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RESONANCES (TE_{omn}, TM_{lmo}) OF A CYLINDRICAL CAVITY CONTAINING A NON-UNIFORM PLASMA COLUMN

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RESONANCES (TE_{omn}, TM_{lmo}) OF A CYLINDRICAL CAVITY CONTAINING A NON-UNIFORM PLASMA COLUMN

by

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A B S T R A C T

Solutions of the wave equation are obtained for TE_{omn} and $TM_{\ell mo}$ modes in cylindrical cavities containing a co-axial, annular plasma column. The radial electron density distribution is assumed to be of the form $n_e = n_o \{ 1-\Upsilon_1(r/a)^2 - \Upsilon_2(a/r)^2 \}$, where n_o, Υ_1, Υ_2 are constants and a is the outer radius of the plasma. The results can be used to determine the effects of density gradients on measurements of electron densities and collision frequencies by the cavity method. Some illustrative examples are presented.

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C O N T E N T S

I. INTRODUCTION

Measurement of the resonant frequency ω and the Q-value of a microwave cavity partially filled with plasma enables information to be obtained about the electron density n_e and collision frequency ν in the plasma[1]. When the electron density is low and the skin depth is large compared to the plasma dimensions, the change in resonant frequency and the Q-value can be determined by a perturbation analysis. This procedure is valid for $\omega_p \leqslant \omega$ for most experimental situations (ω_p is the plasma frequency), although the method can be extended to higher plasma frequencies when the plasma occupies only a small part of the cavity volume. If $\omega_p \gg \omega$, the presence of the plasma significantly changes the distribution of the microwave fields in the cavity and exact solutions of the wave equation for the plasma medium are required. Such calculations have been carried out for cylindrical cavities with a co-axial, uniform plasma column[2,3]. If the plasma spatial distribution is non-uniform and the plasma density is low, then the perturbation theory is still applicable. For high densities the effects of density gradients are more difficult to determine; in general, numerical solution of the wave equation is necessary. In some special cases, the solution can be expressed in terms of tabulated functions. Examples are given by Burman[4] and by Allis, Buchsbaum and Bers[5] for cylindrical dielectrics which have radial variations in dielectric constant. Agdur and Enander[6] have considered the effects of a parabolic density distribution for the TM_{010} mode by obtaining a power series solution of the wave equation.

In this paper we consider an annular plasma column with a radial electron density distribution given by $n_e = n_o\{1-\gamma_1(r/a)^2-\gamma_2(a/r)^2\}$, where n_o, γ_1, γ_2 are constants and a is the outer radius of the plasma. Although the radial density distribution is assumed to have this particular form, the results illustrate the main features of the effects of density gradients on cavity resonances. The distribution chosen does give a good approximation to the electron density distribution in a low pressure arc plasma, when the wall sheath thickness is much smaller that the tube radius.

II. THEORY

In Section A, exact solutions of the wave equation are obtained for TE_{omn} and $TM_{\ell mo}$ modes, allowing the resonant frequency of a cavity-plasma system, such as shown in Fig.1, to be determined. Results for cylindrical plasma $(\gamma_2 = 0)$ are given in Section B and some illustrative examples are presented in Section C. The effect of density gradients on the cavity Q-value is discussed in Section D.

A. Annular plasma, $n_e = n_o\{1 - \gamma_1(r/a)^2 - \gamma_2(a/r)^2\}$

Consider the case of a cold plasma so that the effects of electron thermal motion on the plasma conductivity are negligible. At microwave frequencies the motion of positive ions can be neglected and the current equation in the absence of a magnetistatic field is given by

$$(m_e/n_e e^2)\frac{\partial \tilde{j}}{\partial t} = \tilde{E} - (m_e \nu/n_e e^2)\tilde{j}\ , \qquad \ldots (1)$$

where ν is the effective collision frequency for momentum transfer(1) (electromagnetic units are used).

Maxwell's equations give

$$\nabla \times \tilde{E} = -\frac{\partial \tilde{B}}{\partial t} \quad , \qquad \ldots (2)$$

$$\nabla \times \tilde{B} = 4\pi \tilde{j} + \frac{1}{c^2}\frac{\partial \tilde{E}}{\partial t} \quad . \qquad \ldots (3)$$

Assume time variations of the form $e^{i\omega t}$ and consider only TE_{omn}, $TM_{\ell mo}$ modes for which the electric field is perpendicular to density gradients and $\nabla \cdot \tilde{E} = 0$ (the subscripts ℓ,m,n refer to field variations in the θ,r and z directions respectively[7]).

