}{B}_0$ and therefore, we consider only non-zero azimuthal mode numbers ℓ. Furthermore, since high ℓ values should be fairly adequately described by slab geometry and since the low ℓ values are of greater relevance to experiment we consider only low values of ℓ (mainly $\pm$ 1).

Equation (6) is the required dispersion equation for the potential φ. We now assume a specific form for the density profile, i.e.

$$n_0(r) = N_0\left(1 - \alpha\frac{r^2}{a^2}\right) = N_0\, f(\tfrac{r}{a}) , \qquad \ldots (7)$$

where α is an arbitary dimensionless parameter whose value lies between 0 and 1. With this form for n_0 and introducing the dimensionless variable $\xi = r/a$ the dispersion equation becomes:

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* Since we have neglected the off-diagonal elements of the pressure tensor in equation (1) we have also neglected the term $\underline{v}_0 \cdot \frac{\partial v_1}{\partial \underline{r}}$ since the contributions from these terms are of the same order of magnitude.

$$L\,\varphi - \frac{\left(1 + \frac{i\nu}{\omega}\right)}{\left(1 + \frac{i\nu}{\omega} - \frac{k^2 v_T^2}{\omega^2}\right)}\, 2\,\ell\alpha\, \frac{\omega_p^2}{\omega|\omega_{ce}|}\,\varphi + \frac{k^2 a^2 \omega_p^2\, f(\xi)}{\omega^2\left(1 + \frac{i\nu}{\omega} - \frac{k^2 v_T^2}{\omega^2}\right)}\,\varphi = 0\,, \qquad \ldots (8)$$

where

$$L \equiv \frac{1}{\xi}\,\frac{d}{d\xi}\left(\xi\,\frac{d}{d\xi}\right) - \frac{\ell^2}{\xi^2} - k^2 a^2$$

and ω_p^2 is the electron plasma frequency referred to the axis of the plasma cylinder.

3. THE BOUNDARY CONDITIONS

We consider two sets of boundary conditions, (i) the plasma is bounded by a perfect conductor, and (ii) the plasma is bounded by a vacuum. For case (i), the requirement that the electric field parallel to the wall should vanish gives the condition

$$\varphi(\xi) = 0 \quad \text{at} \quad \xi = 1 \qquad \ldots(9)$$

When the plasma is bounded by a vacuum the boundary conditions are the following (Stratton (1941))

$$E_z \ \text{and}\ E_\theta \ \text{continuous}$$

and

$$(E_r)_{vac} - (E_r)_p = \frac{\sigma}{\varepsilon_0}\,,$$

where σ is the surface charge density and $(E_r)_{vac}$ and $(E_r)_p$ are the radial components of the electric field in the vacuum and plasma respectively. The surface charge density is obtained in terms of the potential φ by expressing n_1 in terms of φ with the aid of equations (1), (2) and (4) and then integrating the resulting expression for n_1 over a small volume at the surface (Stratton (1941)). There is only a surface charge if the density changes discontinuously at the boundary i.e. the density gradient is infinite at the boundary. The boundary conditions for case (ii) can now be written

$$\varphi_{vac} = \varphi_p \quad \text{at} \quad \xi = 1 , \qquad \text{... (10a)}$$

$$\left.\frac{d\varphi}{d\xi}\,\text{vac.}\right|_{\xi=1} - \left.\frac{d\varphi_p}{d\xi}\right|_{\xi=1} = \frac{\ell\omega_p^2}{\omega|\omega_{ce}|}\;\frac{\left(1-\alpha\right)\left(1+\frac{i\nu}{\omega}\right)}{\left(1+\frac{i\nu}{\omega}-\frac{k^2v_T^2}{\omega^2}\right)}\;\varphi(1) \quad \text{.. (10b)}$$

The term on the RHS of equation (10b) is due to the surface charge and disappears if $\alpha = 1$. This is to be expected since for $\alpha = 1$ the density goes to zero at the boundary and there is no discontinuity. One further point about the surface charge is that it also depends on the azimuthal mode number ℓ. In other words, for a surface charge to occur, not only must the density change discontinuously at the boundary but also there must be a radial flow of electrons. The predominant motion perpendicular to $\underline{B}_0$ is the $\underline{E} \times \underline{B}_0$ drift and for this to have a radial component we require an azimuthal electric field E_θ, i.e. a non-zero azimuthal mode number.

