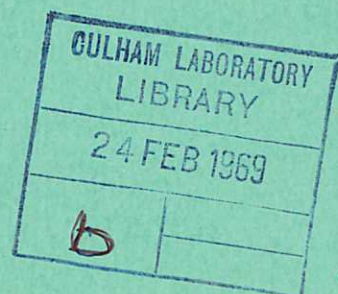
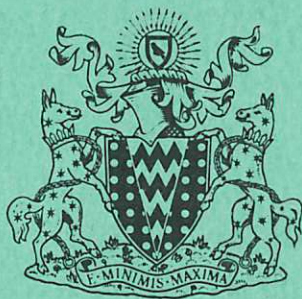


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DYNAMIC STABILISATION OF THE THETA-PINCH

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1968

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DYNAMIC STABILISATION OF THE THETA-PINCH

by

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A B S T R A C T

Previous work has shown the $\beta = 1$ theta-pinch to be stable for the mode $m = 1$ and unstable for $m > 1$. In this paper it is shown that a periodic theta-pinch having a radius $R(z) = R_0(1 + \delta f(z))$, where $\delta \ll 1$, can be dynamically stabilised for $m \ll 1/\delta$.

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1. INTRODUCTION

As a result of the unavoidable end losses from a straight theta-pinch, estimates of the length of a possible thermonuclear power reactor are of the order of several hundred metres. It is possible to remove this loss and to reduce the length by forming a toroidal pinch. The conditions for such toroidal equilibria have been described by Meyer and Schmidt⁽¹⁾, and Morse, Riesenfeld and Johnson⁽²⁾. In such a system the plasma surface has a corrugated form and therefore stabilising and destabilising regions alternate. This configuration is rather complicated theoretically and it is useful to consider a linear analogue. This is provided by a straight axisymmetric pinch in which the pinch radius varies periodically along its length thus introducing regions of favourable and unfavourable curvature.

This linear configuration has been studied theoretically in some detail. The most important result⁽³⁾ is that such a configuration is unstable to a given mode $m(>0)$ if $\beta < 1 / \left(1 + (R/R_w)^{2m} \right)$ everywhere along its length where R and R_w are the radii of the plasma and the surrounding conducting wall. In practice it is almost impossible to avoid the instability regime for $m > 1$ and the condition is very stringent even for $m = 1$.

The purpose of this paper is to describe in detail a method of stabilising low m modes for a $\beta = 1$ plasma, the principle of which was outlined in an earlier publication⁽⁴⁾. The method may be described as follows. Consider a linear pinch with a periodic external magnetic field which produces a surface profile given by $R = R_0(1 + \delta f(z))$, where $f(z)$ is a periodic function and $\delta \ll 1$. In the absence of wall stabilisation this system is unstable for $m > 1$ and marginally stable for $m = 1$ ⁽⁵⁾. Now consider a system in which this magnetic field configuration and surface profile are made to propagate along the pinch with a velocity V_w . It will be shown that if $|V_w| > |V_A|$ then modes $m \ll 1/\delta$ are stable, where $V_A^2 = B^2/\rho$, B being the mean value of the magnetic field at the surface of the plasma and ρ the plasma density.

In order to demonstrate this result it is convenient to consider the problem in the frame of the propagating wave. In this frame the magnetic field is static and the plasma appears to flow. This configuration forms a stationary state. For a given plasma profile, described by specifying the plasma radius R as a function of the axial coordinate z , it is then straightforward to determine the magnetic field and the plasma velocity field.

We shall consider systems in which the variations of the plasma radius along the length of the pinch occur over distances large compared to the radius. For simplicity we

assume an incompressible hydromagnetic model for the plasma. We note that in the analysis of the static pinch the most unstable modes were found to be incompressible.

In Section 2 we consider the propagation of the wave along the pinch. This is done using an expansion based on the small quantity $\varepsilon \sim R \frac{\partial}{\partial z}$. The wave is described by specifying the radius of the plasma as a periodic function of z and t . In the frame of the wave, whose velocity is V_w , R is a function of z only and in leading order the axial flow velocity V is then given by the flux conservation equation

$$R^2 V = \text{constant}.$$

The magnetic field at the surface of the plasma is given by the modified Bernoulli equation

$$V^2 + B^2 = \text{constant},$$

where the density has been taken as unity and we use rationalised units. The radial velocities and magnetic fields are determined in terms of V and B and therefore of R .