Take curl of Eq.(2):

$$\nabla^2 \tilde{E} = i\omega \nabla \times \tilde{B} \ ,$$

and using Eq.(3),

$$\nabla^2 \tilde{E} = 4\pi i \omega \tilde{j} - \frac{\omega^2}{c^2}\tilde{E} \quad . \qquad \ldots (4)$$

Froms Eqs.(1) and (4),

$$\nabla^2 \tilde{E} = \left\{ \left[4\pi n_e e^2 i\omega / m_e (i\omega + \nu) \right] - (\omega/c)^2 \right\} \tilde{E} \qquad \ldots (5)$$

For an annular plasma with dimensions as shown in Fig.2, and $n_e = n_o \left\{ 1 - \Upsilon_1 (r/a)^2 - \Upsilon_2 (a/r)^2 \right\}$, we require solutions of Eq.(5) for the modes under consideration.

1. $TM_{\ell mo}$ modes

We look for solutions of the form $\tilde{E} = (0,0,E_z(r) \cos \ell\theta \, e^{i\omega t})$ and $\tilde{B} = (B_r, B_\theta, 0)$. Taking the z-component of Eq.(5),

$$\frac{d^2E_z}{dx^2} + \frac{1}{x}\frac{dE_z}{dx} - \left[\left(\frac{a\omega_{po}}{c}\right)^2 \left(\frac{i\omega}{i\omega+\nu}\right) (1 - \Upsilon_1 x^2 - \Upsilon_2/x^2) - \left(\frac{a\omega}{c}\right)^2 + (\ell/x)^2 \right] E_z = 0, \qquad \ldots (6)$$

where $x = r/a$ and $\omega_{po}^2 = 4\pi n_o e^2 c^2 / m_e$.

Put $E_z = \varphi(x)/x$ and $\xi = i \left(\frac{a\omega_{po}}{c}\right) \left(\Upsilon_1 \frac{i\omega}{i\omega+\nu}\right)^{\frac{1}{2}} x^2$, then

Eq.(6) becomes

$$\frac{d^2\varphi}{d\xi^2} + \left[-\frac{1}{4} + i \left\{ \left(\frac{a\omega_{po}}{c} \right)^2 \left(\frac{i\omega}{i\omega+\nu} \right) - \left(\frac{a\omega}{c} \right)^2 \right\} \Big/ 4 \left(\frac{a\omega_{po}}{c} \right) \left(\Upsilon_1 \frac{i\omega}{i\omega+\nu} \right)^{\frac{1}{2}} \xi \right.$$

$$\left. + \left\{ \Upsilon_2 \left(\frac{a\omega_{po}}{c} \right)^2 \left(\frac{i\omega}{i\omega+\nu} \right) - (\ell^2-1) \right\} \Big/ 4\xi^2 \right] \varphi = 0 \; .$$

This is Whittaker's equation with solution[8]:

$$\varphi(\xi) = A_1 M_{\kappa', \mu'}(\xi) + A_2 \, W_{\kappa', \mu'}(\xi) \; ,$$

where A_1, A_2 are constants given by the boundary conditions at $r = a$, $r = b$, and

$$\kappa' = i \left\{ \left(\frac{a\omega_{po}}{c} \right)^2 \left(\frac{i\omega}{i\omega+\nu} \right) - \left(\frac{a\omega}{c} \right)^2 \right\} \Big/ 4 \left(\frac{a\omega_{po}}{c} \right) \left(\Upsilon_1 \frac{i\omega}{i\omega+\nu} \right)^{\frac{1}{2}} \; ,$$

$$\mu' = \tfrac{1}{2} \left\{ 1 - \Upsilon_2 \left(\frac{a\omega_{po}}{c} \right)^2 \left(\frac{i\omega}{i\omega+\nu} \right) + (\ell^2-1) \right\}^{\frac{1}{2}} .$$

$$\therefore \; \tilde{E} = \left\{ 0, 0, \frac{1}{(r/a)} \; \varphi(\xi) \cdot \cos \ell\theta \cdot e^{i\omega t} \right\} . \qquad \ldots (7)$$

The B_r, B_θ components of the magnetic field are determined from Eqs.(2) and (7).

2. TE_{omn} modes

Assume solutions of the form $B = (B_r, 0, B_z)$ and $\tilde{E} = \{ 0, E(x) \cdot \sin(kz) \cdot e^{i\omega t}, 0 \}$, where $k = n\pi/L$, with $n = 1, 2, 3 \ldots$

Using this relation for E in Eq.(5),

$$\frac{d^2E(x)}{dx^2} + \frac{1}{x} \frac{dE(x)}{dx} - \left[\left(\frac{a\omega_{po}}{c} \right)^2 \left(\frac{i\omega}{i\omega+\nu} \right) (1 - \Upsilon_1 x^2 - \Upsilon_2/x^2) - \left(\frac{a\omega}{c} \right)^2 + (ak)^2 + \left(\frac{1}{x} \right)^2 \right] E(x) = 0$$

