CLM-P193

This document is intended for publication in a journal, and is made available on the understanding that extracts or references will not be published prior to publication of the original, without the consent of the author.

CULHAM LIBRARY
REFERENCE ONLY

CULHAM LABORATORY
LIBRARY
21 MAY 1969
b.

United Kingdom Atomic Energy Authority

RESEARCH GROUP

Preprint

ADIABATIC CUSP LOSSES

A. S. KAYE

Culham Laboratory
Abingdon Berkshire

1969

Enquiries about copyright and reproduction should be addressed to the
Librarian, UKAEA, Culham Laboratory, Abingdon, Berkshire, England

UNCLASSIFIED
(Approved for Publication)

CLM - P 193

ADIABATIC CUSP LOSSES

by

A.S. KAYE

(Submitted for publication in Physics of Fluids)

ABSTRACT

Previous calculations of $\beta = 1$ cusp losses are extended to the case where there exists a third 'adiabatic' invariant of the particle motion, related to the magnetic moment, and additional to the particle energy and angular momentum. Simple analytic results are obtained for the particle and energy loss rate from a Maxwellian plasma in both plane and axisymmetric geometry, electric fields being assumed zero everywhere. The results are valid for a plasma radius large relative to the ion gyro-radius, in which case the adiabaticity in the sense used reduces the loss rate by a factor $2/3\ \lambda^{\frac{1}{2}}$, to give an overall scaling as $\lambda^{-3/2}$ with external mirror ratio, λ. The absolute value is about a factor two less than the previously accepted result (for $\lambda = 1$ only) for an axisymmetric cusp. The enhanced scaling with λ arises from the absence of a certain class of particles in the sheath, which must be everywhere adiabatic; its applicability to a simple theta-pinch is queried, however. The end loss rate of angular momentum is also obtained, and is non-zero at a point cusp. This is shown to lead to rapid rotational instability in a collisionless theta-pinch.

U.K.A.E.A. Research Group,
Culham Laboratory,
Abingdon,
Berks.

January, 1969 (MEJ)

C O N T E N T S

		Page
1.	INTRODUCTION	1
2.	GENERAL THEORY	3
3.	LOSS CONE	5
4.	DISTRIBUTION FUNCTION	7
5.	PARTICLE LOSS RATE	10
6.	RANGE OF VALIDITY	12
7.	PLANE CUSP	15
8.	ROTATION OF $\beta = 1$ CUSP-ENDED SYSTEMS	16
9.	CONCLUSION	22
10.	ACKNOWLEDGEMENTS	23
11.	REFERENCES	24

1. INTRODUCTION

This paper corrects and extends previous calculations of the steady-state losses from a collisionless, $\beta = 1$ (β being the ratio of plasma pressure to total pressure) plasma through an axi-symmetric cusp hole. Whilst this work is primarily concerned with losses from spindle-cusp and cusp-ended theta-pinch geometries, the results obtained have some relevance to simple theta-pinches. A new mechanism for the production of rotation is also discussed. Electric fields are assumed zero throughout.

The particle loss rate from a $\beta = 1$ spindle-cusp was first considered in a series of papers by Grad et al.[1-3], who show the losses from the point and line cusps to be nearly the same as a result of magnetic flux conservation in the sheath, and to be about equivalent to free effusion through a line cusp of width twice the ion gyro-radius at the cusp hole. This calculation has recently been made more rigorous by Taylor[4] who represents the spindle-cusp plasma by a suitable distribution function which also satisfies the Vlasov equation, the particle energy and angular momentum being assumed the only invariants of the particle motion. The particle loss rate is then given by the appropriate moment of the distribution function evaluated, at the cusp hole where the magnetic field is assumed uniform. Firsov[5] had earlier used a similar approach in plane geometry; he, however, made the additional assumption of no trapped particles in the sheath to obtain a self-consistent transverse electric field and thence the effective hole size as several electron gyro-radii.

In a slowly varying magnetic field, the particle motion in the transverse plane is periodic, and the corresponding action integral

an adiabatic invariant[6]. If the system is also axisymmetric, gradients normal to the magnetic field lead only to an azimuthal drift, the particle motion therefore remaining periodic radially. In a near axial magnetic field, the orbits in the $r-p_r$ phase plane are closed, the area of this curve,

$$\mu = \oint p_r \, dr \qquad \ldots (1)$$

being shown by Grossman[7] to be an adiabatic invariant.

The role of this adiabatic invariant in determining cusp losses has been investigated by Berkowitz et al.[3], in plane geometry and using a rather different formalism, and later by Grossman[7] in both plane and axisymmetric geometry. The effective plane cusp width was found to be $4/3 \cdot a_I$ (a_I being the thermal ion gyro-radius in the vacuum magnetic field) in the absence of any mirror at the cusp hole. Uo[8] has extended the work of Berkowitz et al. to include an external magnetic mirror of ratio λ with the result that the loss rate scales as $\lambda^{-3/2}$. Grossman[7] has numerically evaluated the loss rate through an adiabatic axisymmetric cusp for $\lambda = 1$. His result is, however, less than that of Taylor[4], which is an upper limit for the very reason that additional invariants may exist, only if the plasma radius R is less than about $5a_I$.

In this paper, we shall further investigate adiabatic axial cusp losses, both to resolve this anomaly and to consider the effect of an external mirror, the enhanced $\lambda^{-3/2}$ scaling suggested for a plane cusp leading to a significant decrease in the estimated size and output of projected thermonuclear reactors based on cusp type devices[9]. To this end, the more powerful method of Taylor[4] is slightly reformulated to give a logical and simple extension to an adiabatic cusp in

both plane and axial geometry. Simple analytic results are obtained which show the enhanced mirror scaling and also illustrate its origin, whilst the anomalously high result of Grossman[7] is shown to arise from non-conservation of angular momentum.