In Section 3 the equation for linearised perturbations for which $R \frac{\partial}{\partial z} \ll 1$, is derived. This is done by linearising the modified Bernoulli equation for the plasma surface and applying the surface boundary conditions to the solutions for the perturbed plasma velocity and magnetic field. The equation is derived in terms of the radial surface displacement of the plasma, ξ , and may be written in the form

$$\begin{aligned} \frac{d}{dz} \left[(B^2 - V^2) R^2 \frac{d\xi}{dz} \right] - 2i\omega R^2 V \frac{d\xi}{dz} \\ + \left[\omega^2 R^2 - 4 \left(2 + \frac{V^2}{B^2} \right) V^2 \left(\frac{dR}{dz} \right)^2 \right. \\ \left. - R \frac{d^2 R}{dz^2} \left\{ (m-1) B^2 + (m-3) V^2 \right\} \right] \xi = 0. \end{aligned} \quad \dots (1)$$

For a specified plasma profile $R(z)$ and the 'Alfvén speed', B , and flow velocity, V , given at some value of z , there will be an infinite set of mode numbers m and each of these will have an infinite number of axial modes corresponding to solutions of Eq.(1).

In Section (4) we consider the stability of a periodic plasma defined by $R = R_0(1 + \delta f(z))$ where $1 \gg \delta \gg \varepsilon$. The case of small δ is considered and the stability of modes $m \sim \delta^0$ is determined by an expansion in δ about the cylindrical system $\delta = 0$. The normal modes of this system have the form e^{ikz} and since there is no destabilising mechanism it is positively stable for all $k > 0$ and marginally stable for $k = 0$. Now, for the type of problem we are considering the eigenvalue ω is given by

an equation of the form

$$L\omega^2 + 2M\omega + N = 0$$

where L , M and N are real. Thus

$$\omega = \frac{-M \pm (M^2 - LN)^{1/2}}{L}.$$

The necessary and sufficient condition for stability for a given mode is, therefore, that the discriminant

$$\Delta = M^2 - LN > 0.$$

For a given mode, Δ is expanded in various small parameters and we note that the first non-zero term in the expansion determines the stability of that mode, since the addition of higher order terms cannot affect the sign of Δ .

Now, for a cylinder $\Delta > 0$ for $k > 0$ and $\Delta = 0$ for $k = 0$. For finite δ and $kR \gg \delta$ we can expand Δ in δ and it is clear from the argument just given that these modes will be stable to all orders in δ . Thus, referring to Fig.1, the region marked 1 on the (kR, δ) diagram is stable.

We next investigate $kR \sim \delta$, that is region 3 in Fig.1, by considering the larger regime $1 \gg kR \gg \epsilon$. With this ordering Eq.(1) becomes

$$(B^2 - V^2) \frac{d^2 \xi}{dz^2} - 2i\omega V \frac{d\xi}{dz} + \omega^2 \xi = 0.$$

If R were a constant this equation would describe the stable waves of a cylindrical pinch, their frequency being given by $\omega_{\pm} = -k(V \pm B)$. With R a function of z , then to leading order in ϵ , ω_{\pm} is given by an average of this frequency along the pinch and is still real. Since Δ is given by $\frac{L^2}{4} (\omega_+ - \omega_-)^2$ it is seen that $\Delta > 0$ to leading order in ϵ and therefore to all orders in ϵ . Thus for $1 \gg kR \gg \epsilon$ the system is stable to all orders in ϵ for all δ .

It is seen from Fig.1 that the two regimes shown to be stable, namely $kR \gg \delta$ and $1 \gg kR \gg \epsilon$ overlap. Furthermore the region $kR \ll \delta$ overlaps $1 \gg kR \gg \epsilon$ so that the stability of the $kR \ll \delta$ region determines the stability of the system. This is not surprising since it is this region which is unstable in the case of a static plasma.

Thus we next consider $kR \ll \delta$ and expand Δ first in k and then in δ . The leading term is of order $k^0 \delta^2$ and is given by

$$\Delta_{0,2} = AT \int_0^\lambda \left(\frac{dR}{dz} \right)^2 dz$$

where A is a positive constant and

$$T = 4\left(\frac{V}{B}\right)^4 - \left\{ (m-1) + (m-3) \left(\frac{V}{B}\right)^2 \right\} - \frac{\left\{ (m-1) + (m-3) \left(\frac{V}{B}\right)^2 \right\}^2}{1 - \left(\frac{V}{B}\right)^2},$$

where V and B are zero-order quantities in δ and therefore constants.

