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THE INFLUENCE OF NON-ELECTROSTATIC EFFECTS UPON A HYBRID INSTABILITY

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ABSTRACT

In a recent paper Ashby and Paton[6] demonstrated the existence of a high frequency, high growth rate electrostatic instability in a simple model of a low-β plasma. In this paper we consider the influence of non-electrostatic effects upon their instability. The mode is no longer purely longitudinal and it is shown that provided the appropriate condition is satisfied then the model is stable. The relevance of the model to the cusp and to the implosion phase of the theta pinch is discussed.

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C O N T E N T S

I. INTRODUCTION

There now exist a number of papers discussing the hydromagnetic stability of high-β machines, particularly for cusp and the linear theta-pinch[1-4]. Very little is known, however, about the effect of non-hydromagnetic instabilities on these machines. In a recent feasibility study of a high-β machine as a thermonuclear reactor, Spalding[5] indicates the micro-stability of the plasma boundary to be crucial in determining containment time. Thus any information that can be obtained regarding this problem is of considerable interest.

Recently, Ashby and Paton[6] considered the stability of a collision-free low-β plasma sheath whose thickness is of the order of an ion Larmor radius. To simulate the directed motion of the ions in the sheath, they used a homogeneous model in which a mono-energetic beam of ions moved in straight lines perpendicular to a uniform magnetic field, and through a background of cold electrons. They further neglected all space-charge effects in the unperturbed state. The assumption of straight-line motion for the ions is justified since the authors were concerned with the frequency range $\Omega_i \ll \omega \ll \Omega_e$, Ω_i and Ω_e being the gyro-radii of the ions and electrons respectively. Making the assumption that perturbations are purely electrostatic they obtained a high frequency, high growth rate instability of the two-stream type. In their paper, Ashby and Paton presented experimental results consistent with the characteristics of this instability.

Paton[7] has suggested that this instability could be relevant to a high-β device such as a cusp or theta-pinch. A feature of such a machine is that there is a region where the density falls off in a distance smaller than, or comparable with, the ion Larmor radius. In this case there will be regions where the ions are moving predominantly in a direction perpendicular to the magnetic field. The effect is to produce an ion-diamagnetic current at the 'edge' of the plasma. It follows that the phenomenon discussed by Ashby and Paton may very well occur in the vicinity of the sheath region of a high-β plasma.

The electrostatic assumption requires that the phase velocity of the wave be small compared to the velocity of light, i.e. $v_{ph} \ll c$, and that $\beta \left(\sim \frac{nkT}{B^2}\right)$ be small compared to unity. Although the Ashby-Paton analysis must apply at the very 'edge' of the plasma, it is of considerable interest to determine how their mode is influenced by both the non-electrostatic effect, $v_{ph} \sim c$, and the finite-β effect, $\beta \sim 1$. That finite-β effects can stabilise has been demonstrated by Mikhailovskaya and Mikhailovskii[8] for the drift instability in an inhomogeneous plasma. In order to keep the problem as simple as possible we shall only consider the non-electrostatic effect. It is hoped to consider the effects of finite-β in a later paper. In this paper we reconsider the simple model, but now take the magnetic field and density to vary in a direction perpendicular to both the ion flow and the field. Since we are interested in non-electrostatic conditions it is necessary to solve the fluid equations in conjunction with the full Maxwell equations.

Although the present work has arisen through consideration of the sheath at the 'edge' of a high-β plasma, it is likely that the mode investigated could arise in a variety of circumstances. For example, during the early stages of implosion in a theta-pinch, rapid field diffusion occurs, and the current sheath broadens to a thickness comparable with the plasma radius (two or three ion Larmor radii), and an order of magnitude greater than that computed on the basis of ideal MHD theory[9]. The conditions under which the instability derived in this paper can occur, are consistent with those present during the early stages of a high-β pinch. Such non-classical behaviour can be used to set up a reversed magnetic field in a high-β toroidal pinch, and a field configuration of this type has favorable hydromagnetic stability properties.

II. THE MODEL

The work described in this paper is based on the two-fluid equations, which in c.g.s. units take the form

$$\frac{d\underline{v}_i}{dt} = \frac{e}{M}\left(\underline{E} + \frac{\underline{v}_i \times \underline{B}}{c}\right) \qquad \ldots (1)$$

for the ions, and

$$\frac{d\underline{v}_e}{dt} = -\frac{e}{m}\left(\underline{E} + \frac{\underline{v}_e \times \underline{B}}{c}\right) \qquad \ldots (2)$$

for the electrons, all symbols having their usual meaning. We neglect collisions and take both the ions and electrons to be cold. The assumptions underlying our model will be discussed in Sec.V. The continuity equations for ions and electrons are

$$\frac{\partial n_{i,e}}{\partial t} + \nabla \cdot (n_{i,e}\, \underline{v}_{i,e}) = 0 \; . \qquad \ldots (3)$$

We also require Maxwell's equations, namely,

$$\nabla \times \underline{E} = -\frac{1}{c}\frac{\partial \underline{B}}{\partial t} \; , \qquad \ldots (4)$$

$$\nabla \times \underline{B} = \frac{4\pi}{c}\,\underline{j} + \frac{1}{c}\frac{\partial \underline{E}}{\partial t} \; , \qquad \ldots (5)$$

$$\nabla \cdot \underline{B} = 0 \, , \qquad \ldots (6)$$

and

$$\nabla \cdot \underline{E} = 4\pi\rho \, , \qquad \ldots (7)$$

where the current and charge densities are given by

$$\underline{j} = e(n_i \underline{v}_i - n_e \underline{v}_e) \qquad \ldots (8)$$

and

$$\rho = e(n_i - n_e) \, , \qquad \ldots (9)$$

respectively.