Substituting $E(x) = \varphi(x)/x$, $\xi = i \left(\frac{a\omega_{po}}{c} \right) \left(\Upsilon_1 \frac{i\omega}{i\omega+\nu} \right)^{\frac{1}{2}} x^2$ we obtain

Whittaker's equation as before, giving

$$\tilde{E} = \left\{0, \frac{1}{(r/a)}\left[A_3 M_{\kappa,\mu}(\xi) + A_4 W_{\kappa,\mu}(\xi)\right] \sin\left(\frac{n\pi}{L} z\right) \cdot e^{i\omega t}, 0\right\}, \quad \dots (8)$$

where A_3 and A_4 are constants.

$$\kappa = i\left\{\left(\frac{a\omega_{po}}{c}\right)^2\left(\frac{i\omega}{i\omega+\nu}\right) - \left(\frac{a\omega}{c}\right)^2 + (ak)^2\right\} \Big/ 4\left(\frac{a\omega_{po}}{c}\right)\left(\Upsilon_1 \frac{i\omega}{i\omega+\nu}\right)^{\frac{1}{2}},$$

and

$$\mu = \tfrac{1}{2}\left\{1 - \Upsilon_2\left(\frac{a\omega_{po}}{c}\right)^2\left(\frac{i\omega}{i\omega+\nu}\right)\right\}^{\frac{1}{2}}.$$

Again B_r, B_z are determined by Eq.(2).

The $TM_{\ell mo}$ and TE_{omn} resonant frequencies of a cavity system such as that shown in Fig.1 can be found by applying the appropriate boundary conditions to the fields in each medium. In general this results in rather cumbersome relationships and to illustrate the main effects of plasma non-uniformity we will treat a much simpler plasma-cavity system in the following section. In practice the $TM_{\ell mo}$ modes are not so useful at high densities where the perturbation theory is not applicable. The plasma sheaths which arise at the end walls of the cavity may invalidate the assumption that $\nabla \cdot E = 0$ for these modes, or if the plasma extends through the end walls of the cavity, then the fringing fields cause some uncertainty in determining the plasma parameters(3). These effects are not so important for the TE_{omn} modes since the electric field is azimuthal and $|E_\theta|$ approaches zero at the end walls of the cavity. In the remainder of this paper we shall consider TE_{omn} modes only.

B. Non-uniform cylindrical plasma, $n_e = n_o\left\{1-\Upsilon_1(r/a)^2\right\}$

We now determine the TE_{omn} resonant frequencies of a cylindrical cavity (radius R, length L) containing an axial plasma column of radius a. The electron density distribution is assumed to be $n_e = n_o\left\{1-\Upsilon_1(r/a)^2\right\}$. Initially the effects of collisions are neglected, so that $\nu \ll \omega$, and the region $a < r < R$ is assumed to have a constant dielectric coefficient which we take to be that of vacuum.

For $\nu \ll \omega$, κ and ξ [c.f. Eq.(8)] are imaginary quantities and $\mu = \frac{1}{2}$ (since $\Upsilon_2 = 0$ in this example). It is now convenient to express the Whittaker functions in terms of Coulomb wave functions[8] giving,

$$E_\theta = \frac{-i}{(r/a)} A_5 F_o(\eta,\rho)\cdot\sin(kz) \cdot e^{i\omega t} \qquad \ldots (9)$$

where

$$\eta = \left\{\left(\frac{a\omega_{po}}{c}\right)^2 - \left(\frac{a\omega^2}{c}\right) + (ak)^2\right\}\Big/ 4\,\Upsilon_1^{\frac{1}{2}}\left(\frac{a\omega_{po}}{c}\right),$$

$$\rho = \tfrac{1}{2}\left(\frac{a\omega_{po}}{c}\right)\Upsilon_1^{\frac{1}{2}}(r/a)^2 .$$

From Eq.(2)

$$\left.\begin{aligned} B_z &= \frac{A_5\Upsilon_1^{\frac{1}{2}}}{a\omega}\left(\frac{a\omega_{po}}{c}\right)\cdot F_o'(\eta,\rho)\cdot\sin(kz)\cdot e^{i\omega t}, \\ B_r &= -A_5\left(\frac{k}{\omega}\right)\cdot\frac{1}{(r/a)}\cdot F_o(\eta,\rho)\cdot\cos(kz)\cdot e^{i\omega t}. \end{aligned}\right\} \ldots (10)$$

The vacuum fields are given by Eq.(5) with $n_e = 0$,

$$\left.\begin{aligned} E_\theta &= \frac{-i\omega}{\alpha_1}\left\{A_6\,J_1(\alpha_1 r) + A_7\,Y_1(\alpha_1 r)\right\}\cdot\sin(kz)\cdot e^{i\omega t} \\ B_z &= \left\{A_6\,J_o(\alpha_1 r) + A_7\,Y_o(\alpha_1 r)\right\}\cdot\sin(kz)\cdot e^{i\omega t}, \\ B_r &= -\frac{k}{\alpha_1}\left\{A_6\,J_1(\alpha_1 r) + A_7\,Y_1(\alpha_1 r)\right\}\cdot\cos(kz)\cdot e^{i\omega t} \end{aligned}\right\} \ldots (11)$$

where

$$\alpha_1 a = \left\{ \left(\frac{a\omega}{c}\right)^2 - (ak)^2 \right\}^{\frac{1}{2}} ,$$

A_6, A_7 are constants,

and $J_p(x)$, $Y_p(x)$ are Bessel functions of order p.