T may also be written in the form

$$T = \frac{1}{4} \left[\left(4\left(\frac{V}{B}\right)^2 - \frac{1}{8} \right)^2 + \frac{63}{64} \right] - \frac{1}{1 - \left(\frac{V}{B}\right)^2} \left[(m-1) + (m-3) \left(\frac{V}{B}\right)^2 + \frac{1}{2} \left(1 - \left(\frac{V}{B}\right)^2 \right) \right]^2.$$

It is clear that if $V^2 > B^2$ then $T > 0$, $\Delta_{0,2} > 0$ and therefore $\Delta > 0$ to all orders in δ . Thus for $kR \ll \delta$ the system is stable to all orders in δ and ε .

This result therefore shows that for $\delta \gg \varepsilon$ the plasma is stable to $m \ll 1/\delta$ provided $V^2 > B^2$. In physical terms this means that a static plasma with a periodic profile, for which all modes $m > 1$ would be unstable, will be stable for $m \ll 1/\delta$ if the profile is made to propagate along the plasma as a wave with velocity greater than the 'Alfvén speed'.

Owing to the simplifying assumptions made and the neglect of certain modes the present proposal must be regarded as tentative. In particular, although all low m -modes which were previously unstable appear to have been stabilised it is possible that new modes may have been introduced. Because of the assumed sharp boundary the higher m -modes, which have not been considered, do not fall within the practical range of validity of the present model. It is probable that these modes are not important as they are not observed experimentally in the static configuration⁽⁶⁾.

The result of this paper suggests that for a toroidal theta-pinch the equilibrium profile, necessarily periodic and therefore unstable, may be stabilised if the profile is made to propagate round the torus with sufficiently high velocity.

2. DESCRIPTION OF THE WAVE

In this Section we consider the variation of the properties of the pinch as an azimuthally symmetric wave is made to propagate along it. The plasma is taken to be axisymmetric, perfectly conducting and incompressible. The assumption of incompressibility is made to simplify the calculation but we note that the worst modes in the static system were found to be incompressible. The plasma is completely separated from the confining

field which has no azimuthal component.

The wave is defined by specifying the plasma radius R by

$$R = R_0 (1 + \delta f(z - V_w t)) ,$$

where R_0 and δ are constants and f is a periodic function of period λ . Such a wave may be produced by alternating the current in the coils producing the magnetic field with a frequency V_w/λ and with an axial wavelength λ . The wave is most simply described by moving to the frame in which the wave profile R is static, and the rest of the calculation will be carried out in this frame. We then have

$$R = R_0 (1 + \delta f(z))$$

and the plasma and magnetic field appear to be in a stationary state, the plasma now having a flow velocity in the opposite direction to that of the wave. It is now necessary to determine the magnetic field and plasma flow velocity in terms of R .

The physical quantities are expanded in the small parameter $\varepsilon \sim R/\lambda^{(3),(5)}$. It is assumed that $R_w/\lambda \ll 1$ where R_w is the coil radius so that $\varepsilon \ll 1$. Thus we write for the magnetic field

$$\underline{B} = \underline{B}_0 + \underline{B}_1 + \dots$$

where the subscripts indicate the ε order.

In zero order $\underline{\nabla} \cdot \underline{B} = 0$ gives

$$B_{r0} = \frac{h(z)}{r}$$

where $h(z)$ is an arbitrary function of z . Since, at the interface, we have

$$\frac{dR}{dz} = \frac{B_r(R)}{B_z(R)} \sim \varepsilon , \quad \dots (2.1)$$

it follows that $B_{r0}(R) = h(z)/R = 0$ and hence $B_{r0}(r, z) = 0$.

From $\underline{\nabla} \times \underline{B} = 0$ we have

$$B_{z0} = B_{z0}(z) \quad \text{and} \quad B_{z1} = B_{z1}(z) .$$

To first order $\underline{\nabla} \cdot \underline{B}$ gives

$$B_{r1} = -\frac{1}{2} r \frac{dB_{z0}}{dz} + \frac{g(z)}{r} \quad \dots (2.2)$$

where $g(z)$ remains to be determined.