For our equilibrium we consider a beam of ions with uniform velocity V to be streaming in straight lines perpendicular to a magnetic field B. The magnetic field is assumed to define the z-direction and the ion motion is taken to be in the y-direction (see Fig.1). As mentioned in Sec.I, the assumption of straight line motion for the ions is valid for frequencies very large compared to the ion gyro-frequency. Charge neutrality is provided by a background of cold electrons. The magnetic field and particle densities vary in the x-direction, and are related through the Maxwell equation

$$\frac{dB}{dx} = - 4\pi e n_i \cdot \frac{V}{c} \, . \qquad \ldots (10)$$

In Sec.III we linearise the equations and assume perturbed quantities q to have the form

$$q = \hat{q} \exp[i(k_x x + k_y y + k_z z + \omega t)] \, .$$

The assumption of normal mode solutions should be reasonable provided $\lambda_x \ll L$, that is the wavelength in the direction of the inhomogeneity is small compared to the scale-length. We shall examine stability in the frequency range $\Omega_i \ll \omega \ll \Omega_e$ and assume the following conditions to be valid,

$$k_z \ll k_y, \quad \omega \sim k_y V \sim \omega + k_y V, \quad \frac{\omega_{pe}}{k_y c} \sim 1,$$

$$\frac{\omega}{k_y c} \ll 1, \quad \frac{\omega_{pi}}{k_y c} \ll 1, \quad \Omega_i \ll \omega_{pi},$$

where the ion and electron gyro-frequencies are given by $\Omega_i = \frac{Be}{Mc}$ and $\Omega_e = \frac{Be}{mc}$ respectively, and the ion and electron plasma frequencies are given by $\omega_{pi}^2 = \frac{4\pi e^2 n_i}{M}$ and $\omega_{pe}^2 = \frac{4\pi e^2 n_e}{m}$ respectively. It is also assumed that the perpendicular wavelength λ_y is very small compared to the ion Larmor radius a_i, which is of order V/Ω_i. If the small parameter $\varepsilon = \left(\frac{m}{M}\right)^{\frac{1}{2}}$ is introduced and we take $k_z \sim \varepsilon k_y$, then the above range of conditions can be succinctly expressed by the ordering scheme,

$$\omega^2 \sim \varepsilon^2 k_y^2 c^2 \sim \varepsilon^2 \omega_{pe}^2 = \omega_{pi}^2 \sim \varepsilon^2 \Omega_e^2 = \Omega_e \Omega_i = \frac{1}{\varepsilon^2} \Omega_i^2$$

$$\sim \frac{1}{\varepsilon^2} \left(\frac{V}{a_i}\right)^2 \sim k_y^2 V^2 \sim (\omega + k_y V)^2 .$$

The order of the field gradient may be deduced from Eq.(10). Thus

$$\frac{1}{B}\frac{dB}{dx} \sim \frac{1}{L} \sim \frac{4\pi en}{B} \varepsilon \sim \frac{\omega_{pe}^2}{\Omega_e} \cdot \frac{1}{c} \varepsilon \sim k_y \varepsilon ,$$

and it follows that $\lambda_y \sim \varepsilon L$. Since $\lambda_x \ll L$, we take

$$\lambda_y \sim \lambda_x \sim \varepsilon L .$$

On this ordering, the scale-length of the field variation is of order the ion Larmor radius. Physically, we expect the scale-length of the density variation to have an order comparable with that of the field variation.

This scheme leads to a dispersion equation which is consistent with the adopted ordering. It should be added that it is not essential to introduce an ordering scheme, but it is a means of ensuring that approximations are made correctly.

III. DERIVATION OF DISPERSION EQUATION

We now linearise the equations of the previous section and take normal modes. Using Eq.(4) the perturbed magnetic field may be eliminated from (1), and the following expressions for the perturbed velocity components obtained,

$$v_{ix} = \frac{1}{\Omega_i^2 - (\omega + k_y V)^2} \cdot \frac{e}{M}\left[E_y\left(\Omega_i - \frac{ik_xV}{\omega}\,(\omega + k_yV)\right) + \frac{i}{\omega}\,(\omega + k_yV)^2 E_x\right]$$

$$v_{iy} = \frac{i(\omega + k_yV)}{\Omega_i^2 - (\omega + k_yV)^2} \cdot \frac{e}{M}\left[E_y\left(1 - \frac{i\Omega_i k_x V}{\omega(\omega + k_yV)}\right) + \frac{i\Omega_i}{\omega}\,E_x\right] \qquad \ldots (11)$$

$$v_{iz} = \frac{1}{i(\omega + k_yV)} \cdot \frac{e}{M}\left[E_z\left(1 + \frac{k_zV}{\omega}\right) - \frac{k_zV}{\omega}\,E_y\right],$$

where all terms have been retained. The equivalent expressions for the electrons are

$$v_{ex} = \frac{\frac{e}{m}}{\Omega_e^2 - \omega^2}\left[\Omega_e E_y - i\omega E_x\right]$$

$$v_{ey} = \frac{-\frac{e}{m}}{\Omega_e^2 - \omega^2}\left(i\omega E_y + \Omega_e E_x\right) \qquad \ldots (12)$$

$$v_{ez} = \frac{i}{\omega} \cdot \frac{e}{m}\,E_z \,.$$

By Eqs.(4,5) and (8),

$$\nabla \times (\nabla \times \underline{E}) - \frac{\omega^2}{c^2}\,\underline{E} = -\frac{i\omega 4\pi e}{c^2}\,(n_i'\,V\hat{\underline{y}} + n_i \underline{v}_i' - n_e \underline{v}_e') \,, \qquad \ldots (13)$$

n_i' denoting the perturbed ion density. Using Eqs.(11) and (12), the x-component of (13) may be written in the form,

$$I_x E_x + I_y E_y + I_z E_z = 0 \,, \qquad \ldots . (14)$$

where

$$I_x = 1 + \frac{1}{k_y^2}\left(k_z^2 - \frac{\omega^2}{c^2}\right) - \frac{\omega_{pi}^2}{k_y^2 c^2}\left[\frac{(\omega + k_yV)^2}{\Omega_i^2 - (\omega + k_yV)^2} + \frac{\frac{M}{m}\,\omega^2}{\Omega_e^2 - \omega^2}\right],$$