Using the boundary conditions that the fields are continuous at $r = a$, and $E_\theta = 0$ at $r = R$, the resonance condition is obtained from Eqs.(9), (10) and (11):

$$(\alpha_1 a) \left\{ \frac{J_0(\alpha_1 a)\cdot Y_1(\alpha_1 R) - J_1(\alpha_1 R)\cdot Y_0(\alpha_1 a)}{J_1(\alpha_1 a)\cdot Y_1(\alpha_1 R) - J_1(\alpha_1 R)\cdot Y_1(\alpha_1 a)} \right\} = 2\rho_0 \frac{F_0'(\eta,\rho_0)}{F_0(\eta,\rho_0)} ; \qquad \ldots (12)$$

where

$$\rho_0 = \tfrac{1}{2}\left(\frac{a\omega_{po}}{c}\right) \gamma_1^{\frac{1}{2}} .$$

This result is to be compared with the corresponding expression for a uniform plasma of the same radius (let $\gamma_1 \to 0$ in Eq.(12) and use the asymptotic values of the Coulomb wave functions),

$$(\alpha_1 a) \left\{ \frac{J_0(\alpha_1 a)\cdot Y_1(\alpha_1 R) - J_1(\alpha_1 R)\cdot Y_0(\alpha_1 a)}{J_1(\alpha_1 a).Y_1(\alpha_1 R) - J_1(\alpha_1 R)\cdot Y_1(\alpha_1 a)} \right\} = (\beta_1 a) \frac{J_0(\beta_1 a)}{J_1(\beta_1 a)} \qquad \ldots (13)$$

where

$$(\beta_1 a)^2 = (\alpha_1 a)^2 - \left(\frac{a\omega_p}{c}\right)^2 , \qquad \ldots (14)$$

and $\omega_p^2 = 4\pi\bar{n}_e e^2 c^2/m_e$, $\bar{n}_e$ being the density of the uniform plasma.

Eqs.(12) and (13) have an infinite number of solutions for given plasma parameters and a fixed value of $k(=n\pi/L)$. These determine the resonant frequencies of the TE_{omn} modes. Some special cases are of interest:

(i) Empty cavity

Eq.(11) shows that E_θ = constant $\cdot J_1(\alpha_1 r)$, and since $E_\theta = 0$ at $r = R$, the resonance condition is $\alpha_1 R = \chi_m$, where χ_m is the m^{th} root of $J_1(\chi) = 0$. From the definition of α_1,

$\chi_m^2 = (\frac{R\omega}{c})^2 - (n\pi R/L)^2$, so that the resonant frequency ω_o is given by

$$\omega_o^2 = \frac{c^2}{R^2}\left\{\chi_m^2 + (n\pi R/L)^2\right\}.$$

(ii) High density limit, $\omega_p \gg \omega$

For very large values of electron density, the resonant frequency of the plasma-cavity system approaches the value obtained when the plasma is replaced by a perfectly conducting cylinder of the same radius (i.e. $\omega = \omega_1$, say);

$$\omega_1^2 = \frac{c^2}{R^2}\left\{\chi_m'^2 + (n\pi R/L)^2\right\},$$

where χ_m' is the m^{th} root of $J_1(\frac{a}{R}\chi)\cdot Y_1(\chi) - J_1(\chi)\cdot Y_1(\frac{a}{R}\chi) = 0$. These roots are tabulated[8]. The frequencies ω_o and ω_1 determine the lower and upper bounds of the TE_{omn} mode resonant frequency of the plasma loaded cavity.