The motion of the plasma is taken to be irrotational and since we assume incompressibility

$$\underline{\nabla} \times \underline{V} = \underline{\nabla} \cdot \underline{V} = 0 .$$

In zero order $\underline{\nabla} \cdot \underline{V} = 0$ gives $V_{r0} = H(z)/r$ so that $V_{r0} = 0$. From $\underline{\nabla} \times \underline{V} = 0$ we have

$$V_{z0} = V_{z0}(z) \quad \text{and} \quad V_{z1} = V_{z1}(z) .$$

To first order $\underline{\nabla} \cdot \underline{V} = 0$ gives

$$V_{r1} = -\frac{1}{2} r \frac{dV_{z0}}{dz} . \quad \dots (2.3)$$

Eqs.(2.1), (2.2) and (2.3) together with

$$\frac{dR}{dz} = \frac{V_r(R)}{V_z(R)}$$

now gives

$$g = \frac{1}{2} R^2 B_{z0} \left(\frac{1}{B_{z0}} \frac{dB_{z0}}{dz} - \frac{1}{V_{z0}} \frac{dV_{z0}}{dz} \right) . \quad \dots (2.4)$$

From the conservation of flow we have

$$\int_0^R r V_z(r, z) dr = \text{constant}$$

or to leading order

$$R^2 V_{z0} = \text{constant} . \quad \dots (2.5)$$

Since pressure balance at the interface is given by

$$p(R(z)) = \frac{1}{2} B_{z0}^2 (R(z)) ,$$

we obtain a modified form of Bernoulli's equation

$$V_{z0}^2 + \frac{1}{\rho} B_{z0}^2 = \text{constant} . \quad \dots (2.6)$$

Thus, for specified values of the constants in Eqs.(2.5) and (2.6), V_{z0} and B_{z0} are determined as functions of R .

3. THE PERTURBED MOTION

Having described the propagating wave as a stationary state in the frame of the wave, we shall in this Section consider perturbations to it and obtain an equation for the

perturbed motion of the interface. It will be shown in the next Section that, for the modes of interest, the perturbed motion of the plasma is irrotational. We shall also find that we only need the equation of motion for the case in which the variation of perturbed quantities along z is slow, so that $R \frac{\partial}{\partial z} \sim \epsilon \ll 1$. The quantity ϵ as defined here is not necessarily the same as that for the equilibrium quantities. However it is convenient to use the same symbol for both cases and to keep terms to the same order in both quantities. If then conditions are chosen such that the two ϵ 's are of the same order the equations will be valid. If, on the other hand, conditions are such that in a particular equation any terms are much smaller than the others the remaining equation will still be correct. The case of $R \frac{\partial}{\partial z}$ not much less than unity will be discussed in the next Section. Since the plasma is incompressible it is convenient to set $\rho = 1$.

The velocity potential ϕ satisfies the ordered Laplace equation

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right) - \frac{m^2}{r^2} \phi = 0,$$

where perturbations are taken to vary as $\exp i(m\theta + \omega t)$ and we only consider $m > 0$. We shall take $m \sim \epsilon^0$ and we note that for higher m the present sharp boundary model would not be appropriate. The bounded solution is

$$\phi = A(z)r^m \quad \dots (3.1)$$

and hence

$$v'_r = v'_r(R) \left(\frac{r}{R} \right)^{m-1}$$

and

$$v'_z(R) = \frac{R}{m} \left(\frac{dv'_r}{dz} - \frac{m-1}{R} \frac{dR}{dz} v'_r \right)_{r=R}, \quad \dots (3.2)$$

where $\frac{d}{dz}$ means the total derivative with respect to z of a quantity evaluated on the surface $r=R$.

Similarly the scalar magnetic potential has the form

$$\psi = C(z)r^{-m}$$

giving

$$B'_r = B'_r(R) \left(\frac{R}{r} \right)^{m+1},$$

and

$$B'_z(R) = - \frac{R}{m} \left(\frac{dB'_r}{dz} + \frac{m+1}{R} \frac{dR}{dz} B'_r \right)_{r=R}. \quad \dots (3.3)$$

We note that $v'_z \sim \epsilon v'_r$ and $B'_z \sim \epsilon B'_r$.

The normal velocity of the surface, u' , is given by

$$u' = \underline{n}' \cdot \underline{V} + \underline{n} \cdot \underline{\tilde{V}}, \quad \dots (3.4)$$

where

$$\underline{\tilde{V}} = \underline{V}' + \underline{\xi} \cdot \underline{\nabla} V,$$

and \underline{n} is the unit vector normal to the surface. Its perturbed z component is given by $n'_z = -\frac{d\xi}{dz}$ where $\xi = \xi_r(R)$ and n'_r is of order ϵ^2 .

