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# ADIABATIC CUSP LOSSES

## PART 2



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# ADIABATIC CUSP LOSSES-II

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## A B S T R A C T

Previous calculations of collisionless, axisymmetric, adiabatic, $\beta = 1$ cusp losses are extended to cover separately a simple transverse electric sheath field and arbitrary $\beta$. Expressions are given for the point cusp loss rate from a monoenergetic plasma for arbitrary electric field strength, E. The $\lambda^{-3/2}$ enhanced scaling with applied mirror ratio found previously for $E = 0$ is shown to decrease with increasing inward electric field, vanishing for a potential greater than about $kT/e$ across an ion gyroradius in the sheath at maximum plasma radius. Useful analytic expressions are also given for the loss rate from a Maxwellian plasma of arbitrary $\beta$ through point and line cusps. Decreasing $\beta$ leads to a rapid increase in loss rate, with a corresponding decrease in the net mirror enhancement. The enhanced mirror scaling is shown not to be expected in theta-pinches; the effect is however valid for spindle-cusp and other devices over a narrow range of parameters.

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## C O N T E N T S

## I. INTRODUCTION

In a recent paper[1], (paper I), the author has extended earlier theories[2-6] of cusp losses from a collisionless, steady-state, axisymmetric, $\beta = 1$ ($\beta$ being the ratio of plasm pressure to total pressure) plasma in which the electric field was everywhere zero. If the motion of only those particles trapped in the sheath was everywhere adiabatic, that is, if there existed a third invariant, $\mu$, of the particle motion in addition to the energy $H$ and momentum, $p_\theta$, then the spindle cusp particle loss rate at both line and point cusps was shown to scale as $\lambda^{-3/2}$ with the magnetic mirror ratio $\lambda$ at the holes. In addition, such devices with non-mirror symmetric point cusps were shown to be rotationally unstable.

In this paper, the above work on an axisymmetric cusp will be extended firstly to include a transverse electric field of a particularly simple assumed profile in the sheath, and secondly, to consider the effect of reducing $\beta$ by the introduction of trapped parallel magnetic field. In both cases, expressions are given for the particle loss rate from both adiabatic and non-adiabatic systems, it being shown that the mirror enhancement in the former occurs only over a narrow regime of weak electric fields and high $\beta$, and thus is not to be expected in simple theta-pinches. The onset of rotation by the mechanism of paper I is shown to be delayed by these electric fields, a sheath width much less than the ion gyroradius being required for stability, however.

It was shown in paper I that the end loss rate of any quantity $Q(H, p_\theta, \mu)$ from a plasma represented by the distribution function $f(H, p_\theta, \mu)$ is given by

$$F(Q) = 4\pi \iiint f \cdot Q \cdot dH \cdot dp_\theta \quad d\mu \qquad \ldots (1)$$

integrated over the loss cone, that is, over the accessible region of phase space at the cusp hole, taken as the plane of maximum magnetic field.

The invariants of the particle motion are the energy,

$$H = \frac{1}{2}(v_r^2 + v_\theta^2 + v_z^2) + e \quad \Phi(r, z)$$

the angular momentum,

$$p_\theta = rv_\theta + \psi(r, z)$$

where $\psi = \frac{e}{c} \cdot r \cdot A_\theta$, and $\underline{A}$ is the vector potential, and, where relevant, the adiabatic invariant[6]

$$\mu = \frac{1}{2} \oint p_\perp d\ell_\perp$$

integrated round the closed orbit in the $p_\perp - \ell_\perp$ phase plane, $\ell_\perp$ being orthogonal to both $\underline{B}$ and $\theta$.

We further restate the fundamental assumption of paper I, that all particles present which are able to contribute to the end loss have access to the central, high $\beta$ plasma reservoir which is taken to be non-adiabatic. We can then write the distribution function

$$f(H, p_\theta, \mu) = g(H, p_\theta) \cdot h(H, p_\theta, \mu) \qquad \ldots (2)$$

where $g$ is the distribution function in the central contained plasma, and $h$ is 1 or 0 depending on whether the escaping particle $(H, p_\theta, \mu)$ at the cusp hole penetrates the high $\beta$ plasma or remains trapped in the sheath on reversal of its velocity. An 'adiabatic' cusp is now one in which $\mu$ is invariant for any particle orbit whilst trapped in the sheath, penetrating orbits having been assumed non-adiabatic remote from the hole, an assumption valid except, perhaps, in theta-pinch geometry, where the plasma radius may be slowly varying everywhere. In a non-adiabatic cusp, $H$ and $p_\theta$ are the only invariants, $h$ and $f$ then being independent of $\mu$ but Eq.(1) being otherwise still valid.