Asymmetry in the angular momentum loss cone is then shown to lead to substantial nett loss of angular momentum at a point cusp, which gives rise to rapid rotational instability in a theta-pinch.

2. GENERAL THEORY

We now consider a general axisymmetric, steady state, collisionless plasma. Such a plasma is described by the Vlasov equation, which is well known to be satisfied by any function of the constants of the particle motion. Here, we have immediately the two invariants, the particle energy

$$H = \frac{1}{2} (v_r^2 + v_\theta^2 + v_z^2) \qquad \ldots (2)$$

and the particle angular momentum

$$p_\theta = r v_\theta + \psi(r,z) \qquad \ldots (3)$$

where

$$\psi(r,z) = \frac{e}{c} r A_\theta$$

and A_θ is the only non-zero component of the vector potential.

There may exist other invariants. In particular, if the magnetic field is slowly varying longitudinally relative to the characteristic dimension of the particle orbit, then Grossman's adiabatic invariant[7] is generalised to

$$\mu = \frac{1}{2} \oint p_\perp \, d\ell_\perp \qquad \ldots (4)$$

The factor $\frac{1}{2}$ is included for convenience later so that the integration is performed only once between the limits of radial motion, whilst $\ell_\perp$ is a co-ordinate orthogonal to both $\underline{B}$ and θ.

We shall assume these to be the only possible invariants.

The Vlasov equation is now satisfied by any function $f(H,p_\theta,\mu)$, whereupon the flux of any quantity $Q(H,p_\theta,\mu)$ across any r-θ plane becomes

$$F(Q) = 2\pi \int_0^\infty r dr \iiint f Q dv_r\, dv_\theta\, dv_z \qquad \ldots (5)$$

the limits being determined by the accessible region of phase space at the plane of integration. We now depart from Taylor[4] by transforming to the variables (H,p_θ,μ) using

$$\frac{\partial(H,p_\theta,\mu)}{\partial(v_r,v_\theta,v_z)} = - \; r v_z \left[\frac{\partial \mu}{\partial v_r}\right]_{v_\theta,v_z} + \; r v_r \left[\frac{\partial \mu}{\partial v_z}\right]_{v_r,v_\theta} \qquad \ldots (6)$$

Considering initially a point cusp, we have at the cusp hole $\underline{B} \simeq B_z$ and

$$\mu = \int_{r_1}^{r_2} p_r \cdot dr \qquad \ldots (7)$$

where r_1, r_2 are the limits of the radial motion, from which

$$\left[\frac{\partial \mu}{\partial v_z}\right]_{v_r,v_\theta} = 0 \qquad \ldots (8)$$

In addition,

$$\left[\frac{\partial \mu}{\partial v_r}\right]_{v_\theta,v_z} = v_r \left[\frac{\partial \mu}{\partial H}\right]_{p_\theta,p_z} = p_r \int_{r_1}^{r_2} \frac{dr}{p_r} \qquad \ldots (9)$$

Now, from Eqs.(6), (8) and (9), including a factor 2 to allow for the double mapping of v_r in the transformation and simply reversing the order of integration, Eq.(5) becomes:

$$F(Q) = 4\pi \iiint f(H,p_\theta,\mu) \cdot Q(H,p_\theta,\mu)\, dH\, dp_\theta\, d\mu \qquad \ldots (10)$$

where the limits are determined by the loss cone i.e. by the accessible region of phase space at the cusp hole. Thus by using (H,p_θ,μ) as independent co-ordinates, and using Grossman's action integral form for μ, a particularly simple formula for the total flux of any quantity Q results. If μ is not an invariant, the formula Eq.(10) is still valid if evaluated at the plane of the cusp hole (defined as the position of maximum magnetic field) except that in this case, f is independent of μ, which is now merely a convenient notation.

The total radial loss through a ring cusp can be similarly shown to be given by the same formula Eq.(10), with, of course, limits appropriate to this cusp. We are, however, still left with the fundamental problem of obtaining the distribution function and the loss cone, these two items containing all the physics of the problem. We consider the latter first.

3. LOSS CONE

Following previous authors[3-5,7,8], we assume a uniform magnetic field, λB_o and zero electric field at the cusp holes, B_o being the vacuum field required to contain the $\beta = 1$ plasma and λ the mirror ratio at the hole. Thus, at a point cusp, we have

$$\psi = \frac{1}{2} \lambda\, \Omega_o\, r^2$$

where

$$\Omega_o = \frac{e}{c} B_o$$

giving

$$p_r^2 = 2H - p_z^2 - \frac{1}{r^2} (p_\theta - \frac{1}{2} \lambda \Omega_o r^2)^2 .$$

Direct evaluation of Eq.(7) now gives

$$\mu = \frac{\pi}{\lambda\Omega_o} (H - \frac{1}{2} p_z^2) + \frac{\pi}{2} (p_\theta - |p_\theta|) \qquad \ldots (11)$$

whilst the condition $p_r^2 \geqslant 0$ somewhere for the orbit to exist gives

$$\mu \geqslant 0$$

and hence

$$p_\theta \geqslant - \frac{1}{\lambda\Omega_o} (H - \frac{1}{2} p_z^2) .$$

It is apparent that μ is closely related to the magnetic moment, differing in a uniform magnetic field only by a quantity which is itself invariant. Noting μ to be monotonically decreasing with increasing p_z, the point cusp loss cone is now

$$0 < \mu < \begin{cases} \frac{\pi H}{\lambda\Omega_o} & \text{for } p_\theta > 0 \\ \frac{\pi H}{\lambda\Omega_o} + \pi p_\theta & \text{for } -\frac{H}{\lambda\Omega_o} < p_\theta < 0 \\ 0 & \text{for } p_\theta < -\frac{H}{\lambda\Omega_o} \end{cases} \qquad \ldots (12)$$

The strong asymmetry in the loss cone arises from the canonical term in the angular momentum being always positive, large negative values of p_θ thus requiring a large velocity and hence energy.