Writing Eq.(3.4) in its explicit form, we have

$$u' = i\omega\xi + O(\omega\epsilon^2\xi) = n'_r V_{r1} + n'_z V_{z0} + n_r v'_r + n_z v'_z + n_r \xi_r \frac{\partial V_{r1}}{\partial r}.$$

Retaining leading order terms the equation becomes

$$v'_r = \left(i\omega + V_{z0} \frac{d}{dz} - \frac{\partial V_{r1}}{\partial r} \right) \xi. \quad \dots (3.5)$$

Similarly, since the surface must remain a stream surface,

$$\underline{n}' \cdot \underline{B} + \underline{n} \cdot \underline{\tilde{B}} = 0, \quad \dots (3.6)$$

where

$$\underline{\tilde{B}} = \underline{B}' + \underline{\xi} \cdot \underline{\nabla} B,$$

so that

$$n'_r B_{r1} + n'_z B_{z0} + n_r B'_r + n_z B'_z + n_r \xi_r \frac{\partial B_{r1}}{\partial r} = 0.$$

Thus to leading order

$$B'_r = B_{z0} \frac{d\xi}{dz} - \frac{\partial B_{r1}}{\partial r} \xi. \quad \dots (3.7)$$

Since we are considering irrotational motion Bernoulli's equation can be applied to the surface of the plasma which must be a stream surface, thus

$$-\frac{\partial \phi}{\partial t} + p + \frac{1}{2} V^2 = 0.$$

Linearising and using the pressure balance relation

$$p' = \underline{B} \cdot \underline{\tilde{B}}$$

the modified Bernoulli equation is

$$-\frac{\partial \varphi}{\partial t} + \underline{B} \cdot \underline{\tilde{B}} + \underline{V} \cdot \underline{\tilde{V}} = 0 .$$

We now consider the orders of the terms which comprise this equation. It can be written in the form

$$\begin{aligned} -i\omega\varphi' + B_{z0} (B'_z + \xi_r \frac{\partial B_{r1}}{\partial z}) + V_{z0} (v'_z + \xi_r \frac{\partial V_{r1}}{\partial z}) \\ \dots (3.8) \\ + B_{r1} (B'_r + \xi_r \frac{\partial B_{r1}}{\partial r}) + V_{r1} (v'_r + \xi_r \frac{\partial V_{r1}}{\partial r}) = 0 . \end{aligned}$$

From Eqs.(3.5) and (3.7) we observe that $\omega R \sim \varepsilon V_{z0}$ and $B'_r \sim \varepsilon \xi$ respectively.

Using the equation for the total derivative at the surface,

$$\frac{dy(R(z))}{dz} = \left(\frac{\partial y}{\partial z} + \frac{dR}{dz} \frac{\partial y}{\partial r} \right)_{r=R} ,$$

and the surface equation

$$\frac{dR}{dz} = \frac{B_{r1}}{B_{z0}} = \frac{V_{r1}}{V_{z0}} , \quad \dots (3.9)$$

Eq.(3.8) becomes

$$\begin{aligned} i\omega\varphi' + B_{z0} B'_z + B_{z0} \frac{dB_{r1}}{dz} \xi_r + B_{r1} B'_r \\ \dots (3.10) \\ + V_{z0} v'_z + V_{z0} \frac{dV_{r1}}{dz} \xi_r + V_{r1} v'_r = 0 . \end{aligned}$$

Since $v'_r \propto r^{m-1}$ we have

$$\varphi' = - \int v'_r dr = - \frac{R}{m} v'_r(R) .$$

Substituting for v'_z and B'_z from Eqs.(3.2) and (3.3), and using Eqs.(2.6) and (3.9), equation (3.10) becomes,

$$i\omega R v'_r - B_{z0} \frac{d}{dz} (R B'_r) + V_{z0} \frac{d}{dz} (R v'_r) + m(B_{z0}^2 + V_{z0}^2) \frac{d^2 R}{dz^2} \xi = 0 . \quad \dots (3.11)$$

Using Eqs.(2.5) and (3.9), and

$$\left(\frac{\partial V_{r1}}{\partial r} \right)_{r=R} = \frac{V_{r1}}{R}$$

and

$$\left(\frac{\partial B_{r1}}{\partial r} \right)_{r=R} = - \frac{B_{r1}}{R} - \frac{dB_{z0}}{dz} ,$$