(iii) Plasma-filled cavity, a = R

Eqs.(12) and (13) become,

$$\left.\begin{aligned} F_o(\eta,\rho_o) &= 0 \qquad \text{non-uniform plasma,} \\ J_1(\beta_1 R) &= 0 \qquad \text{uniform plasma} \end{aligned}\right\} \quad \ldots (15)$$

C. Numerical Examples

To illustrate the effects of radial density gradients more clearly, the TE_{011} resonant frequency of a cylindrical cavity has been calculated using Eqs.(12) and (13) for the following conditions:

$$L = \pi R/2$$

$$R = 2a$$

and
$$\begin{cases} \Upsilon_1 = 1.0\ , \quad \Upsilon_2 = 0\ , \quad \text{i.e.} \quad n_e = n_o\{1-(r/a)^2\} \\ \Upsilon_1 = 0\ , \quad \Upsilon_2 = 0\ , \quad \text{i.e.} \quad n_e = \bar{n}_e\ . \end{cases}$$

For comparison of these results, the plasma frequency ω_p is defined in terms of the average electron density

$$\text{i.e.} \quad \omega_p^2 = 4\pi\bar{n}_e\, e^2c^2/m_e\ .$$

With $\Upsilon_1 = 1.0$, $\bar{n}_e = n_o/2$ so that $\omega_p^2 = \omega_{po}^2/2$. Fig.3 shows the dimensionless resonant frequency ω/ω_o expressed as a function of the normalised average electron density ω_p^2/ω^2 for $\Upsilon_1 = 0$ and $\Upsilon_1 = 1.0$.

The results show that, in this example, the non-uniform plasma gives a smaller frequency shift than a uniform plasma with the same average density and radius. This is as expected, since the electric field is zero at the axis and, for the assumed density distribution and $a/R = 0.5$, the electron density in the region of highest electric field is smaller than the average value. For smaller values of a/R this behaviour is more pronounced. When a/R approaches unity and the plasma density is low, a parabolic distribution produces a greater frequency shift than a uniform plasma of the same average density. In this case the electron density in the region of high electric field ($r = R/2$ for the TE_{011} mode and small densities) is larger than the average value. At high plasma densities however, the fields are confined to the outer regions of the cavity and the frequency shift is again smaller for the non-uniform plasma.

This is illustrated in Fig.4, where we have chosen $R = a$, $L = R/2$ and the resonant frequencies have been calculated from Eq.(15).

Perturbation Theory Results

For small densities the perturbation theory[1] can be used for both uniform and non-uniform plasmas giving (with $\nu \ll \omega$),

$$\frac{\omega}{\omega_o} - 1 = \tfrac{1}{2}\,\frac{\int_V (\omega_p(r)/\omega)^2\, E_o^2\, dV}{\int_V E_o^2\, dV}$$

where $\omega_p(r)$ is the spatially dependent plasma frequency, E_o is the electric field in the cavity in the absence of plasma and the integrations are over the volume of the cavity.

Since $E_\theta(r)$ = constant $J_1(3.8317\ r/R)$ for the TE_{011} mode, and using the density distribution $n_e = n_o\{1-\Upsilon_1(r/a)^2\}$,

$$\frac{\omega}{\omega_o} - 1 = \frac{1}{2}\,(\omega_{po}/\omega)^2 \int_0^a \{1-\Upsilon_1(r/a)^2\} J_1^2(3.8317\ r/R)\cdot r\cdot dr \Bigg/ \int_0^R J_1^2(3.8317\ r/R)\cdot r\cdot dr$$

The average density is $\bar{n}_e = n_o(1-\Upsilon_1/2)$, and so $\omega_p^2 = (1-\Upsilon_1/2)\omega_{po}^2$

$$\therefore\ \frac{\omega}{\omega_o} - 1 = \frac{2(\omega_p/\omega)^2 (a/r)^2}{(2-\Upsilon_1)J_o^2(3.8317)} \int_0^1 (1-\Upsilon_1 x^2) J_1^2(3.8317(\tfrac{a}{R})x)\cdot x\cdot dx$$

Now[9],

$$\int x\, J_1^2(x)\, dx = \frac{x^2}{2}\left[J_1^2(x) - J_o(x)J_2(x)\right]$$

and

$$\int x^3\, J_1^2(x)\, dx = \frac{x^4}{6}\left[J_1^2(x) + J_2^2(x)\right],$$

so that

$$\frac{\omega}{\omega_o} - 1 = \frac{(\omega_p/\omega)^2\ (a/r)^2}{(2-\Upsilon_1)J_o^2(3.8317)} \left\{\left[J_1^2(\alpha) - J_o(\alpha)\cdot J_2(\alpha)\right] \frac{\Upsilon_1}{3}\left[J_1^2(\alpha) + J_2^2(\alpha)\right]\right\},$$

... (16)

where $\alpha = 3.8317(a/R)$.

Some special cases are:

(i) Small plasma column, $\alpha \ll 1$

Eq.(16) gives,

$$\frac{\omega}{\omega_o} - 1 = \frac{(3.8317)^2}{24J_o^2(3.8317)} (a/R)^4 (\omega_p/\omega)^2 \cdot \left\{ \frac{3-2\Upsilon_1}{2-\Upsilon_1} \right\}$$

A uniform plasma $(\Upsilon_1 = 0)$ with density $\bar{n}_e$ gives a frequency shift which is 1.5 times larger than that produced by a non-uniform plasma with $\Upsilon_1 = 1.0$, i.e. $n = n_o\{1-(r/a)^2\}$. This result has been given previously by Agdur and Enander[6].