At a line cusp at radius R^*

$$\psi = \lambda \cdot \Omega_o \cdot R^* \cdot z$$

for which one finds, after some algebra,

$$\mu = \frac{\pi}{\lambda\Omega_o} (H - \frac{1}{2} p_z^2) > 0$$

for all p_θ. The line cusp loss cone is therefore

$$0 < \mu < \frac{\pi H}{\lambda\Omega_o} \qquad \ldots (13)$$

independent of p_θ.

4. DISTRIBUTION FUNCTION

It remains to determine f, which requires more specific definition of the system. The contained plasma is assumed field free ($\beta=1$) and of maximum radius R much greater than the sheath thickness δ, of the order or less than a_I. Penetrating orbits remote from the cusp hole are presumed to be strongly non-adiabatic, which may not be the case in a simple theta-pinch. Particles trapped in the sheath which are capable of escaping through the ends will now do so in a time of the order of the ion transit time through the device. However, particles penetrating the $\beta = 1$ region are contained for the order of $R/\delta \gg 1$ transit times. We therefore assume that at the time of interest, all particles present in the system able to contribute to the end loss have access to the $\beta = 1$ plasma. The distribution function can then be written in the form

$$f(H),p_\theta,\mu) = g(H,p_\theta) \cdot h(H,p_\theta,\mu)$$

where g is the distribution function in the central $\beta = 1$ plasma, and h is 1 or 0 according to whether or not the orbit (H,p_θ,μ) in the sheath penetrates this region, it being through this function that the adiabaticity enters. It is observed that some trapped

particles are still allowed, thus retaining a degree of arbitrariness in the sheath structure.

Considering firstly the function g, the assumed non-adiabaticity implies isotropy in the v_r, v_z phase plane, whilst in symmetric devices such as the spindle-cusp, one also has symmetry in p_θ (i.e. no rotation). Here we further assume that the distribution is completely Maxwellian in this region, it being shown below that a regime exists where this is consistent.

The function h is determined, for our purposes, by considering an escaping particle at the cusp hole, typical orbits for such particles being sketched in Fig.1. If, on reversing its velocity, the particle at some time penetrates the $\beta = 1$ plasma (orbits I and II, Fig.1) it is to be included in the summation whilst if it either passes right through the device in the sheath (orbit III, Fig.1) or is reflected out again, it is to be excluded. In general the determination of this criterion requires computation of the particle orbits. However, two limits are amenable to solution.

In the first case, if the sheath fields are sufficiently non-uniform for all regions of phase space allowed for given (H, p_θ) to be in fact accessible to particles from the plasma, then the necessary and sufficient criterion for a particle to be penetrating is

$$|p_\theta| < R\,(2H)^{\frac{1}{2}} \qquad \ldots (14)$$

giving the non-adiabatic distribution function

$$f(H,p_\theta) = A \cdot \exp(-H/kT) \cdot U(R(2H)^{\frac{1}{2}} - |p_\theta|) \qquad \ldots (15)$$

where

$$A = n/(2\pi kT)^{3/2}$$

and

$$U(x) = 1 \quad \text{for} \quad x > 0$$
$$= 0 \quad \text{for} \quad x < 0$$

This is the case considered by Taylor[4].

If, on the other hand, the motion of trapped particles is everywhere completely adiabatic, simple results can also be obtained. It is now necessary, however, to specify the field profiles in the sheath. We therefore follow previous authors[3,7,8] and assume a discontinuous step in the magnetic field from zero to its vacuum value at the sheath. Considering initially particles with $p_\theta > 0$, from Eq.(11) the invariant for trapped particles in this sheath is just the magnetic moment. Now at the point of locally minimum radius r_{min} of a particle gyrating in the sheath,

$$\mu = \frac{\pi}{2\Omega_o}\left(\frac{p_\theta - \psi_{min}}{r_{min}}\right)^2 .$$

For this particle to be trapped, we require $r_{min} > r_o$, $\psi_{min} > 0$, where r_o is the local $\beta = 1$ plasma radius, giving the criteria

$$\mu \geqslant \frac{\pi}{2\Omega_o}\left(\frac{p_\theta}{r_o}\right)^2$$

and

$$p_\theta < r_o(2H)^{\frac{1}{2}}$$

for the particle to be penetrating at plasma radius r_o.

If $p_\theta < 0$, the particle orbit encloses the axis, and can therefore be trapped only if its gyro-radius is larger than the plasma radius. Such a possibility is ignored. The adiabatic cusp distribution function is therefore

$$f(H,p_\theta,\mu) = A \cdot \exp(-H/kT) \cdot U\left(R(2H)^{\frac{1}{2}} - |p_\theta|\right) \cdot U\left(\mu^{\frac{1}{2}} - \left(\frac{\pi}{2\Omega_o}\right)^{\frac{1}{2}} \frac{p_\theta}{R}\right). \quad \ldots (16)$$

It is noted that adiabaticity may manifest itself in other ways, for example by loss cone depletion as considered by Kilb[10], or by differing parallel and perpendicular temperatures. These, however, require anisotropic distribution (and adiabaticity) in the central $\beta = 1$ regions and are not discussed here.