In order to specify the problem completely we must also obtain the solution to equation (8) in vacuum. The solution is

$$\varphi = B\,K_\ell\,(k\,a\,\xi), \qquad \text{... (11)}$$

Where K_ℓ is the modified Bessel function of the second kind of order ℓ and we have rejected the other solution since it diverges at infinity. With the aid of (11) the two conditions (10) can be expressed as one condition

$$\left.\frac{d\varphi_p}{d\xi}\right|_{\xi=1} = -\left\{|\ell| + ka\,\frac{K_{\ell-1}(ka)}{K_\ell(ka)} + \frac{\ell\omega_p^2}{\omega|\omega_{ce}|}\;\frac{(1-\alpha)\left(1+\frac{i\nu}{\omega}\right)}{\left(1+\frac{i\nu}{\omega}-\frac{k^2v_T^2}{\omega^2}\right)}\right\}\varphi_p(1) \qquad \text{... (12)}$$

The solution of equation (8) which satisfies the conditions given by either equation (9) or (12) specifies the relation between ω and k

for perfectly conducting or vacuum boundaries respectively.

4. THE EIGEN-FREQUENCIES

To obtain the eigen-frequencies we must solve equation (8). We were not able to find an analytic solution of this equation in the general case. However, analytic solutions can be found for certain special cases:

4.1 $\underline{k} \cdot \underline{B}_0 = 0$, n_0 = constant

Under these conditions equation (8) reduces to

$$\frac{1}{\xi} \frac{d}{d\xi} \left(\xi \frac{d\varphi}{d\xi} \right) - \frac{\ell^2}{\xi^2} \varphi = 0 \qquad \dots (13)$$

whose solution in the plasma is

$$\varphi_p = A \, \xi^{|\ell|} \qquad \dots (14)$$

where the other solution has been discarded since it diverges at the origin. There are no wave solutions for this case when the plasma is bounded by a perfect conductor. However, when the plasma is bounded by a vacuum, the boundary conditions (10) result in the following dispersion relation

$$\omega = - \frac{1}{2} \frac{\ell}{|\ell|} \frac{\omega_p^2}{|\omega_{ce}|} \qquad \dots (15)$$

where n_0 = constant corresponds to $\alpha = 0$ and k is of course zero. Equation (15) corresponds to azimuthal waves where the eigen-frequency is independent of mode number and electron mass. This azimuthal wave has only one sense of rotation and results in a mode similar to that discussed by Trivelpiece and Gould (1959) for negative ℓ but having a finite cut-off for $k = 0$ instead of passing through the origin as was indicated in Fig.8 of their paper.

4.2 $\underline{k} \cdot \underline{B}_o = 0$, $n_o \neq$ constant

For this case the dispersion relation becomes,

$$\frac{1}{\xi} \frac{d}{d\xi}\left(\xi \frac{d\varphi}{d\xi}\right) - \frac{\ell^2}{\xi^2}\varphi - 2\ell\alpha \frac{\omega_p^2}{\omega|\omega_{ce}|}\varphi = 0 \quad \ldots (16)$$

As for the previous case, there is no solution of equation (16) for $\ell \geqslant 0$, which satisfies the boundary conditions (9) or (12). However for $\ell < 0$ the solution of (16) is

$$\varphi = A\, J_\ell\, (p\xi)$$

where A is a constant and $p^2 = 2|\ell|\alpha\omega_p^2/\omega|\omega_{ce}|$ and J_ℓ is the ℓ^{th} order Bessel function of the first kind. In contrast to the previous case we can now find a wave solution which satisfies the boundary condition (9). If $z_{\ell s}$ is the s^{th} zero of the ℓ^{th} order Bessel function then the eigen frequencies for a perfectly conducting boundary are

$$\omega = \frac{2|\ell|\ \alpha\omega_p^2}{z_{\ell s}^2\ \ |\omega_{ce}|} \quad \ldots (17)$$

where the eigen-frequencies are again independent of the electron mass but where the waves of different azimuthal mode number ℓ are no longer degenerate. Equations (15) and (17) give the condition for neglect of ion motion i.e., $\dfrac{\omega_{pe}}{\omega_{ce}} \gg \sqrt{\dfrac{m_e}{m_i}}$.