(ii) $\Upsilon_1 = 1.0$, i.e. $n_e = n_o\{1-(r/a)^2\}$

By putting $\Upsilon_1 = 1.0$ and $\Upsilon_1 = 0$ in Eq.(16) we see that a uniform plasma with density $\bar{n}_e$ gives a frequency shift which is $f(a/R)$ times that produced by the non-uniform plasma, where

$$f(a/R) = 1/2 \left\{ 1 - \frac{1}{3}\left[J_1^2(\alpha) + J_2^2(\alpha)\right] \Big/ \left[J_1^2(\alpha) - J_o(\alpha)\cdot J_2(\alpha)\right] \right\}$$

For densities such that the perturbation theory is valid, $\frac{\omega}{\omega_o} - 1$ is proportional to n_e, for a uniform density distribution. Consequently we can regard $f(a/R)$ as an 'effective density factor' so that a uniform plasma of density $\bar{n}_e/f(a/R)$ gives the same frequency shift as the non-uniform plasma of the assumed distribution. The function $f(a/R)$ is shown in Fig.5, and it can be seen that the 'effective density' of the non-uniform plasma is less than the average density for all values of a/R less than 0.77. As discussed earlier, this will be true for densities outside the range of the perturbation theory.

The perturbation theory results are shown in Fig.4 for a plasma-filled cavity, and in Fig.6 for $a/R = 0.5$.

D. The Cavity Q-value

When the effects of collisions are included, the cavity fields can be calculated from Eq.(8) and the power dissipated per unit volume found from $j \cdot E$, with j given by Eq.(1). In general these calculations are rather cumbersome and we restrict the following discussion to low plasma densities and to the case $\nu \ll \omega$. The change $\Delta(1/Q)$ in the $1/Q$ value of the cavity can then be obtained from the perturbation theory expression[1] (to first order in ν/ω),

$$\Delta\left(\frac{1}{Q}\right) = \frac{\nu}{\omega} \int_V (\omega_p(r)/\omega)^2 \, E_o^2 \, dV \Big/ \int_V E_o^2 \, dV \,, \qquad \ldots (17)$$

where ν is assumed to be constant throughout the plasma. The integrals in Eq.(17) are identical with those used in Section C when determining the change of resonant frequency produced by the plasma. Again we find that a uniform plasma with density $\bar{n}_e$ gives a change in the $1/Q$ value which is $f(a/R)$ times that produced by a non-uniform plasma with $n_e = n_o\{1-(r/a)^2\}$, where $f(a/R)$ is shown in Fig.5.

III. DISCUSSION

It has been shown that the $TM_{\ell mo}$, TE_{omn} resonant frequencies of a cylindrical microwave cavity loaded with a non-uniform plasma column may be determined for electron density distributions of the form $n_e = n_o\{1-\Upsilon_1(r/a)^2 - \Upsilon_2(a/r)^2\}$. This distribution gives a good approximation to many laboratory plasmas and can be used to assess the magnitude of errors caused by the neglect of density gradients in the analysis of experimental results.

Particular care should be taken when operating in the high density regime, where the skin depth is much less than the plasma radius. Only the surface layers of the plasma are important in determining the electromagnetic field distributions and it is necessary to use a density distribution in the theoretical model which closely matches the experimental situation near the plasma surface.

The effects of a steady magnetic field have not been included in this analysis. However for the TE_{011} mode, the results for an annular density distribution can be applied to a plasma with an axial, current-carrying conductor. The static B_θ field does not affect the high frequency currents for this mode and removes the degeneracy with the TM_{111} mode.

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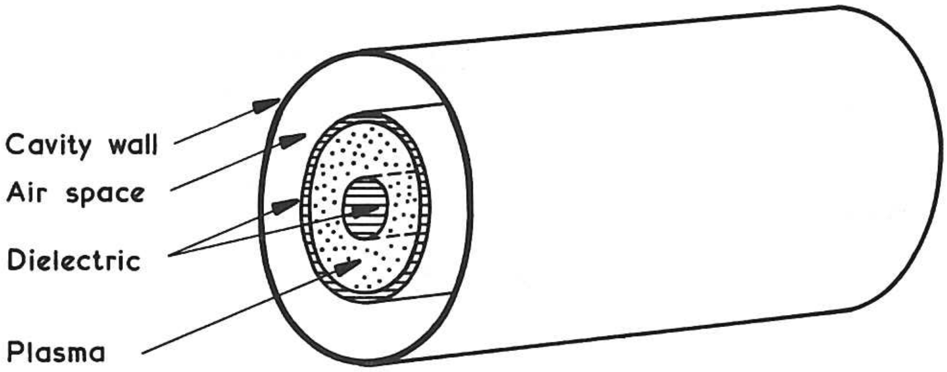