5. PARTICLE LOSS RATE

The particle loss rate through a non-adiabatic point cusp is now given from Eq.(10), with $Q = 1$, the distribution function (15), and the loss cone (12) as

$$F = 4\pi A \int_{}^{\infty} e^{-\frac{H}{kT}} \cdot dH \left\{ \int^{R(2H)^{\frac{1}{2}}} (\pi H/\lambda\Omega_o) dp_\theta + \int_{-H/\lambda\Omega_o}^{o} (\pi H/\lambda\Omega_o + \pi p_\theta) \cdot dp_\theta \right\}$$

$$= F_o \cdot \left(1 + (\frac{8}{27\pi})^{\frac{1}{2}} \cdot \frac{a}{\lambda R}\right) \qquad \ldots (17)$$

where

$$F_o = \frac{\pi}{2} \frac{n \cdot R \cdot a}{\lambda} \cdot (3kT)^{\frac{1}{2}} \quad \text{and} \quad a = \frac{1}{\Omega_o} \cdot (3kT)^{\frac{1}{2}}$$

which is just Taylor's result as expected.

For the adiabatic cusp the loss cone and the region populated by penetrating orbits are sketched in Fig.2. Integration of Eq.(10), using the distribution function (16), over the shaded area now gives:

$$F = F_o \cdot \left(\frac{2}{3\lambda^{\frac{1}{2}}} + (\frac{8}{27\pi})^{\frac{1}{2}} \cdot \frac{a}{\lambda R} \right) . \qquad \ldots (18)$$

It is seen that for R/a large, the loss rate through an adiabatic point cusp is a factor $2/3\ \lambda^{\frac{1}{2}}$ less than that through a non-adiabatic cusp. The adiabatic loss rate also shows the same $\lambda^{-3/2}$

enhanced scaling with mirror ratio as found by Uo[8] in a plane cusp. Its absolute magnitude is, however, about a factor two less than the numerical results of Grossman[7] at large R/a. This factor has its origin in the limitation on p_θ in the loss cone (12), which effectively eliminates all particles with $p_\theta < 0$. This was not found by Grossman as his expression for μ is valid only for penetrating particles, which in the limit $r_o \to 0$ includes only particles with $p_\theta = 0$ for which his loss cone is indeed correct. Further, all particles considered have originated in the central, isotropic $\beta = 1$ region, and their motion having left this region is completely adiabatic. Any further reduction of this loss rate therefore requires either additional invariants or relaxation of the assumption of isotropy.

It is noted that simple analytic results for a Maxwellian distribution have been obtained by avoiding explicit consideration of the penetrating orbits which has led to considerable complexity previously[7]. However, for the reason just stated, namely that all particles considered have originated in the $\beta = 1$ plasma and have allowed orbits at the hole, the two methods as used are equivalent. It is apparent, however, that both are subject to the possibility that particles with allowed orbits at either end of the nozzle, may in fact be forbidden and reflected in intermediate regions.

It is also apparent from this derivation, that the enhanced mirror effect arises from the absence of a certain class of particles in the sheath. For this effect to occur, therefore, the sheath must be everywhere adiabatic, not merely near the hole.

The line cusp loss rate is similarly given from Eqs.(10), (13)

and (15) as

$$F = 2 \cdot F_o \qquad \ldots (19)$$

for a non-adiabatic cusp, again as given by Taylor[4], and using Eq. (16), as

$$F = \frac{4}{3\lambda^{\frac{1}{2}}} \cdot F_o \qquad \ldots (20)$$

for an adiabatic cusp. Similar conclusions therefore apply to this cusp, the near equality of the line and the two point cusp losses being preserved.

The flux of energy through a cusp hole is similarly calculated by putting $Q = H$ in Eq.(10). In all cases considered, the energy flux per particle is $5/2$ kT to order $a/5R$. The angular momentum flux is similarly given with $Q = p_\theta$ and is non-zero at the point cusp. This can lead to rapid rotational instability in devices such as the theta-pinch or cusp-ended theta-pinch which do not have mirror symmetric point cusps and is discussed in Sec.8 below.

6. RANGE OF VALIDITY

We have assumed above that the plasma radius is large relative to the sheath width, which is readily shown to be of the order a_I for $E = 0$ for the distribution functions used, giving

$$R \gg a_I \qquad \ldots (21)$$

For the assumption $\beta = 1$ to be valid, we require that the magnetic flux at the plasma boundary be small relative to the typical particle angular momentum, $R(kT)^{\frac{1}{2}}$, which leads to

$$(1-\beta)^{\frac{1}{2}} \ll \frac{a_I}{R}$$

which is rather demanding in view of (21).

An estimate of the containment time resulting from the above loss rate is obtained from

$$\tau_c \sim n \cdot V/F$$

where $V = \pi R^2 L$ is the plasma volume, and L the plasma length. Using Eq.(18), we find

$$\tau_c \sim \lambda^{3/2} \cdot \frac{R}{a_I} \cdot \tau_t \qquad \ldots (22)$$

where τ_t is the ion thermal transit time. Thus for (21) satisfied, the containment time is long relative to the transit time, justifying the steady state assumption. It is noted that (22) can be written

$$\tau_c \sim \frac{a_I}{R} \cdot \frac{V}{a_I^2 \cdot (kT)^{1/2}}$$

for which, for both V and a/R constant, one gets the $B^2/T^{3/2}$ scaling of Grossman[7], which is, however, seen to be a consequence of the implicit T/B^2 scaling of L rather than of the adiabaticity.