Eqs.(3.5) and (3.7) become

$$RV'_r = i\omega R \xi + \frac{d}{dz} (RV_{z0} \xi)$$

and

$$RB'_r = \frac{d}{dz} (RB_{z0} \xi) .$$

Substituting these into Eq.(3.11), we obtain after some algebra,

$$\begin{aligned} & \frac{d}{dz} \left((B^2 - V^2) R^2 \frac{d\xi}{dz} \right) - 2i\omega VR^2 \frac{d\xi}{dz} \\ & + \left[\omega^2 R^2 + BR \frac{d}{dz} \left(R \frac{dB}{dz} \right) - R(V^2 + B^2)(m-1) \frac{d^2 R}{dz^2} \right] \xi = 0, \end{aligned}$$

where the suffices on V_{z0} and B_{z0} have now been dropped. Using Eqs.(2.5) and (2.6) this equation can be written in the alternative form

$$\begin{aligned} & \frac{d}{dz} \left((B^2 - V^2) R^2 \frac{d\xi}{dz} \right) - 2i\omega VR^2 \frac{d\xi}{dz} \\ & + \left[\omega^2 R^2 - 4 \left(2 + \frac{V^2}{B^2} \right) V^2 \left(\frac{dR}{dz} \right)^2 - \left\{ (m-1)B^2 + (m-3)V^2 \right\} R \frac{d^2 R}{dz^2} \right] \xi = 0 . \quad \dots (3.12) \end{aligned}$$

Given the values of V and B at some z , their values are determined everywhere as a function of R from Eqs.(2.5) and (2.6). Thus for a specified $R(z)$ Eq.(3.12) describes the perturbed motion of the surface of the plasma. In the next Section we shall use this equation in analysing the stability of the system.

4. STABILITY ANALYSIS

We are concerned with the stability of a pinch having a profile which, in the frame of the wave, is given by

$$R = R_0 (1 + \delta f(z)) ,$$

where f is a periodic function and $\delta \ll 1$. Although Eq. (3.12) is valid for higher order in δ we shall restrict our attention to $m \sim \delta^0$. Stability will be determined by an expansion in δ about the cylindrical system $\delta = 0$. The normal modes of the cylindrical system have the form e^{ikz} and since this system has no destabilising

mechanism it is positively stable for all $k > 0$ and marginally stable for $k = 0$ ⁽⁷⁾.

Now, for the type of problem we are considering, it can be shown⁽⁸⁾ that the eigenvalue ω is given by an equation of the form

$$L\omega^2 + 2M\omega + N = 0$$

where L, M and N are real. Thus

$$\omega = \frac{-M \pm (M^2 - LN)^{\frac{1}{2}}}{L}.$$

The necessary and sufficient condition for stability for a given mode is, therefore, that the discriminant

$$\Delta = M^2 - LN > 0.$$

In what follows we shall be making use of expansions in various small parameters. We note that if Δ is expanded in these parameters then the sign of the first non-zero term in the expansion determines stability since the addition of higher order terms cannot affect the sign.

Now, for the cylinder we have $\Delta > 0$ for $k > 0$ and $\Delta = 0$ for $k = 0$. For finite δ and $kR \gg \delta$ we can expand Δ in δ and it is clear from the argument just given that these modes will be stable to all orders in δ . Thus, referring to Fig.1, the region marked 1 on the (kR, δ) diagram is stable.

When $kR \ll \delta$ this argument cannot be used since the expansion of Δ must first be made in the smallest parameter which is no longer δ . This corresponds to region 2 on the diagram and this is the crucial region in that it is those modes which are unstable for a static system. We shall ultimately obtain a stability criterion for this region. However to prove stability for the system we must demonstrate stability for all k and the region 3, $kR \sim \delta$, must also be covered. This part of the proof will be carried out next.

In this Section we shall use the symbol ε to refer to the characteristic lengths of the equilibrium only, so that $\varepsilon \sim R/\lambda$ where λ is the period of $f(z)$. We shall consider only systems for which $1 \gg \delta \gg \varepsilon$ so that the region $kR \sim \delta$ is part of the region $1 \gg kR \gg \varepsilon$. Eq.(3.12) is valid for this region but before using it we must demonstrate the validity of the earlier assumption $\underline{\nabla} \times \underline{v}' = 0$.