Fig. 1 Plasma-cavity system (CLM-P190)

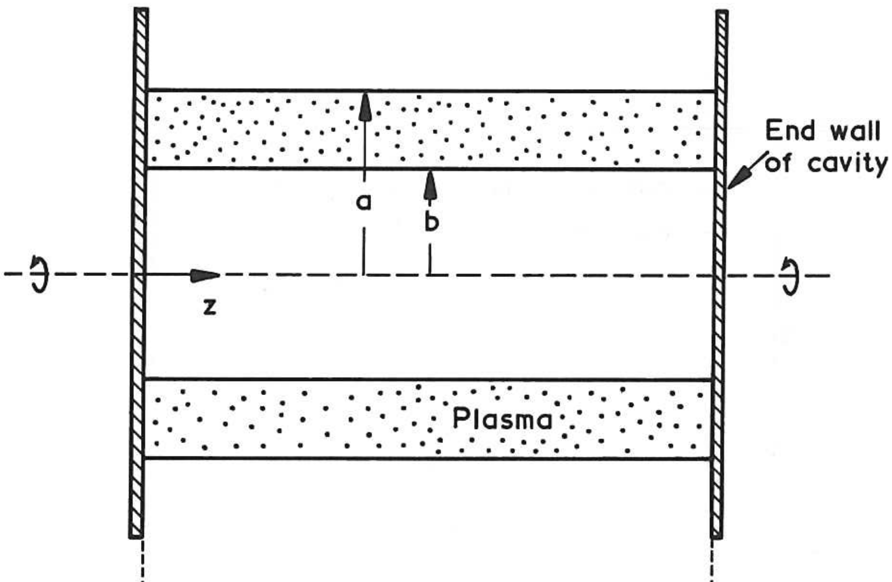

Fig. 2 Dimensions of annular plasma and cavity (CLM-P190)

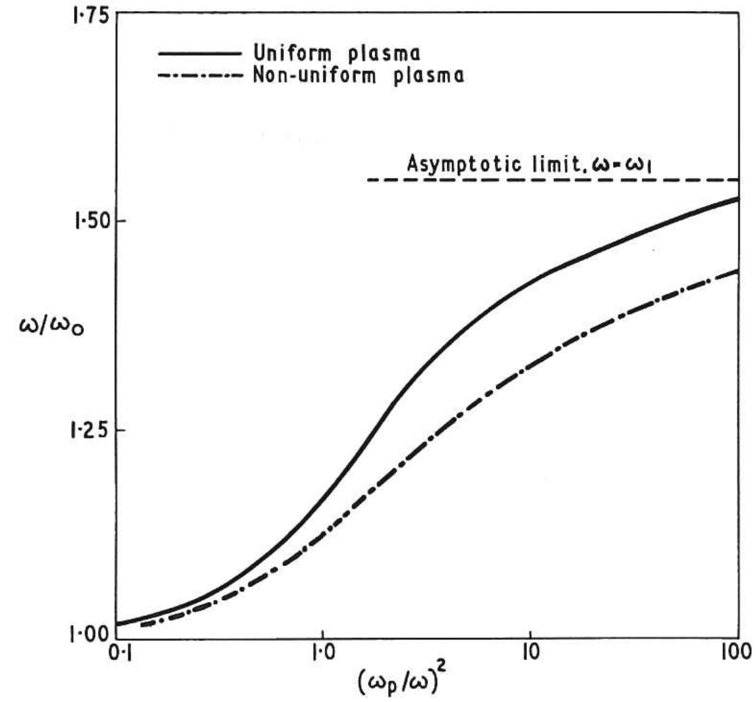

Fig. 3 (CLM-P 190)
Resonant frequency of a TE_{011} mode cavity containing an axial plasma column. $R = 2a$, $L = \pi R/2$