Comparing this wave with the low frequency drift wave $(\omega \ll \omega_{ci})$ which propagates perpendicularly to $\underline{B}_o$ we notice that whereas the phase velocity of the low frequency wave is proportional to the temperature of the ions (Mikhailovskii (1967)) the phase velocity of the high frequency wave is independent of the electron temperature. The high frequency wave also propagates in the same direction as the low frequency drift wave i.e. opposite to the electron diamagnetic drift and the ratio of the phase velocity of the high frequency wave to the diamagnetic drift velocity

$$\left|\frac{v_p}{v_o}\right| = \frac{1}{z^2_{\ell s}} \frac{a^2}{\lambda^2_D} \qquad \ldots (18)$$

where λ_D is the electron Debye length. i.e. $|v_p| \gg |v_o|$

We can obtain a similar solution to equation (17) for the case when the plasma is bounded by a vacuum except that the quantities $z_{\ell s}$ are replaced by $y_{\ell s}$ (say) which were obtained graphically. Comparing the solutions for a vacuum boundary for n_o constant or varying we see that the effect of the density gradient is to split the degeneracy of the different azimuthal modes.

To compare the eigen-frequencies for perfectly conducting and vacuum boundary conditions the ratio of the eigen-frequencies has been tabulated against ℓ. ω refers to a vacuum boundary and ω' to a perfectly conducting boundary. The large factor between the two sets of results can be understood by the result for a constant density when ω is finite for a vacuum boundary and zero for a perfectly conducting boundary. In Fig.1, ω has been plotted against α. The different dependence on α is due to the fact that there is a surface charge at the vacuum boundary but not at the perfectly conducting boundary.

$-\ell$	ω/ω'
1	5·8
2	4·6
3	4·4
4	4·5

for $\alpha = 0{\cdot}5$ and $\dfrac{\omega_p}{\omega_{ce}} = 0{\cdot}1$

4.3 $\underline{k} \cdot \underline{B}_0 \neq 0$

For a constant density, equation (8) is identical with the dispersion equation of Trivelpiece and Gould (1959) for $\omega \ll \omega_{ce}$ and $\omega_{pe}^2 \ll \omega_{ce}^2$. Thus our analysis contains the Trivelpiece and Gould modes as a special case*.

When the density variation is included analytic solutions of equation (6) were only obtained for special density profiles and then for only one frequency. These solutions provided useful checks for the numerical solutions.

The eigen-frequencies for the general case were obtained numerically using a programme written by McNamara (1966). The method employed was to express the differential equation and the boundary conditions as a homogeneous set of linear algebraic equations (using the method of finite differences). Setting the determinant of this set of equations equal to zero specifies a function whose complex zeros approximate the required eigen-values.

Figure 2 shows the result of computations for the case when collisions are neglected and the frequencies are thus all real. The curves for both vacuum and perfectly conducting boundary conditions are given together for comparison, both for $|\ell| = 1$. The effect of the density gradient is to split the ± 1 modes even for perfectly conducting boundaries (for a uniform plasma these modes are degenerate). The occurrence of two branches to the dispersion diagram for finite ka and given $|\ell|$ is analogous to the two branches which occur for the

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*The boundary condition used by Trivelpiece (1959) is equivalent to ours since the term $n\varepsilon_2$ in equation (11) of that paper is identical with the term we obtain through allowance for a surface charge.

low frequency drift wave for finite ka and $c_s \ll c_A$ (Mikhailovskii 1967), Kadomtsev (1965)). However, for the high frequency wave the lower branch has the same sense of rotation as the electron diamagnetic drift whereas for the low frequency wave the upper branch rotates with the electrons. Figure 2 shows that the phase velocity for both branches and both boundary conditions is much greater than the electron thermal velocity ($\omega/k \geqslant 20\, v_T$). This is just the condition for Landau damping to be negligible and therefore we were justified in using the electron fluid equation to describe the electron motion along $\underline{B}_0$. Fig.3 shows the $\ell = -1$ mode for a vacuum boundary for three different density profiles. The profile for $\alpha = 1$ gives no surface charge and results in the eigen-frequencies being lower. In the experiment (Harding and Pigache (1968)) designed to check the theory of this high frequency wave it was found that the density profile corresponded most nearly to $\alpha = 1$. Consequently the remaining results given will be for this profile.

The curves in Figs.4-5 show the dependence on the parameter ω_p/ω_{ce} for the $\ell = \pm 1$ modes for both vacuum and perfectly conducting boundaries. The -1 curves become flatter as ω_p/ω_{ce} increases due to the fact that for $ka = 0$ the wave frequency is proportional to ω_p^2 whereas for large ka, the effect of the density gradient is small and ω tends to become proportional to ω_p. In contrast to the -1 branch, which is very sensitive to the boundary conditions for long axial wavelengths, the $+1$ branch is not very dependent on these conditions since $\omega \to 0$ as $ka \to 0$ for both perfectly conducting and vacuum boundaries.