$$I_y = -\frac{k_x}{k_y} + \frac{\omega_{pi}^2}{k_y^2 c^2}\left[-\frac{1}{\Omega_i^2 - (\omega + k_y V)^2}\left(i\omega\Omega_i + k_x V(\omega + k_y V)\right) - \frac{M}{m}\cdot\frac{i\omega\Omega_e}{\Omega_e^2 - \omega^2}\right],$$

and

$$I_z = -\frac{k_x k_z}{k_y^2}.$$

Using Eqs.(3), (11) and (12), the y-component of (13) may be written as,

$$II_x E_x + II_y E_y + II_z E_z = 0 , \qquad \ldots (15)$$

where

$$II_x = -\frac{k_x}{k_y} + \frac{i\omega\omega_{pi}^2}{k_y^2 c^2}\left\{V\left(\frac{1}{n}\frac{dn}{dx} - 2\Omega_i\frac{d\Omega_i}{dx}\cdot\frac{1}{\Omega_i^2 - (\omega + k_y V)^2} + ik_x\right)\frac{\omega + k_y V}{\omega((\omega + k_y V)^2 - \Omega_i^2)}\right.$$

$$\left. + \frac{M}{m}\cdot\frac{\Omega_e}{\Omega_e^2 - \omega^2} + \frac{\Omega_i}{(\omega + k_y V)^2 - \Omega_i^2}\right\},$$

$$II_y = \frac{k_x^2}{k_y^2} + \frac{k_z^2}{k_y^2} - \frac{\omega^2}{k_y^2 c^2} + \frac{\omega_{pi}^2}{k_y^2 c^2}\left\{\frac{V(\omega\Omega_i - ik_x V(\omega + k_y V))}{(\omega + k_y V)[(\omega + k_y V)^2 - \Omega_i^2]}\left(\frac{1}{n}\frac{dn}{dx} - \frac{2\Omega_i\cdot\frac{d\Omega_i}{dx}}{\Omega_i^2 - (\omega + k_y V)^2} + ik_x\right)\right.$$

$$- \frac{M}{m}\cdot\frac{\omega^2}{\Omega_e^2 - \omega^2} - \frac{\omega^2}{\Omega_i^2 - (\omega + k_y V)^2}\left(1 + \frac{\Omega_i k_x V}{i\omega(\omega + k_y V)}\right)$$

$$\left. - \frac{\omega V\frac{d\Omega_i}{dx}}{(\omega + k_y V)(\Omega_i^2 - (\omega + k_y V)^2)} + \frac{k_z^2 V^2}{(\omega + k_y V)^2}\right\},$$

$$II_z = -\frac{k_z}{k_y} - \frac{\omega_{pi}^2}{k_y^2 c^2}\cdot\frac{k_z V}{\omega + k_y V}.$$

To obtain a further equation relating the perturbed electric field components, we use Poisson's equation. Using Eqs.(3), (9), (11) and (12), Equation (7) may be written in the form,

$$III_x E_x + III_y E_y + III_z E_z = 0 , \qquad \ldots (16)$$

where

$$III_x = \frac{\omega_{pi}^2}{\omega} \cdot \frac{\omega + k_y V}{(\omega+k_y V)^2 - \Omega_i^2} \left(\frac{ik_x}{k_y} - \frac{2\Omega_i}{k_y} \cdot \frac{d\Omega_i}{dx} \cdot \frac{1}{\Omega_i^2 - (\omega+k_y V)^2} + \frac{1}{k_y} \cdot \frac{1}{n}\frac{dn}{dx} \right)$$

$$- \frac{\omega_{pe}^2}{\Omega_e^2 - \omega^2} \left(\frac{ik_x}{k_y} - \frac{2\Omega_e}{k_y} \frac{d\Omega_e}{dx} \cdot \frac{1}{\Omega_e^2 - \omega^2} + \frac{1}{k_y} \cdot \frac{1}{n}\frac{dn}{dx} \right)$$

$$+ \frac{\Omega_i \omega_{pi}^2}{\omega(\Omega_i^2 - (\omega+k_y V)^2)} + \frac{\Omega_e}{\omega} \cdot \frac{\omega_{pe}^2}{\omega^2 - \Omega_e^2} - \frac{ik_x}{k_y} ,$$

$$III_y = \frac{\omega_{pi}^2}{i\omega(\omega+k_y V)} \cdot \frac{\omega\Omega_i - ik_x V(\omega+k_y V)}{(\omega+k_y V)^2 - \Omega_i^2} \left\{ \frac{ik_x}{k_y} - \frac{2\Omega_i}{k_y} \frac{d\Omega_i}{dx} \cdot \frac{1}{\Omega_i^2 - (\omega+k_y V)^2} + \frac{1}{k_y} \cdot \frac{1}{n}\frac{dn}{dx} \right\}$$

$$+ \frac{\omega_{pi}^2}{(\omega+k_y V)^2 - \Omega_i^2} \left(i + \frac{\Omega_i k_x V}{\omega(\omega+k_y V)} \right) + \frac{k_z^2 V \, \omega_{pi}^2}{i\omega \, k_y} \cdot \frac{1}{(\omega+k_y V)^2} - i$$

$$+ \frac{\omega_{pe}^2 \Omega_e}{i\omega(\Omega_e^2 - \omega^2)} \left[\frac{ik_x}{k_y} - \frac{2\Omega_e}{k_y} \cdot \frac{d\Omega_e}{dx} \frac{1}{\Omega_e^2 - \omega^2} + \frac{1}{k_y n} \frac{dn}{dx} \right] + \frac{i\omega_{pe}^2}{\omega^2 - \Omega_e^2}$$

$$+ \frac{i\omega_{pi}^2}{(\omega+k_y V)} \circ \frac{1}{k_y} \cdot \frac{d\Omega_i}{dx} \cdot \frac{1}{\Omega_i^2 - (\omega+k_y V)^2} - \frac{i\omega_{pe}^2}{\omega k_y} \cdot \frac{d\Omega_e}{dx} \cdot \frac{1}{\Omega_e^2 - \omega^2} ,$$

$$III_z = \left[\frac{\omega_{pi}^2}{\omega(\omega+k_y V)} + \frac{\omega_{pe}^2}{\omega^2} - 1 \right] \frac{ik_z}{k_y} .$$