The fraction of time spent in the sheath by a particle is of order a_I/R, giving the total time spend in the sheath as

$$\tau_s \sim \lambda^{3/2} \tau_t$$

which is required to be short relative to the collision time for trapped particles to be absent. This gives

$$\ell \gg \lambda^{3/2} L$$

where ℓ is the ion-ion mean-free-path. The collision time may still be short relative to the containment time, however, justifying the Maxwellian assumption if

$$\lambda^{3/2} \frac{R}{a_I} \gg \frac{\ell}{L} \ .$$

These results can be combined to give

$$(1-\beta)^{-\frac{1}{2}} \gg R/a_I \gg \ell/\lambda^{3/2} L \gg 1 \qquad \ldots (23)$$

arising in order from the assumptions of $\beta = 1$, Maxwellian distribution, and no trapped particles (i.e. collisionless).

The time scale for depletion of the loss cone is the transit time. It is therefore apparent from (23) that particle collisions are not capable of maintaining an isotropic distribution, depletion of the loss cone in the $\beta = 1$ region therefore occurring if this region is adiabatic. At high β, such depletion has been shown to lead to lack of equilibrium and instability[10]. It is therefore required that penetrating orbits be non-adiabatic remote from the cusp hole, as assumed, if the mirror enhancement is to be observed i.e.

$$\frac{dr_o}{dz} \sim 1$$

somewhere. This may well not be satisfied in simple theta-pinches, for which however, the non-adiabatic results Eq.(17) will be a useful approximation. If, on the other hand, collisions are sufficiently frequent to populate the loss cone, then trapped orbits will also be occupied giving a somewhat greater loss than suggested by Eqs.(17) or (18) as appropriate. Such devices are in any case shown below to be rotationally unstable under conditions for which mirror enhancement is found.

Finally, we remark that the range of β of interest falls above that for which the stability has been recently queried[11,12].

7. PLANE CUSP

We now briefly consider the plane cusp. Taking the z-axis parallel to $\underline{B}$ at the cusp hole and the x-axis parallel to the line of the cusp, the invariants of the particle motion are the energy Eq.(2), the x-momentum

$$p_x = v_x + \frac{e}{c} \cdot A_x$$

and possibly

$$\mu = \tfrac{1}{2} \oint p_y \cdot dy \ .$$

Using an approach analogous to that above, the loss rate per unit length of the cusp is found to be

$$F = 2 \cdot \iiint f \cdot Q \cdot dH \ dp_x \ d\mu$$

integrated over the loss cone. In the uniform magnetic field at the cusp hole

$$\mu = \frac{\pi}{\lambda \Omega_o} \ (H - \tfrac{1}{2} p_z^2)$$

giving the loss cone

$$0 < \mu < \frac{\pi H}{\lambda \Omega_o}$$

Also as above, the conditions that an adiabatic particle orbit penetrate the $\beta = 1$ plasma become

$$\mu > \pi \ p_x^2 / 2\Omega_o$$

and

$$|p_x| < (2H)^{\frac{1}{2}}$$

Taking a monoenergetic distribution for comparison purposes,

$$f(H, p_x, \mu) = \frac{n}{4\pi} \ (2H_o)^{-\frac{1}{2}} \cdot \delta(H - H_o) \cdot U((2H)^{\frac{1}{2}} - |p_x|)_o U(\mu - \frac{\pi}{2\Omega_o} p_x^2)$$

one gets

$$F = \frac{4 \cdot n \cdot H_o}{3 \cdot \lambda^{3/2} \cdot \Omega_o} \cdot \qquad \ldots (24)$$

If one expands Grossman's result[7] for a plane cusp in powers of a/R, it is found to be less than Eq.(24) only by terms of order $(a/R)^2$ or less, arising from his exclusion of particles crossing the plane of integration in the sheath. This confirms the assertion of the equivalence of the two methods to this order.

8. ROTATION OF $\beta = 1$ CUSP-ENDED SYSTEMS

A familiar feature of theta-pinch experiments is the observation of rotating flute instabilities after about an Alfvén transit time, many mechanisms suggested for the production of this rotation being recently surveyed by Haines[13]. We shall here discuss a further mechanism which appears from a simple extension of the above work.

The stability of a low β rotating plasma column has been considered by Rosenbluth et al.[14,15] who, for rigid rotation, obtain the criterion

$$1 + M^{1/2} > 2 \cdot \frac{\omega}{\omega_d} > 1 - M^{1/2} \qquad \ldots (25)$$

for stability to the M^{th} flute mode for $M \geqslant 2$, where ω is the guiding centre angular velocity and ω_d that associated with the diamagnetic currents. The $M = 1$ mode is presumed stabilised by other means, such as a conducting wall[16]. For non-rigid rotation, shearing instabilities can arise[15], no simple criterion being available; however, Taylor[17] has shown (25) to have some validity at higher values of β. We here assume this criterion to be valid in $\beta = 1$ narrow sheath geometry with ω, ω_d taken as the equivalent

rigid body angular velocities to give the same average angular momentus per ion-electron pair, $\bar{p}_{\theta r}$ and $\bar{p}_{\theta d}$ respectively.