Curling and linearizing the equation of motion gives

$$\frac{\partial \underline{w}'}{\partial t} + \underline{v} \cdot \underline{\nabla} \underline{w}' - \underline{w}' \cdot \underline{\nabla} \underline{v} = 0$$

where $\underline{w}' = \nabla \times \underline{v}'$. To zero order in δ and indicating orders in δ by subscripts,

$$i(\omega_0 + kV_{z0})\underline{w}'_0 = 0 ,$$

where the solutions have been Fourier analysed. Thus apart from the uninteresting solution $\omega/k = -V_{z0}$ we have $\underline{w}'_0 = 0$. Since $\underline{w}' \cdot \underline{W}$ now has no first order part we obtain

$$\frac{d\underline{w}'_1}{dt} = 0 ,$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + V_{z0} \frac{\partial}{\partial z}$ is the time derivative moving with velocity V_{z0} . Since the system is periodic this shows that there are no instabilities for $\underline{w}'_1 \neq 0$. Thus we consider only $\underline{w}'_1 = 0$ and then, by induction, \underline{w}' is zero to all orders in δ .

We now return to Eq.(3.12) and for $1 \gg kR \gg \epsilon$ the equation becomes

$$(B^2 - V^2) \frac{d^2 \xi}{dz^2} - 2i\omega V \frac{d\xi}{dz} + \omega^2 \xi = 0 .$$

Defining a new variable $k(z)$ by $\xi = \exp i \int k dz$ we obtain

$$(B^2 - V^2) \left(-k^2 + i \frac{dk}{dz} \right) + 2\omega V k + \omega^2 = 0 .$$

Assuming that $\left| \frac{dk}{dz} \right| \ll |k^2|$,

$$k(z) = \frac{-\omega}{V(z) + B(z)} .$$

Since this k , and therefore ξ , is periodic with wavelength λ it is necessary that

$$\int_0^\lambda k(z) dz = 2\pi n$$

where n is an integer, thus

$$\omega_\pm = - \frac{2\pi n}{\int_0^\lambda \frac{dz}{V \pm B}} ,$$

so that ω is real. We notice that the neglect of the $\frac{dk}{dz}$ term was justified.

To leading order in ϵ the discriminant is given by

$$\Delta = \frac{1}{4} L^2 (\omega_+ - \omega_-)^2$$

and this is clearly positive. It will therefore be positive to all orders in ε and we conclude that for $1 \gg kR \gg \varepsilon$ the system is stable to all orders in ε for all δ .

It remains now to consider the case $kR \ll \delta$. Since we have shown stability for all other k the stability of the system depends upon this case. First we need to determine Δ . Multiplying Eq.(3.12) by ξ^* and integrating over a period in z we obtain

$$L\omega^2 + 2M\omega + N = 0 ,$$

where

$$L = \int R^2 \xi^2 dz ,$$

$$M = -i R^2 V \int \xi^* \frac{d\xi}{dz} dz$$

and

$$N = \int \left\{ (V^2 - B^2) R^2 \left(\frac{d\xi}{dz} \right)^2 - \left[4 \left(2 + \frac{V^2}{B^2} \right) V^2 \left(\frac{dR}{dz} \right)^2 + ((m-1) B^2 + (m-3) V^2) R \frac{d^2 R}{dz^2} \right] \xi^2 \right\} dz .$$

It is straightforward to show that the coefficients L , M and N are real. Solving for ω we have

$$\omega = \frac{-M \pm [M^2 - LN]^{\frac{1}{2}}}{L} .$$

The discriminant is therefore

$$\Delta = M^2 - LN .$$

In principle the method now will be to expand Δ first in k and then in δ and to determine the largest term in the expansion. Thus we write $\Delta = \sum_{m,n} \Delta_{mn}$ where the subscripts denote the order in k and δ , so that $\Delta_{mn} \sim k^m \delta^n$. It turns out, however, that there is a non-zero term for $k = 0$ and it is sufficient to consider only $\Delta_{0,n}$. Thus, from Eq.(3.12) we have $\xi_0 = \text{constant}$ and $\omega_0 = 0$ where subscripts denote the order in δ . To first order in δ Eq.(3.12) becomes,

$$(B_0^2 - V_0^2) R_0 \frac{d^2 \xi_1}{dz^2} - \frac{d^2 R_1}{dz^2} ((m-1) B_0^2 + (m-3) V_0^2) \xi_0 = 0$$

and the solution for $k = 0$ is

$$\xi_1 = \frac{(m-1) B_0^2 + (m-3) V_0^2}{B_0^2 - V_0^2} \frac{R_1}{R_0} \xi_0 .$$