Now the condition that Eqs.(14), (15) and (16) have a non-trivial solution, is obtained by setting the determinant Δ formed from their coefficients, equal to zero. It now becomes convenient to use the ordering scheme. We observe that the leading terms of Δ are of order $1/\varepsilon$. Thus forming Δ to <u>all orders</u> from only those coefficients of Eqs.(14) to (16) such that their products have <u>leading-order</u> $1/\varepsilon$, we obtain the determinantal equation,

$$\Delta(1/\varepsilon) = \Big(I_x(1) II_y(1) - I_y(1) II_x(1) \Big) III_z(1/\varepsilon) = 0 , \qquad \ldots (17)$$

where for example, $III_z(1/\varepsilon)$ denotes those terms from the coefficients of E_z in Eq.(16) which are of leading-order $1/\varepsilon$. Now in fact the terms comprising Eq.(17) cancel exactly. In other words, cancellation occurs to all orders and so there are no terms to be carried through to $\Delta(1)$. This is an indication, as remarked earlier, that an ordering scheme is not strictly necessary. Similarly, the determinant $\Delta(1)$ is given by

$$\Delta(1)=I_z(\varepsilon)[II_x(1)III_y(1/\varepsilon)-II_y(1)III_x(1/\varepsilon)]+II_z(\varepsilon)[I_y(1)III_x(1/\varepsilon)- - I_x(1)III_y(1/\varepsilon)] - III_z(1/\varepsilon)[I_y(1)II_x(\varepsilon)+I_y(\varepsilon)II_x(1)] = 0. \quad \ldots (18)$$

As before, the terms in Eq.(18) cancel exactly and there is no condition to be carried through to next order. Finally, we consider $\Delta(\varepsilon)$, and it is this order which leads to the dispersion equation for our problem. Neglecting terms of order ε^2 or higher, the dispersion equation is given by

$$\begin{aligned}\Delta(\varepsilon)=&I_x(1)[II_y(1)III_z(\varepsilon)+II_y(\varepsilon^2)III_z(1/\varepsilon)-III_y(1)II_z(\varepsilon)]\\&+III_z(1/\varepsilon)[I_x(\varepsilon^2)II_y(1)-I_y(\varepsilon^2)II_x(1)]+\\&+I_y(\varepsilon)[III_x(1/\varepsilon)II_z(\varepsilon)-II_x(\varepsilon)III_z(1/\varepsilon)]\\&+I_y(1)[III_x(1)II_z(\varepsilon)-II_x(1)III_z(\varepsilon)-II_x(\varepsilon^2)III_z(1/\varepsilon)]\\&+I_z(\varepsilon)[II_x(1)III_y(1)+II_x(\varepsilon)III_y(1/\varepsilon)-III_x(1)II_y(1)] = 0 .\end{aligned} \quad \ldots (19)$$

After considerable algebra, this can be expressed in the form

$$\frac{\mu}{\omega^2}+\frac{2\alpha}{(\omega+k_yV)^2}-\frac{2\Upsilon}{\omega}-\left(\frac{\eta}{\omega}+\frac{\chi}{\omega^2}\right)\frac{1}{(\omega+k_yV)^2} = Q , \quad \ldots (20)$$

where

$$\mu = 1+\frac{k_x^2}{k_y^2}\left(1+\frac{\omega_{pi}^2}{k_z^2c^2}\right) , \quad 2\alpha = \frac{m}{M}\cdot\frac{k_y^2}{k_z^2}\left\{\left(1+\frac{k_x^2}{k_y^2}\right)^2+\frac{\omega_{pe}^2}{k_y^2c^2}\right\} ,$$

$$2\Upsilon = \frac{1}{\Omega_e}\left(1+\frac{k_x^2}{k_y^2}\right)\frac{k_y}{k_z^2}\left\{\frac{1}{n}\cdot\frac{dn}{dx}-\frac{1}{B}\cdot\frac{dB}{dx}\right\} ,$$

$$\eta = 2\,\frac{\omega_{pi}^2}{k_yV}\cdot\frac{k_x^2V^2}{k_z^2c^2} , \quad \chi = \omega_{pi}^2\,\frac{k_x^2V^2}{k_z^2c^2}$$

and

$$Q = \frac{k_y^2}{k_z^2}\left\{\frac{1}{k_y^2c^2}\left(1+\frac{k_x^2}{k_y^2}\right)+\left(\frac{1}{\omega_{pe}^2}+\frac{1}{\Omega_e^2}\right)\left(1+\frac{k_x^2}{k_y^2}\right)^2\right.$$

$$\left.+\frac{\omega_{pe}^2}{\Omega_e^2}\cdot\frac{1}{k_y^2c^2}\left[2\left(1+\frac{k_x^2}{k_y^2}\right)+\frac{\omega_{pe}^2}{k_y^2c^2}\right]\right\} .$$

The dispersion Eq.(20) is rather complicated to discuss analytically and so we re-order with respect to a further small parameter δ. This is given by $\delta = \frac{k_x^2}{k_y^2}$ and is such that $\varepsilon \ll \delta \ll 1$. In other words, on this ordering we take $\lambda_x \gg \lambda_y$. The dispersion equation now becomes

$$\frac{1}{\omega^2} + \frac{2\alpha}{(\omega + k_y V)^2} - \frac{2\Upsilon}{\omega} = Q \; , \qquad \ldots (21)$$

where

$$2\alpha = \frac{m}{M}\cdot\frac{k_y^2}{k_z^2}\left[1+\frac{\omega_{pe}^2}{k_y^2c^2}\right] , \qquad \ldots (22)$$

$$2\Upsilon = \frac{1}{\Omega_e}\cdot\frac{k_y}{k_z^2}\left\{\frac{1}{n}\frac{dn}{dx} - \frac{1}{B}\frac{dB}{dx}\right\} \qquad \ldots (23)$$

and

$$Q = \frac{m}{M}\cdot\frac{k_y^2}{k_z^2}\left[1+\frac{\omega_{pe}^2}{k_y^2c^2}\right]\left(\frac{1}{\omega_o^2}+\frac{1}{\Omega_e\Omega_i}\cdot\frac{\omega_{pe}^2}{k_y^2c^2}\right) , \qquad \ldots (24)$$

the lower hybrid frequency ω_o being defined by

$$\frac{1}{\omega_o^2} = \frac{1}{\omega_{pi}^2} + \frac{1}{\Omega_i\Omega_e} . \qquad \ldots (25)$$