Taylor[17] has also shown that at the densities of interest in high β devices, the canonical contribution to the nett angular momentum is negligible as a result of charge neutrality, whilst the nett angular momentum per unit length due to diamagnetic currents can be written, now with explicit inclusion of the particle masses, as

$$P_d = \int_o^\infty (B(r) - B_o) \cdot r \cdot dr \cdot \sum_i \frac{c}{qi} m_i \eta_i$$

where q_i is the particle charge of and η_i the fraction carried by the i^{th} species. Ignoring the contribution of the sheath to the particle line density and using pressure balance, we now get

$$\bar{p}_{\theta d} = -4 a_I \left(\frac{m_I kT}{3} \right)^{\frac{1}{2}} + O\left(\frac{a_I}{R}\right). \qquad \ldots (26)$$

The equivalent angular velocity of rigid rotation is

$$\omega_d \sim \left(\frac{a_I}{R}\right)^2 \Omega_I \qquad \ldots (27)$$

where $\Omega_I = \Omega_o / m_I$ is the ion gyro-frequency in the vacuum field. We are therefore interested in rotational velocities of order a_I/R of the ion thermal velocity, which can be shown to modify the end loss rates given above by terms only of the same order a_I/R, which are neglected.

Now, from Eq.(10) with $Q = p_\theta$, (12) and (15) the flux of angular momentum for the i^{th} species through a non-adiabatic point cusp is

$$F_i(p_\theta) = \left(\frac{8\pi}{3}\right)^{\frac{1}{2}} nR^2 \cdot a_i \cdot kT \cdot \frac{q_i}{\lambda |q_i|} \left[1 - \left(\frac{a_i}{R}\right)^2 \right]$$

giving the loss per particle as

$$\bar{p}_{\theta zi} = \frac{q_i}{|q_i|} \cdot R \cdot \left(\frac{32\ kT\ m_i}{9\pi} \right)^{\frac{1}{2}} + O\left(\frac{a_I}{R}\right) \qquad \ldots (28)$$

The equivalent calculation at a line cusp gives zero nett flux. To maintain charge neutrality, the electron and ion particle losses must be equal, giving the angular momentum loss per ion-electron pair as

$$\bar{p}_{\theta z} = R \cdot \left(\frac{32 \cdot kT \cdot m_i}{9\pi} \right)^{\frac{1}{2}} + O\left(\frac{m_e}{m_I}\right)^{\frac{1}{2}} \qquad \ldots (29)$$

which is immediately observed to be of opposite sign to and much greater than (26) the average per contained pair, a result of the strong asymmetry of the loss cone. The convention used gives $p_\theta < 0$ in the direction of rotation of the ions in the vacuum magnetic field, the change in sign between Eqs.(26) and (29) being readily confirmed by simple physical considerations.

To investigate the rotation of any device, it is necessary to average Eq.(29) over all cusp holes. In some systems, the axial component of magnetic field is oppositely directed in different regions. This is taken into account by the transformation $p_\theta \rightarrow - p_\theta$ in the above work in the relevant sections. Thus, in a spindle-cusp the two point cusps are mirror symmetric, the contributions (29) from either end then exactly cancelling to give zero nett end loss and no rotation by this mechanism. In a theta-pinch, however, the two ends are complementary with a consequently large average loss per particle. The same applies to the cusp-ended theta-pinch, any induced rotation now being in the opposite direction relative to the central theta-pinch field, however.

To estimate the effect of this large angular momentum flux, we consider a quasi-steady state theta-pinch. The conservation equations

$$\frac{d}{dt}(NL) = -2F$$

and

$$\frac{d}{dt}(NL\bar{H}) = -5\,kT\cdot F$$

where N is the ion line density and $\bar{H}$ the mean energy per ion-electron pair, have the linear solutions for small times t,

$$\frac{n}{n_o} = 1 + \frac{2t}{3\tau_c}$$

$$\frac{T}{T_o} = 1 - \frac{2t}{3\tau_c} \qquad \ldots (30)$$

$$\frac{R}{R_o} = 1 - \frac{5t}{6\tau_c}$$

where

$$\tau_c = \frac{N_o L}{2F}$$

is the containment time, and the subscript zero implies initial values.

Angular momentum conservation now gives

$$\frac{d}{dt}(NL\bar{p}_\theta) = -2\cdot F\cdot\bar{p}_{\theta z} \qquad \ldots (31)$$

where

$$\bar{p}_\theta = \bar{p}_{\theta d} + \tfrac{1}{2}\, m_I\, R^2\omega\ .$$

Using Eq.(30), Eq.(31) can be written

$$\frac{d\omega}{dt} = -\frac{4F\,\bar{p}_{\theta z}}{N\cdot R^2\cdot L\, m_I}\left[1 - \frac{\bar{p}_{\theta d}}{\bar{p}_{\theta z}} - \frac{8}{3}\frac{\bar{p}_{\theta r}}{\bar{p}_{\theta z}} + O\left(\frac{t}{\tau_c}\right) + O\left(\frac{a_I}{R}\right)\right].$$

From (26), (27) and (29), the second and third terms are of order (a_I/R), giving the linear solution, with $\omega = 0$ at $t = 0$,

$$\frac{\omega}{\Omega_I} = -2.53\left(\frac{R}{a_I}\right)^2\frac{t}{\tau_{inst}}$$

where

$$\tau_{inst} = 0.46 \frac{a_I}{\bar{p}_{\theta z}} (m_I kT)^{\frac{1}{2}} \tau_c \qquad \ldots (32)$$