Using the periodicity of $R(z)$ and noting that $\frac{d\xi_0}{dz} = 0$ it is easily seen that $\Delta_{0,0} = \Delta_{0,1} = 0$. The next term is finite and is given by

$$\begin{aligned} \Delta_{0,2} = & \int R_0^2 \xi_0^2 dz \int \left\{ 4 \left(2 + \frac{V_0^2}{B^2} \right) V_0^2 \left(\frac{dR_1}{dz} \right)^2 \xi_0^2 \right. \\ & + 2 \left((m-1) B_0 B_1 + (m-3) V_0 V_1 \right) R_0 \frac{d^2 R_1}{dz^2} \xi_0^2 \\ & + \left((m-1) B_0^2 + (m-3) V_0^2 \right) \left[R_1 \frac{d^2 R_1}{dz^2} \xi_0^2 \right. \\ & + 2 R_0 \frac{d^2 R_1}{dz^2} \xi_0 \xi_1 \left. \right] \\ & \left. - (V_0^2 - B_0^2) R_0^2 \left(\frac{d\xi_1}{dz} \right)^2 \right\} dz . \end{aligned}$$

From Eqs. (2.5) and (2.6) we have

$$V_1 = - 2V_0 \frac{R_1}{R_0} ,$$

and

$$B_1 = 2 \frac{V_0^2}{B_0} \frac{R_1}{R_0}$$

so that

$$\Delta_{0,2} = \lambda \xi_0^4 B_0^2 R_0^2 T \int \left(\frac{dR_1}{dz} \right)^2 dz ,$$

where

$$T = 4U^4 - ((m-1) + (m-3)U^2) - \frac{((m-1) + (m-3)U^2)^2}{1 - U^2}$$

and

$$U^2 = V_0^2 / B_0^2 .$$

The necessary and sufficient condition for stability is that $T > 0$. If $T > 0$ the system is stable to all orders in δ and ε . We may rewrite T in the form

$$T = \frac{1}{4} \left[\left(4U^2 - \frac{1}{8} \right)^2 + \frac{63}{64} \right] - \frac{1}{1-U^2} \left((m-1) + (m-3)U^2 + \frac{1}{2} (1-U^2) \right)^2 .$$

It is now clear that for $U^2 > 1$, that is for the wave velocity V_w greater than the Alfvén speed B , all $m \sim \delta^0$ ($m \neq 0$) are stable. This is the crucial result of this paper. We can also ask more generally what values of m and U are stable. This is

easily determined from T and the results are shown graphically in Fig.2.

5. DISCUSSION

In this paper a method of dynamically stabilising the low m -modes of a corrugated theta-pinch has been suggested. However, owing to the simplifying assumptions made and the neglect of certain modes, the suggestion must remain tentative.

We have only considered the $\beta = 1$ case, but since this was found to be positively stable, there must be at least a range of β just less than 1, for which the system is stable. The modes $m > \delta^0$ have not been discussed but this is probably not a serious restriction since these modes are not observed in experimental theta-pinch. The reason for their absence is thought to be the influence of finite Larmor radius effects which of course are neglected in the present calculation. The mode $m = 0$ is not included in the calculation and although this is a very stable mode in the static system, it is possible that it may be made unstable by the presence of the imposed wave. Finally, it is possible that new modes, not covered by the asymptotic expansions in this paper, may be introduced.

Following the original publication⁽⁴⁾ of the proposed method of dynamic stabilisation, a letter was published by Troyon⁽⁹⁾ pointing out the mistake of the authors in not drawing attention to the limitation of the calculation to $m \sim \delta^0$ and in wrongly claiming stability for all m . Troyon also considers the stability of modes $m \sim 1/\delta$ and concludes that for sufficiently large m there will be instability. In deriving this result it was assumed that ω^2 is real so that the boundary of stability is $\omega = 0$ ⁽¹⁰⁾. This is not necessarily, and not in general, correct, since ω is determined by the solution of a full quadratic. It is also stated in Troyon's paper that the proposed method depends upon the fact that the average destabilising force along the pinch vanishes. This is not so since the average destabilising force along the pinch does not vanish and the corrugated pinch is unstable in the absence of dynamic stabilisation⁽⁵⁾. Finally, the relevance of the linear analogue of a bumpy torus is questioned. This is indeed an assumption, but it is interesting to note that in the description of toroidal theta-pinch equilibria given by Morse, Riesenfeld and Johnson⁽²⁾, the leading order perturbation to the cylindrical configuration is azimuthally symmetric.

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10. On the basis of this assumption the result given for the region of unstable m appears to be too high by a factor of 4.