IV. ANALYSIS OF DISPERSION EQUATION

For completeness we begin by analysing the low-β case. Mathematically, this limit may be obtained by allowing $c \to \infty$. More precisely, we consider the ordering

$$\omega^2 \sim \varepsilon^4 \; k_y^2c^2 \sim \varepsilon^2\Omega_e^2 \sim \varepsilon^2\omega_{pe}^2 \; ,$$

and it follows from this, that

$$\frac{1}{B}\frac{dB}{dx} \sim \frac{4\pi en}{B}\,\varepsilon \sim \frac{\omega_{pe}^2}{\Omega_e}\cdot\frac{1}{c}\,\varepsilon \sim k_y\varepsilon^2 \;.$$

Thus for low-β, the appropriate dispersion equation is given by

$$D_o(\omega) = \frac{1}{\omega^2} + \frac{2\alpha_o}{(\omega + k_y V)^2} = Q_o \;, \qquad \ldots (26)$$

where

$$2\alpha_o = \frac{m}{M}\cdot\frac{k_y^2}{k_z^2} \quad \text{and} \quad Q_o = \frac{m}{M}\cdot\frac{k_y^2}{k_z^2}\cdot\frac{1}{\omega_o^2} \;. \qquad \ldots (27)$$

We observe that the above dispersion equation is of the two-stream type and applicable to a homogeneous plasma. In Fig. 2 we give a plot of $D_o(\omega)$ against ω. If the line Q_o intersects with $D_o(\omega)$ at four real points then the system is stable, otherwise it is unstable. The necessary and sufficient condition for stability is given by

$$Q_o > D_o(\omega_c) \;, \qquad \ldots (28)$$

where ω_c is the frequency corresponding to the minimum of $D_o(\omega)$. Now elementary algebra shows ω_c to be given by

$$\omega_c = -\frac{k_y V}{1 + (2\alpha)^{1/3}} \;,$$

and it follows that

$$D_o(\omega_c) = \frac{1}{k_y^2 V^2}\left(1 + (2\alpha)^{1/3}\right)^3 \;.$$

Condition (28) can be written in the form

$$\frac{1}{\omega_o^2} > \frac{1}{s}\left(1 + s^{1/3}\right)^3 \frac{1}{\eta} \;, \qquad \ldots (29)$$

where

$$S = \frac{m}{M}\frac{k_y^2}{k_z^2} \;, \quad c_1 = \frac{\omega_{pe}^2 V^2}{c^2} \quad \text{and} \quad \eta = k_y^2 V^2 \;.$$

It is clear from Equation (29) that for sufficiently small η the low-β case must be unstable. Alternatively, Equation (29) may be written as

$$\lambda_y < \lambda_o = \frac{2\pi V}{\omega_o} \cdot \frac{s^{1/2}}{(1+s^{1/3})^{3/2}} \, .$$

Thus all wavelengths λ_y longer than the critical wavelength λ_o are unstable. This instability was first discussed by Ashby and Paton.

In this paper we are particularly interested in the influence of non-electrostatic effects upon this instability and so we now return to the dispersion equation (21). By Eq.(10), $\frac{dB}{dx} < 0$, and since we expect on physical grounds that $\frac{dn}{dx}$ have the opposite sign, then for $k_y > 0$, it follows that Υ must be positive. This fact is useful since it makes sketching the left-hand side of (21), which we denote by $D(\omega)$, a straightforward matter (see Fig.3). For $k_y < 0$, the dispersion equation is of the same form as (21) but with ω replaced by $-\omega$. Since we know that an unstable root must occur with its complex conjugate, the effect of changing the sign of ω is to interchange the stable and unstable roots leaving the growth rate unaltered. Thus it is only necessary to consider $k_y > 0$.

As for the low-β case, stability is determined by the number of intersections that the Q line makes with $D(\omega)$. Unfortunately, to set up a necessary and sufficient condition for stability for the non-electrostatic case is difficult, as determining $D(\omega_c)$ now entails solving a quartic equation, the coefficients of which involve many parameters. It is considerably easier to derive a <u>sufficient</u> condition for stability. We do this by replacing the curve $D(\omega)$ in the range $-k_y V \leqslant \omega \leqslant 0$ by a curve with a similar shape $D_s(\omega)$, but whose magnitude is everywhere greater than $D(\omega)$ and whose minimum $D_s(\omega_c)$ we can determine (see Fig.4). Thus since $-\frac{2\Upsilon}{\omega} \leqslant \frac{1}{\omega^2} + \frac{1}{\Upsilon^2}$, we consider the dispersion equation

$$D_s(\omega) = \frac{2}{\omega^2} + \frac{2\alpha}{(\omega + k_y V)^2} + \Upsilon^2 = Q \ , \qquad \ldots (30)$$

and a sufficient condition is given by

$$Q > D_s(\omega_c) \ , \qquad \ldots (31)$$

which becomes

$$Q > \frac{2}{k_y^2 V^2} \left(1 + \alpha^{1/3}\right)^3 + \Upsilon^2 \ . \qquad \ldots (32)$$

Using Eqs.(22) to (25), condition (32) may be written as

$$\frac{1}{\omega_o^2} > \frac{1}{s(\eta + c_1)^2} \left\{ \frac{1}{4} \cdot \frac{1}{\Omega_e^2} s^2 \left(\frac{M}{m}\right)^2 \eta V^2 \left[\frac{1}{n}\frac{dn}{dx} - \frac{1}{B}\frac{dB}{dx}\right]^2 + 2\left(\eta^{1/3} + \left[\frac{s}{2}(\eta + c_1)\right]^{1/3}\right)^3 \right\} + \frac{1}{\omega_{pi}^2} \cdot \frac{c_1}{\eta + c_1} \ . \qquad \ldots (33)$$

To simplify the discussion we consider a more stringent condition which we obtain by replacing $\frac{c_1}{\eta + c_1}$ by 1, $\eta^{1/3} + \left(\frac{s}{2}(\eta + c_1)\right)^{1/3}$ by $(\eta + c_1)^{1/3}\left(1 + (\frac{s}{2})^{1/3}\right)$, and $\frac{\eta}{(\eta + c_1)^2}$ by its maximum value $\frac{1}{4c_1}$.