$$\approx 0.5 \lambda \tau_t \qquad \ldots (33)$$

is the time at which the stability criterion (25) is marginally satisfied for the $M = 2$ mode. Thus a collisionless $\beta = 1$ theta-pinch with an ion gyro-radius sheath is rotationally unstable after the order of one ion transit time, much less than the containment time with the rotation in the direction of the ion motion in the vacuum field. The steady-state theory is only accurate for vatiations on a scale long relative to the transit time. In practice, a torsional Alfvén wave would propagate from the end, its velocity in $\beta = 1$ system being near the ion thermal speed. The rapidity of onset of the predicted instability is, however, noteworthy, and arises from the negligible contribution of the electrons to Eq.(29). This, in turn, arises as in the absence of diffusion the electrons flow through a hole related in area to their own gyro-radius, with trapped electrons providing charge neutrality in the sheath. Such trapped electrons may be introduced by wall shorting of transverse electric fields[17,18]. However, the simultaneous acquisition of angular momentum associated with these currents is not required for this mechanism, instabilities during or immediately after the creation of the plasma being capable of providing the trapped electrons which are the essential feature. If, on the other hand, the electrons broadening the sheath arise from a dynamic balance between radial diffusion and end loss, the nett end loss of angular momentum per ion-electron pair will be of the order of Eq.(26), rotation then occurring only after a considerable fraction of the plasma has escaped. This is the mechanism considered and eliminated by Rostoker and Kolb[19].

The rate of diffusion required to stabilise this mechanism can be estimated by observing that diffusion of an ion-electron pair through a distance δr is equivalent to a transfer of a quantity $\psi(r+\delta r) - \psi(r)$ of angular momentum from the ion to the electron if the energy transfer is negligible, as in unlike particle collisions. The total rate of transfer by diffusion is therefore

$$\left(\frac{\partial P_{\theta I}}{\partial t}\right)_{diff} = -\int_0^\infty n(r)\, A(r)\, \bar{v}_r\, \frac{d\psi}{dr}\, dr \qquad \ldots (34)$$

where $A(r)$ is the surface area,

$$\bar{v}_r = \frac{D}{n}\frac{dn}{dr} \approx \frac{D}{\delta}$$

is the radial drift velocity, D is the diffusion co-efficient and δ the sheath width. Now Eq.(34) can be written approximately

$$\left(\frac{\partial P_{\theta I}}{\partial t}\right)_{diff} \approx -\, n\, \pi R^2 L\, m_I\, \Omega_I\, D$$

which, for diffusion to stabilise this mechanism, must be greater than the nett end loss, $2 \cdot F \cdot \bar{p}_{\theta z}$ (carried by the ions), giving, from Eqs.(29) and (17),

$$\frac{D}{D_c} \gtrsim 7 \left(\frac{m_e}{m_I}\right)^{\frac{1}{2}} \frac{\ell}{\lambda L} \qquad \ldots (35)$$

where $D_c = \frac{1}{2 \cdot 2} \cdot a_e^2 \cdot \nu_{eI}$ is the classical diffusion co-efficient for binary collisions[20], with ν_{eI} the electron-ion collision frequency. Thus to stabilise the theta-pinch to this mechanism, the effective mean-free-path for collisions has to be very much shorter than the plasma length. This criterion is observed to be equivalent to demanding that radial diffusion be sufficiently rapid for the electrons to diffuse a distance of the order of the ion gyro-radius in λ ion transit times, and thus be able to maintain an ion gyro-radius sheath.

In an adiabatic cusp, using Eq.(16) the point cusp loss rate of angular momentum is found to be reduced by a factor 2λ, with a corresponding increase in the stable time, τ_{inst}, and decrease in the critical D/D_c condition (35). However, except at very large λ, the stability condition (35) is inconsistent with the 'collisionless' requirement (23) for the observation of any enhanced mirror effect. Thus, in addition to the equilibrium problems discussed above which rule out the appearance of mirror enhancement in theta-pinches, it is further concluded that for rotational stability, any device showing the effect must have mirror symmetric point cusps, and hence an odd number of line cusps. Both the mirror enhancement and the rotational stability are modified by transverse electric fields and trapped magnetic fields ($\beta < 1$). It appears, however, that the former appears only in a narrow regime of weak electric fields and high β when the rotation is strongest[21].

9. CONCLUSION

Previous calculations of the particle loss rate from a $\beta = 1$ cusp have been extended to the case when a third 'adiabatic' invariant of the particle motion exists, in addition to the energy and momentum. Simple analytic results have been obtained for the loss from a Maxwellian plasma in both plane and axisymmetric geometry. In both cases, the adiabaticity of particles trapped in the sheath reduces the particle flux by a factor $2/3\ \lambda^{\frac{1}{2}}$ to give an overall $\lambda^{-\frac{3}{2}}$ scaling with mirror ratio. For the axisymmetric cusp, the magnitude of this loss is about half the previously accepted value (for $\lambda = 1$ only). The enhanced scaling was shown to arise from the absence of a certain class of particles in the sheath, which must therefore be

adiabatic everywhere, whilst the mean-free-path must also be long relative to the plasma length. As a consequence, the validity of the results for simple theta-pinches is queried.

The large flux of angular momentum at a point cusp, due to the preferential loss of ions rotating in a particular direction, was shown to lead to rapid rotation of devices, such as the theta-pinch, with non-mirror symmetric point cusps. This rotation occurs whenever the sheath is significantly broadened by trapped electrons and is therefore stabilised by radial diffusion. An effective mean-free-path much less than the plasma length is required, however.

10. ACKNOWLEDGEMENTS

The author wishes to express his appreciation to his supervisors, Drs. L.C. Woods and T.K. Allen, and also to acknowledge useful discussions with Drs. I.J. Spalding and J.A. Wesson.

11. REFERENCES

1. H. Grad, in Progress in Nucl. Energy, Series XI, Plasma Physics, C.L. Longmire et al., Eds. (Pergamon Press, London, 1963). 2, 189.