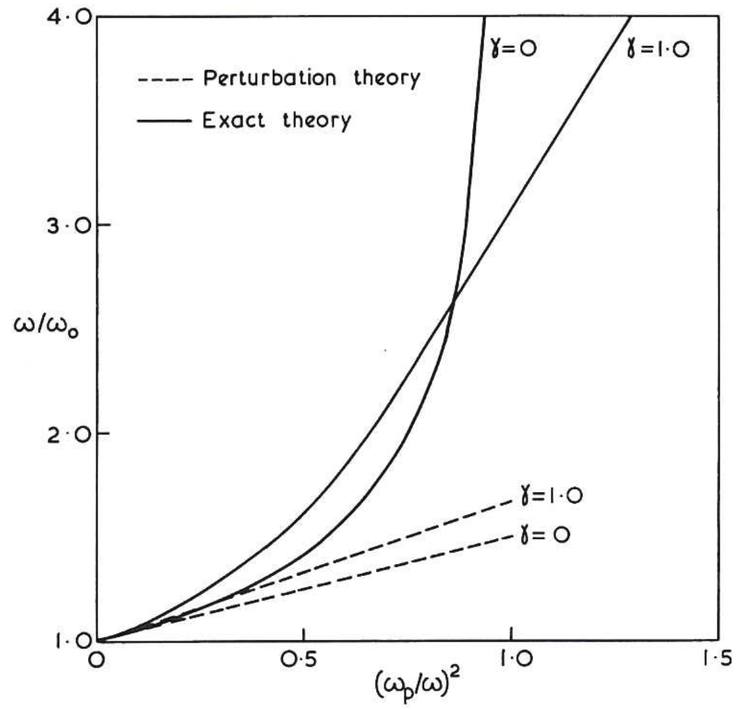

Fig. 4 (CLM-P 190)
Resonant frequency of TE_{011} mode plasma-filled cavity $R = a$; $L = \pi R/2$; $n_e = n_0\{1-\gamma_1(r/a)^2\}$

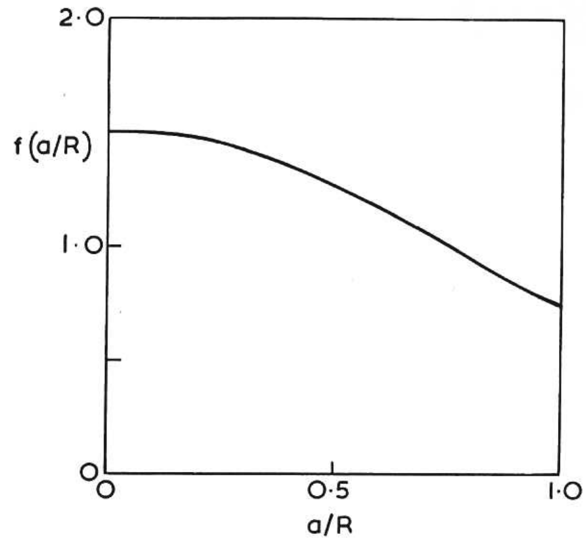

Fig. 5 (CLM-P 190)
'Effective density factor' for TE_{011} mode
$(\omega_p \ll \omega)$; $n_e = n_o \{1-(r/a)^2\}$

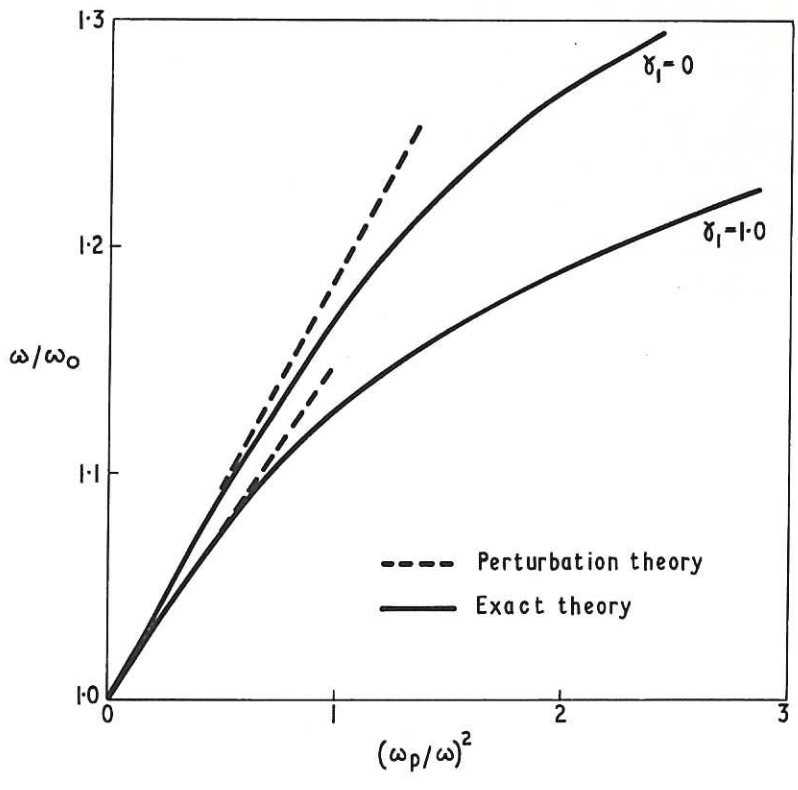

Fig. 6 (CLM-P 190)
Resonant frequency of a TE_{011} mode cavity containing an axial plasma column, $n_e = n_o \{1-\gamma_1 (r/a)^2\}$ $R = 2a$, $L = \pi R/2$