Condition (33) now becomes

$$\frac{1}{\omega_o^2} > \frac{1}{16} \circ \frac{1}{\Omega_e^2} \cdot \frac{s}{c_1}\left(\frac{M}{m}\right)^2 V^2 \left[\frac{1}{n}\frac{dn}{dx} - \frac{1}{B}\frac{dB}{dx}\right]^2 + \frac{2}{s(\eta + c_1)}\left(1 + (\frac{s}{2})^{1/3}\right)^3 + \frac{1}{\omega_{pi}^2} \ . \qquad \ldots (34)$$

Unlike the low-β case, this condition can always be satisfied, however small η, since the non-electrostatic effect produces a cut-off. Introducing the Alfvén speed V_A, which is given by $V_A^2 = \frac{c^2}{\omega_{pe}^2} \Omega_i \Omega_e$, the above sufficient condition for stability can be written as

$$V^2 > \frac{\frac{2}{s}\left(1 + (\frac{s}{2})^{1/3}\right)^3 V_A^2}{1 - \frac{1}{16} s \frac{V_A^2}{\Omega_i^2}\left(\frac{1}{n}\frac{dn}{dx} - \frac{1}{B}\frac{dB}{dx}\right)^2} \ , \qquad \ldots (35)$$

where the denominator must be positive. Clearly, the inhomogeneity of the plasma has a destabilising effect.

We now wish to establish a <u>necessary</u> condition for stability. We replace the curve $D(\omega)$ in the range $-k_y V \leqslant \omega \leqslant 0$ by a curve of similar form D_N but whose magnitude is everywhere less than $D(\omega)$ and whose minimum we can easily determine (see Fig.4). Thus we consider the dispersion equation

$$D_N = \frac{1}{\omega^2} + \frac{2\,\alpha}{(\omega + k_y V)^2} = Q \;, \qquad \dots (36)$$

and a sufficient condition for instability is

$$Q < D_N(\omega_c) \;.$$

Using the explicit forms for Q and α this becomes

$$\frac{1}{\omega_o^2} < \frac{1}{s(\eta + c_1)^2} \left\{ \eta^{1/3} + s^{1/3} (\eta + c_1)^{1/3} \right\}^3 + \frac{1}{\omega_{pi}^2} \cdot \frac{c_1}{\eta + c_1} \;. \qquad \dots (37)$$

A more stringent condition is given by

$$\frac{1}{\omega_o^2} < \frac{1}{s} \cdot \frac{\eta}{(\eta + c_1)^2} \left\{ 1 + s^{1/3} \right\}^3 \;,$$

or alternatively,

$$\eta^2 + \eta \left[2c_1 - \frac{\omega_o^2}{s} \left(1 + s^{1/3} \right)^3 \right] + c_1^2 < 0 \;. \qquad \dots (38)$$

This can be written as

$$\left[\eta - \frac{1}{2}\,(a + b) \right] \left[\eta - \frac{1}{2}\,(a - b) \right] < 0 \;, \qquad \dots (39)$$

where

$$a = \frac{\omega_o^2}{s} \left(1 + s^{1/3} \right)^3 - 2c_1$$

and

$$b = +\sqrt{\left(2c_1 - \frac{1}{s}\,\omega_o^2\left(1 + s^{1/3}\right)^3\right)^2 - 4c_1^2} \,.$$

Now if we assume

$$\frac{\omega_o^2}{s}\left(1 + s^{1/3}\right)^3 - 4c_1 > 0 \,, \qquad \ldots (40)$$

this ensures that b is real and that both a and $a - b$ are positive. It follows at once that (39) is satisfied for all η in the range

$$\frac{1}{2}\,(a - b) \leqslant \eta \leqslant \frac{1}{2}\,(a + b) \,.$$

Since the sufficient condition (for instability) is automatically satisfied by the requirement that (40) be satisfied, this implies that the condition (40) is also sufficient. Introducing the Alfvén speed, V_A, condition (40) may be expressed in the form

$$V^2 < \frac{1}{4}\,V_A^2 \cdot \frac{1}{1 + \dfrac{\Omega_i\Omega_e}{\omega_{pi}^2}} \cdot \frac{1}{s}\left(1 + s^{1/3}\right)^3 \,. \qquad \ldots (41)$$

This is a sufficient condition for instability. It is a simple matter to demonstrate that the right-hand side of (41) is smaller than the right-hand side of (35), as it should be.

V. DISCUSSION

In this Section, we discuss the conditions under which the model is expected to be valid, and then consider its relevance to laboratory plasmas. The analysis of this paper is based upon the two-fluid equations, which are appropriate provided that $k_\perp a_i \gg 1$ and $k_\perp a_e < 1$, a_i and a_e being the ion and electron Larmor radii respectively. Assuming $k_x \ll k_y$, the first condition is satisfied, since by our ordering, $k_y a_i \sim \frac{1}{\varepsilon}$. The second condition is trivially satisfied since the electrons are taken to be cold, this being valid if $\frac{\omega^2}{k_z^2} \gg \frac{kT_e}{m}$. Now the ion drift velocity, V, has been assumed larger than the ion thermal speed, that is, $V \gg \frac{kT_i}{m}$. The validity of this assumption is difficult to assess, although there are circumstances, for example the implosion phase of theta pinch(9), where it is known to be reasonable. Since we have considered $\omega \sim k_y V$ and $\frac{k_y^2}{k_z^2} \sim \frac{M}{m}$, it follows that $V^2 \gg \frac{kT_e}{M}$. Thus our model should be consistent with the situations $T_i \gg T_e$ and $T_i \sim T_e$. A detailed study of temperature effects should be made using the Vlasov equation, and it is hoped to consider this problem in a further paper.