2. H. Grad, Phys. Rev. Letters 4, 222 (1960).

3. J. Berkowitz, K.O. Friedrichs, H. Goertzel, H. Grad, J. Killeen and E. Rubin, Proc. 2nd. U.N. International Conference on the Peaceful Uses of Atomic Energy, 31, 171 (1958).

4. J.B. Taylor, Culham Laboratory Report, CLM- R 58 (1966).

5. O.B. Firsov, in Plasma Physics and the Problem of Controlled Thermonuclear Reactions, M.A. Leontovich, Ed. (Pergamon Press, London, 1959) 3, 386.

6. C.S. Gardner, Phys. Rev. 115, 791, (1959).

7. W.B. Grossman, Phys. Fluids 9, 2478 (1966).

8. K. Uo, Unpublished information.

9. I.J. Spalding, Nucl. Fusion 8, 161 (1968).

10. R.W. Kilb, Proc. A.P.S. Topical Conference on Pulsed High Density Plasmas, LA-3770, Paper No.B5, (1967).

11. R.L. Morse, Phys. Fluids 11, 1558 (1968).

12. J.P. Friedberg and W. Grossman, Phys. Fluids 11, 2476 (1968).

13. M.G. Haines, Advances in Physics 14, 167 (1965).

14. M.N. Rosenbluth, N.A. Krall and N. Rostoker, Nucl. Fusion Supplement Pt.1, 143 (1962).

15. M.N. Rosenbluth and A. Simon, Phys. Fluids 8, 1300 (1965).

16. F.A. Haas and J.A. Wesson, Phys. Fluids 9, 2472 (1966).

17. J.B. Taylor, J. Nucl. Energy Pt.C 4, 401 (1962).

18. K.V. Roberts, in H.A.B. Bodin et al., Nucl. Fusion Supplement Pt.2, 521 (1962).

19. N. Rostoker and A.C. Kolb, Phys. Rev. 124, 965 (1961).

20. L. Spitzer, Physics of Fully Ionized Gases (New York, Interscience, (1962), 2nd ed.

21. A.S. Kaye, To be published.

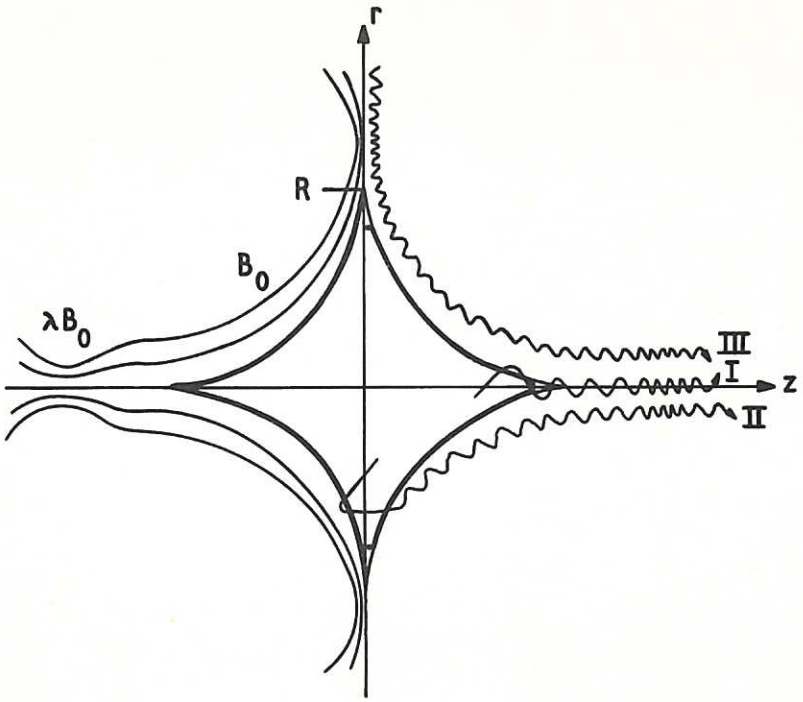

Fig. 1 (CLM-P 193)

Spindle-cusp device – heavy lines indicate the limits of the $\beta = 1$ region on the left and the non-adiabatic region on the right. Also shown are typical magnetic field lines and particle orbits for:
I - $p_\theta < 0$; II - $p_\theta > 0$, penetrating; III - $p_\theta > 0$, trapped

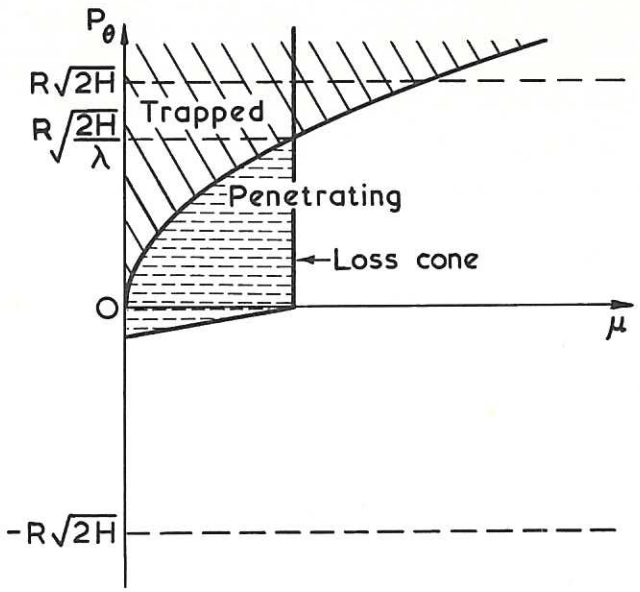

Fig. 2 (CLM-P 193)

The loss cone and the phase space region occupied by particles trapped in an adiabatic sheath