5. ATTENUATION

The attenuation of the high frequency wave was calculated by including the effect of collisions on the motion of the electrons along the magnetic field. The damping of the motion across the magnetic field was neglected but this was justified since it is proportional to ν/ω_{ce} whereas the former damping is proportional to ν/ω.

The curves of Imk/Rek (Figs. 6-7) show that the attenuation per wavelength is proportional to the collision frequency for $ka \sim 1$ for both ± 1 curves. In this region the damping of the two branches differs by only a small amount. The biggest difference between the two branches occurs at small values of ka which is the region where the two branches are most widely separated. The attenuation of the $+1$ curve is always greater than the -1 curve. This is to be expected since for a fixed wavelength ν/ω is larger for the $+1$ branch.

6. CONCLUSION

The high frequency electrostatic wave described in this paper requires the presence of an axial magnetic field. The existence of azimuthal waves for negative mode numbers has been demonstrated, even for the case of a uniform plasma. In a uniform plasma these waves can only exist in the presence of a surface charge due to the discontinuous change in the plasma density at the plasma vacuum interface. In general, these azimuthal waves exist in the absence of surface charge provided there is a radial density gradient. (In fact, the azimuthal waves for a uniform plasma are due to the infinite density gradient at the surface).

When $ka \lesssim \omega/c$ the $\ell = 0$ and positive ℓ modes can no longer be described by the electrostatic approximation. However, the negative ℓ modes retain their electrostatic character even in the limit of $ka \to 0$.

The electron temperature is found to have very little influence on these high frequency waves but its effect was included because of its importance for the low frequency drift waves. The main similarity with the low frequency waves is that in the electrostatic approximation there are two branches for a given $|\ell|$. However, the azimuthal phase velocity of the high frequency waves is much greater than the electron diamagnetic drift velocity and the lower branch rotates in the same sense as the electron drift in contrast to the low frequency drift waves (Kadomtsev (1965)). Both perfectly conducting and vacuum boundary conditions are considered and for $ka = 0$ the difference between these two cases is very large for negative ℓ values. This is due to the fact that for a uniform plasma the azimuthal waves can not exist for a perfectly conducting boundary. This fact may be of significance for the low frequency drift waves.

Finally, Mikhailovskii and Pashitskii (1966) have shown that there is a high frequency instability in the range of frequencies we have considered when $\eta \gg 1$ where $\eta = \dfrac{d\, \ell n\, T}{d\, \ell n\, n_0}$. However, for small temperature gradients the high frequency wave described above is stable.

7. ACKNOWLEDGEMENTS

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APPENDIX

THE ELECTROSTATIC APPROXIMATION

The analysis given above was greatly simplified by neglecting perturbations of the uniform magnetic field and assuming

$$\underline{E} = -\underline{\nabla}\, \varphi$$

At low frequencies it is the Alfven mode which perturbs the magnetic field. However, at high frequencies, this mode can be neglected since the ion motion is negligible. Instead, we must show under what conditions the helicon mode can be neglected since it is this mode which will cause the greatest magnetic field perturbations. Introducing the electromagnetic potentials $\underline{A}$ and φ by the relations

$$\underline{E}_1 = -\underline{\nabla}\varphi + i\omega\underline{A} \qquad \text{... A.1}$$

$$\underline{H}_1 = \frac{1}{\mu_o}\underline{\nabla} \times \underline{A} \qquad \text{... A.2}$$

and choosing the Lorentz gauge

$$\underline{\nabla} \cdot \underline{A} - i\omega\mu_o\varepsilon_o\varphi = 0 \qquad \text{... A.3}$$

we obtain

$$\underline{A} = \frac{\mu_o \underline{J}_1}{(k^2 + \omega^2/c^2)} \qquad \text{... A.4}$$

where $\qquad k^2 = k_\perp^2 + k_z^2$.

(NB: The spatial components of the gradient operator perpendicular to $\underline{B}_o$ have been approximated by $i\,\underline{k}_\perp$ since we are only interested in comparing the magnitudes of various terms.)