A further consequence of our ordering is that the field and density scale-lengths are comparable with an ion Larmor radius. Low-β analysis is based on the assumptions that the phase velocity is small compared to the velocity of light and that $\beta \ll 1$. In the present work we have taken $\frac{\omega}{k_z} \sim c$, and neglected the finite-β effects.

We now consider the applicability of our model to laboratory plasmas. In typical cusp experiments the sheath region at the 'edge' of the plasma has been shown to have a thickness of order the ion gyro-radius(10). The plasma density scale length is of comparable size. Measurements indicate $\omega_{pi} \gg \Omega_i$, at least in the central region of these machines. It is also known that for typical high-β experiments $T_i \gg T_e$. Thus we expect our theory to be a reasonable description of a hybrid instability occurring in the sheath region of such a plasma. Observations(10) of the cusp compression of colliding plasma blobs are consistent with the hypothesis that the strong space-charge electric fields associated with a collision-free sheath are short-circuited. In some experimental work[11], however, no evidence of wall short-circuiting has been found. In the latter circumstance, the relevance of our model is not clear. For completeness, we shall apply conditions (35) and (41) to Centaur[12]. Since there is some doubt as to the values to be ascribed to the physical quantities in (35) and (41), the results should only be regarded as qualitative. In Centaur, the sheath thickness is of order 1 cm and the density rises to 10^{16} cm^{-3} at the centre of the plasma. In the outside region the magnetic field rises to about 16 kG. Taking $n_i = 5 \times 10^{15}$ cm^{-3} and $B = 8$ kG, the Alfvén velocity is $V_A = 2{\cdot}5 \times 10^7$ cm sec^{-1}. Putting $s = 1$, we find from (35) that the plasma is stable to the lower hybrid mode, if, $V > 4 \times 10^7$ cm sec^{-1}. By (41), the plasma is unstable if $V < 3{\cdot}5 \times 10^7$ cm sec^{-1}.

As a second example, we consider the applicability of our model to the implosion phase of a theta pinch[9]. During the early stages of discharge in the Culham 8-metre theta pinch, rapid field diffusion occurs, accompanied by a broadening of the current sheath to a thickness comparable with the plasma radius (two or three ion Larmor radii). This is an order of magnitude larger than that computed on the basis of ideal MHD theory. During these early stages β is of order $0{\cdot}1$ to $1{\cdot}0$ and so finite-β effects are expected to be important. During the period of instability the sheath thickness L satisfies the following inequalities,

$$a_i \sqrt{\frac{m}{M}} + a_e < L < a_i + \sqrt{\frac{M}{m}}\, a_e \; ,$$

a_i and a_e being the ion and electron Larmor radii respectively. Since we have taken $a_e = 0$, these inequalities are not inconsistent with our model. It is deduced from the experiment that V is four or fives times greater than the ion sound speed, which is consistent with our assumption that $V^2 \gg \frac{kT_i}{M}$. It is found that the growth rate of the instability is of order the lower-hybrid frequency, ω_o. Thus it appears that the instability described in this paper could explain the non-classical behaviour observed during the implosion phase of the theta pinch.

VI. CONCLUSIONS

Ashby and Paton[6] considered a purely electrostatic instability with characteristic frequency ω_o and such that $\frac{k_z}{k_y} \sim (\frac{m}{M})^{\frac{1}{2}}$. In particular, their mode is always unstable for sufficiently small λ_y. In the present work we have considered the influence of non-electrostatic effects upon their instability. Unlike the electrostatic case, we have found a necessary condition for stability and a sufficient condition for stability. Our model should be relevant to the sheath region of a cusp or theta pinch. It may also explain the anomalous field diffusion occurring during the early stages of the implosion phase of a theta pinch.

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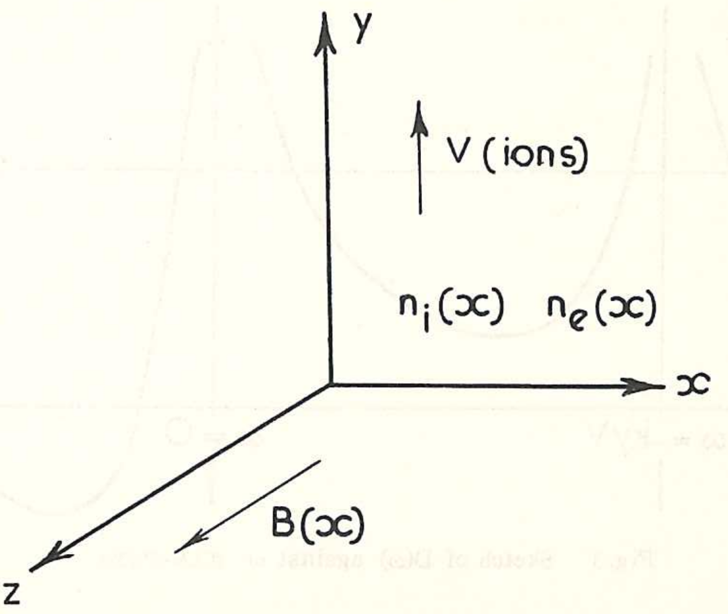

Fig. 1 The equilibrium (CLM-P203)

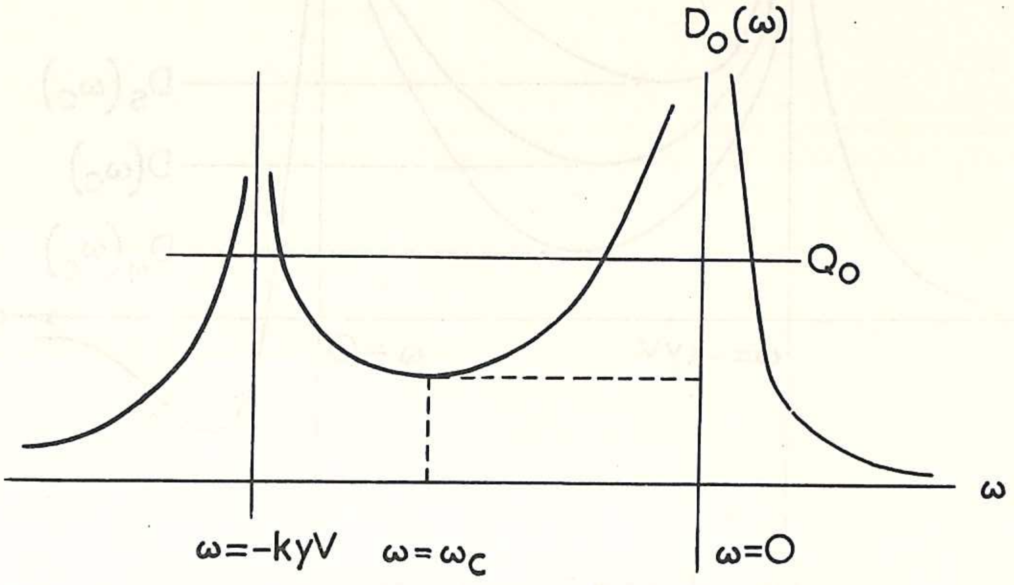

Fig. 2 Sketch of $D_0(\omega)$ against ω (CLM-P203)

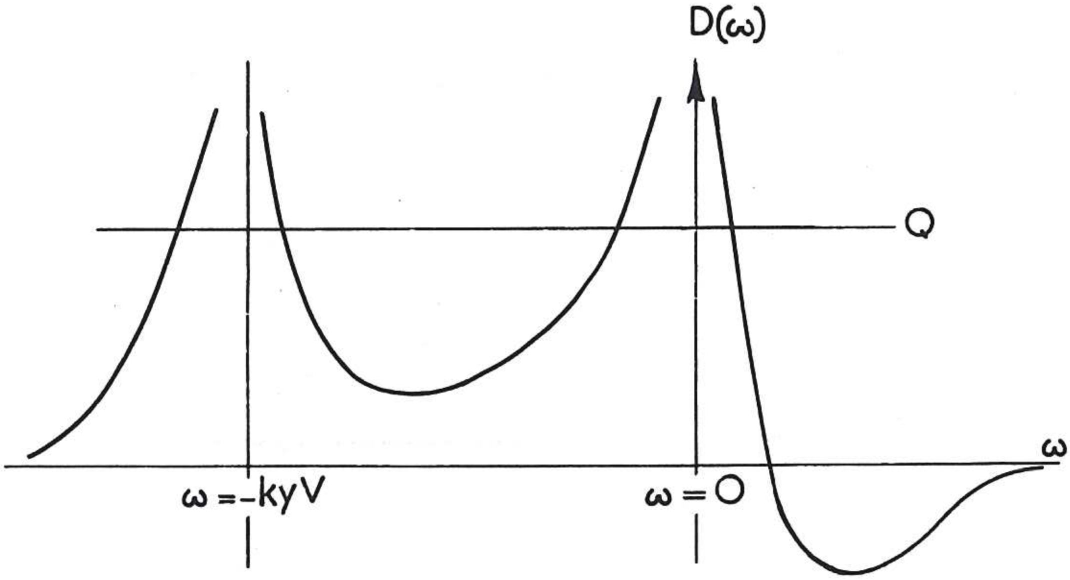

Fig. 3 Sketch of $D(\omega)$ against ω (CLM-P 203)

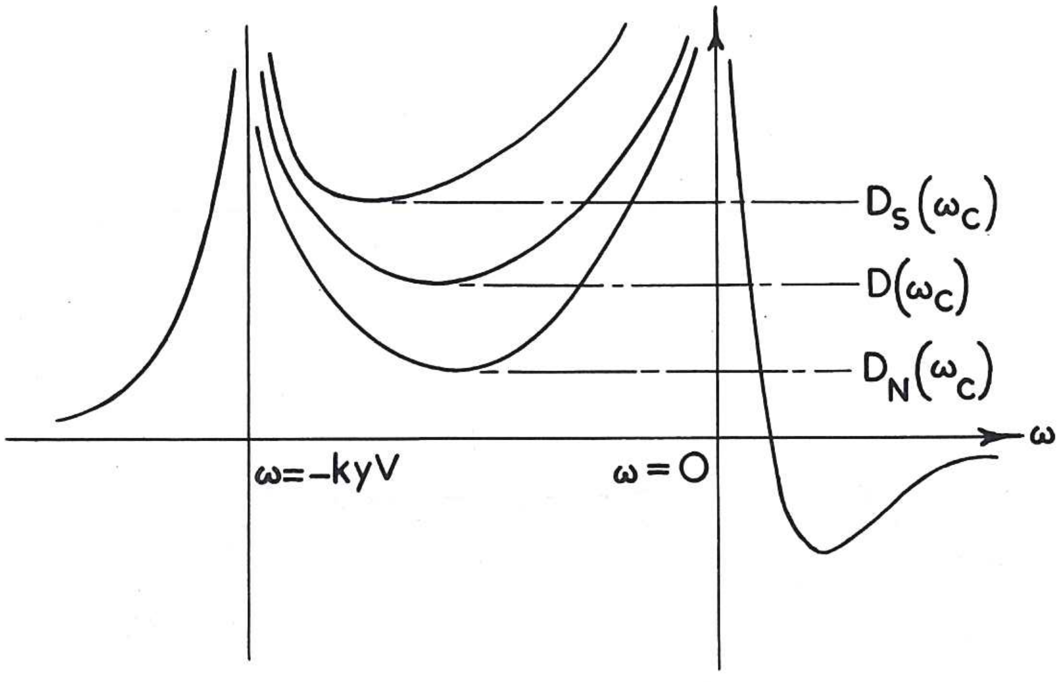

Fig. 4 Sketch of $D(\omega)$, $D_s(\omega)$ and $D_N(\omega)$ (CLM-P 203)