To discover the importance of magnetic perturbations we simply compare the part of $\underline{E}_1$ which arises from these perturbations with the total electric field. For the field perpendicular to $\underline{B}_o$ we need to evaluate

$$\left|\frac{i\,\omega\,\underline{A}_\perp}{\underline{E}_\perp}\right| \qquad \text{... A.5}$$

From equation A.4, equation (1) and $\underline{J} = -\,ne\underline{v}$ we obtain:

$$\frac{\omega|\underline{A}_\perp|}{|\underline{E}_\perp|} = \frac{\omega_p^2}{c^2(k^2+\omega^2/c^2)}\;\frac{\omega}{|\omega_{ce}|} + \frac{\beta}{2}\,\frac{\omega|\omega_{ce}|}{\omega_p^2}\;\frac{k^2}{k^2+\omega^2/c^2} \quad \text{.. A.6}$$

where the second term was obtained assuming $\underline{E} = -\,\underline{\nabla}\varphi$. We see immediately from the first term on the RHS of A.6 that, as expected, for helicon waves

$$\omega\,\frac{|\underline{A}_\perp|}{|\underline{E}_\perp|} \approx 1$$

since for this case

$$k^2 \simeq \frac{\omega_p^2}{c^2}\;\frac{\omega}{|\omega_{ce}|} \gg \frac{\omega^2}{c^2}$$

when $\omega \ll |\omega_{ce}|$ and $\dfrac{\omega_p^2}{\omega|\omega_{ce}|} \gg 1$

Note also that this term is independent of β where $\beta \equiv \dfrac{n_o \kappa T}{\frac{1}{2} B_o^2/\mu_o}$

Thus, for the neglect of helicons we require $\dfrac{\omega_p^2}{\omega_{ce}^2} \ll \dfrac{c^2k^2}{\omega|\omega_{ce}|}$ and for the second term to be negligible $\beta \ll 1$. For the azimuthal waves ($k_z = 0$) to be electrostatic $k_\perp^2 \gg \dfrac{\omega^2}{c^2}$ and $\omega_p \lesssim ck_\perp$.

Finally, we compare the E_z fields using equations (1), (2) and A.4 we obtain

$$\left|\frac{\omega A_z}{E_z}\right| = \frac{\omega_p^2}{c^2k^2}\;\frac{\left(\dfrac{\omega^2}{k_z^2 v_T^2} + \dfrac{\omega}{|\omega_{ce}|}\,\dfrac{\underline{K}_n\cdot\underline{k}_\perp}{k_z^2}\right)}{\left\{1 - \dfrac{\omega^2}{k_z^2 v_T^2}\right\}}$$

where collisions have been neglected, $\underline{K}_n \equiv \dfrac{1}{n_o}\underline{\nabla}\, n_o$ and $(\underline{E}_\perp \times \underline{B}_o)/E_z$ was approximated by $(\underline{k}_\perp \times \underline{B}_o)\,\varphi/k_z\varphi$. Since $\omega/k_z \gg v_T$ the final condition for the validity of the electrostatic approximation is

$$\omega_p^2 \ll c^2\, k^2$$

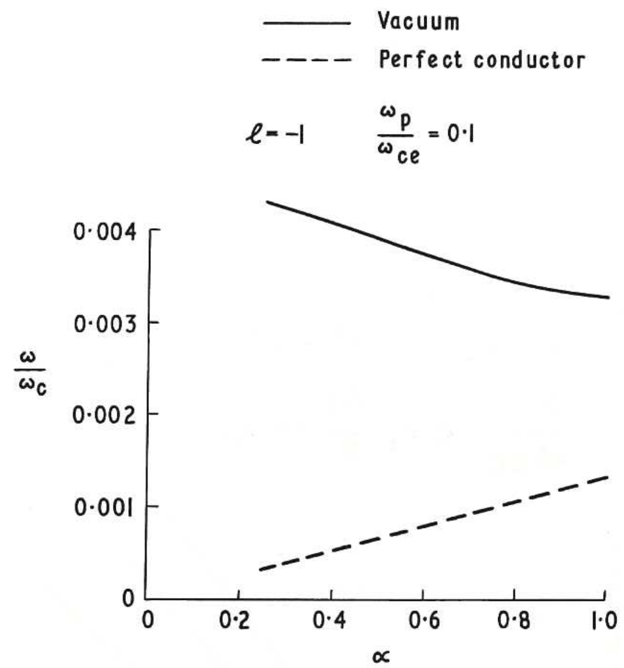

Fig. 1 (CLM-P 186)
Dependence of eigen frequencies of high frequency electrostatic wave on α for $\underline{k}.\underline{B}_0 = 0$ and $\ell = -1$. The full line ——— is for a vacuum boundary and the dotted line - - - - - for a perfectly conducting boundary

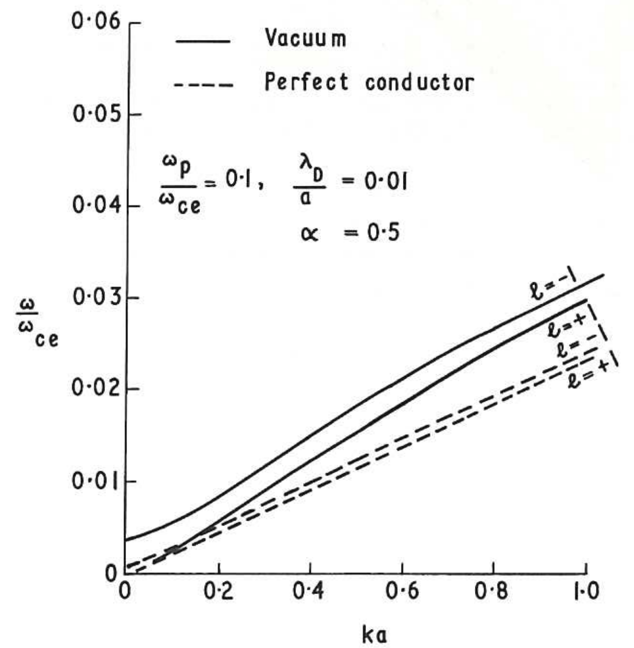

Fig. 2 (CLM-P 186)
Computed dispersion relation for high frequency electrostatic wave for $\underline{k}.\underline{B}_0 \neq 0$. The full line ———— is for a vacuum boundary and the dotted line - - - - - for a perfectly conducting boundary

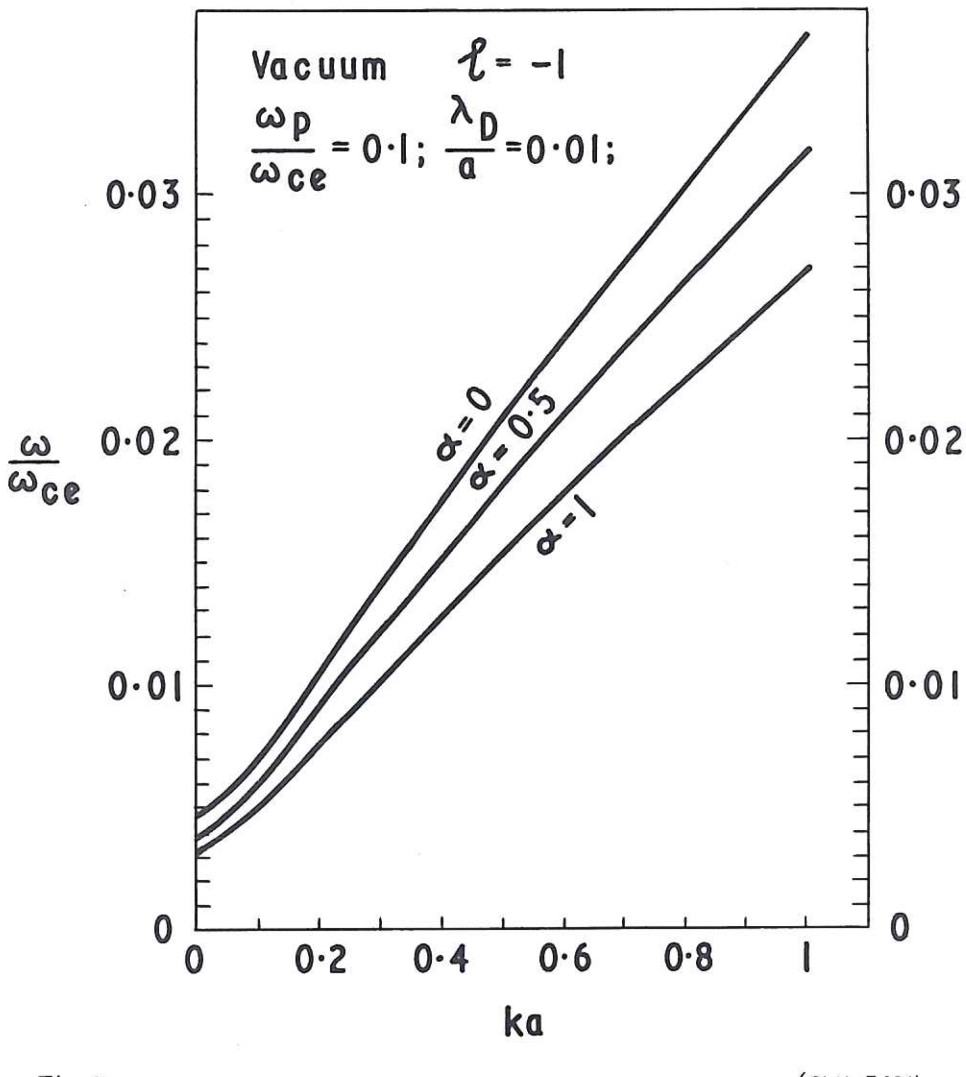

Fig. 3 (CLM-P 186)

Computed dispersion relation for $\ell = -1$ branch of high frequency wave and $\underline{k} \cdot \underline{B}_0 \neq 0$. The three curves are for different equilibrium density profiles and vacuum boundary

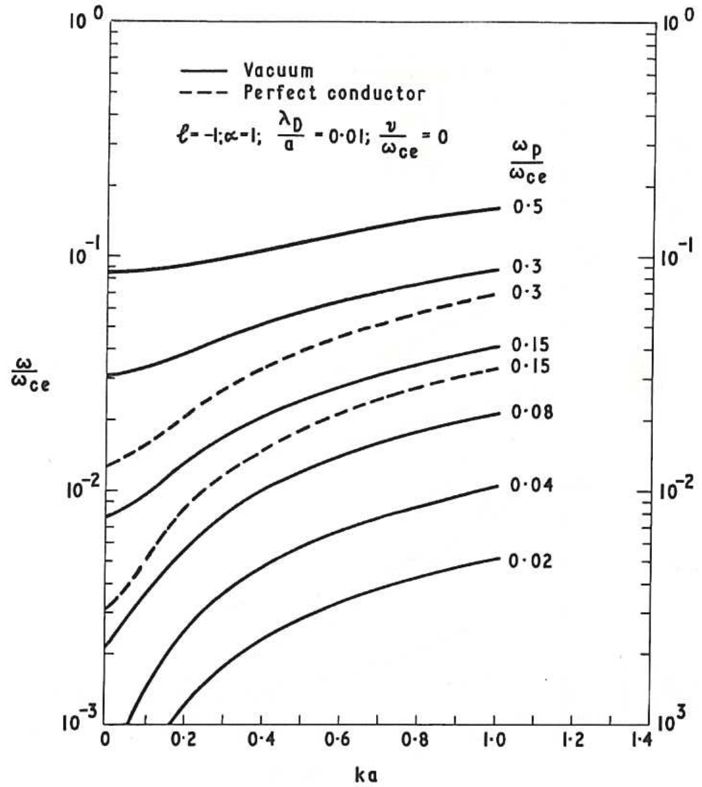

Fig. 4 (CLM-P 186)

Computed dispersion relations for $\underline{k}.\underline{B}_0 \neq 0$ and different values of ω_p/ω_{ce} for $\ell = -1$ branch. The full line ——— is for a vacuum boundary and the dotted line - - - - - for a perfectly conducting boundary

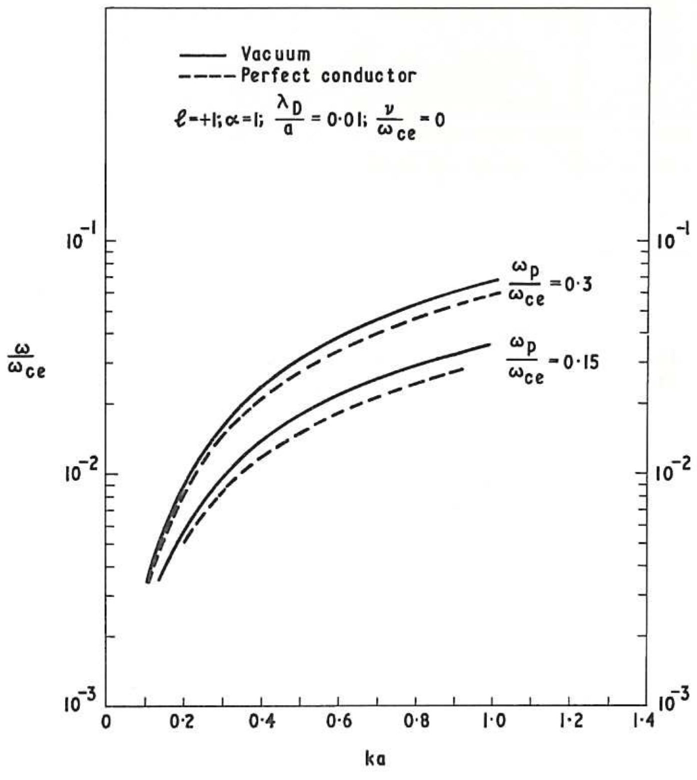

Fig. 5 (CLM-P 186)

Computed dispersion relations for $\underline{k}.\underline{B}_0 \neq 0$ and different values of ω_p/ω_{ce} for $\ell = +1$ branch. The full line ——— is for a vacuum boundary and the dotted line - - - - - for a perfectly conducting boundary

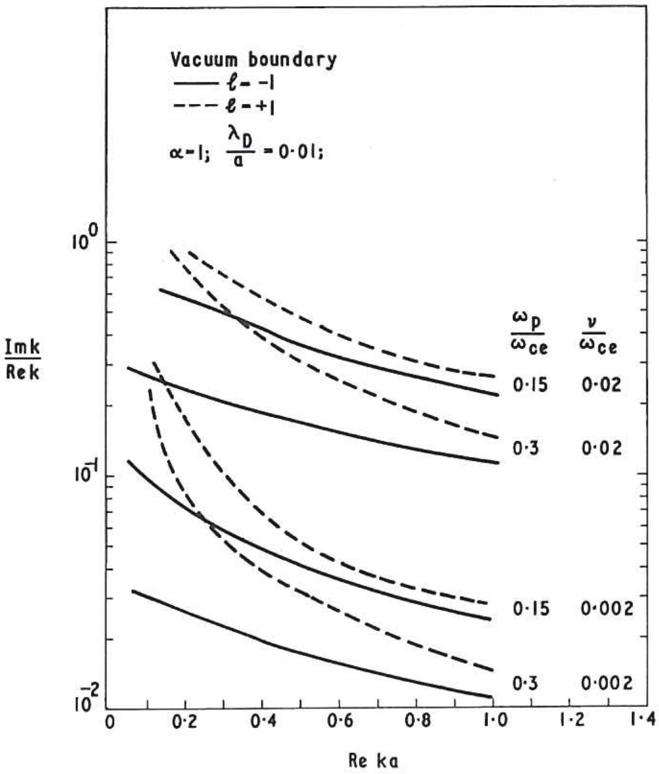

Fig. 6 (CLM-P 186)

Computed attenuation curves for various values of ω_p/ω_{ce} and ν/ω_{ce} for a vacuum boundary. The full line ——— is for $\ell = -1$ and the dotted line - - - - - is for $\ell = +1$

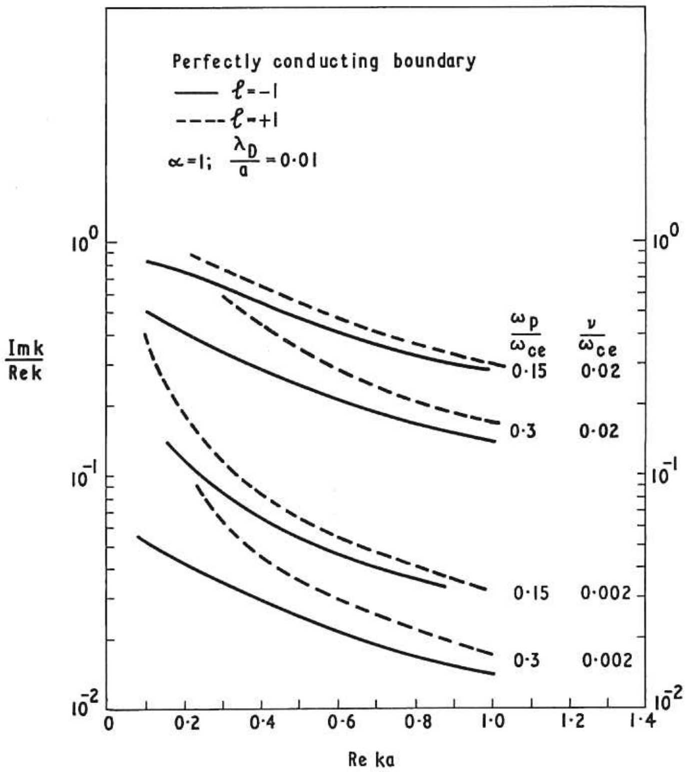

Fig. 7 (CLM-P 186)

Computed attenuation curves for various values of ω_p/ω_{ce} and ν/ω_{ce} for a perfectly conducting boundary. The full line ——— is for $\ell = -1$ and the dotted line - - - - - for $\ell = +1$