## II. LOSSES WITH A TRANSVERSE ELECTRIC FIELD

The influence of a transverse electric on $\beta = 1$ cusp losses is firstly considered. We assume an electric field such that the magnetic flux lines are equipotential and that the electric potential is given by

$$\Phi(r\,,z) = \frac{c}{e}\frac{E_0}{B_0 R}\,|\psi(r\,,z)| \qquad \ldots (3)$$

where $E_0$ is a constant, $B_0$ the magnetic field strength required to contain the $\beta = 1$ plasma, and $R$ the maximum $\beta = 1$ plasma radius. Also assuming, after previous authors[4,6], a discontinuous step in magnetic field at the sheath from zero to its vacuum value $B_0$, this potential gives a near uniform electric field $E_0$ at maximum plasma radius, and a linearly increasing field at a point cusp.

### A. THETA-PINCH LOSSES

We consider initially theta-pinch geometry in which $B_r \ll B_z$, giving for $r > r_0$, $r_0$ being the local $\beta = 1$ plasma radius,

$$\psi(r\,,z) = \frac{1}{2}\,\Omega\,(r^2 - r^2)$$

$$\Phi(r\,,z) = \frac{E_0}{2R}\frac{B}{B_0}\,(r^2 - r_0^2) \qquad \ldots (4)$$

where $\Omega = \frac{eB}{c}$. We now calculate $\mu$ for arbitrary $r_0$, $B$, both the loss cone and the function $h(H\,,p_\theta\,,\mu)$ then being given in the appropriate limits.

## A.1 Adiabatic Invariant for Trapped Particles

For a particle gyrating entirely within the sheath

$$p_r^2 = \frac{1}{r^2} (- C_1 + C_2 r^2 - C_3 r^4) \qquad \ldots (5)$$

where

$$C_1 = (p_\theta + \frac{1}{2} r_o^2 \Omega)^2$$

$$C_2 = 2H - P_z^2 + p_\theta \Omega + \frac{1}{4} r_o^2 \Omega^2 \left(1 + \frac{1}{\varepsilon} 2\right)$$

$$C_3 = \frac{\Omega^2}{4\varepsilon^2}$$

$$\varepsilon = \left(1 + \alpha \frac{\Omega_o}{\Omega}\right)^{-\frac{1}{2}} \qquad \ldots (6)$$

and $\alpha = \frac{4eE_o}{R\Omega_o^2}$ is introduced for later convenience. Straightforward integration now gives

$$\mu = \frac{1}{2} \oint p_r dr = \frac{\pi}{2} \left(\frac{1}{2} C_2 \cdot C_3^{-\frac{1}{2}} - C_1^{\frac{1}{2}}\right) \qquad \ldots (7)$$

subject to the following conditions that the orbit should both exist and be trapped:

(1) $p_r^2 \leqslant 0$ somewhere, giving from (5) to (7), on elimination of $p_z$

$$\mu \geqslant 0 \qquad \ldots (8)$$

(2) $p_z^2 \geqslant 0$ giving similarly

$$\mu \leqslant \frac{\pi}{2} \left[ \varepsilon \cdot \frac{2H}{\Omega} - p_\theta (1 - \varepsilon) + \frac{1}{4} r_o^2 \Omega \; \varepsilon (1 - \varepsilon)^2 \right] + \mu_o \qquad \ldots (9)$$

(3) $r > r_o$ everywhere giving both

$$\mu \geqslant \frac{\pi}{2} \cdot \frac{1}{\varepsilon} \left[ \frac{1}{2} \cdot r_o^2 \Omega \; (1 - \varepsilon) - \varepsilon p_\theta \right] + \mu_o \qquad \ldots (10)$$

and

$$\mu \leqslant \frac{\pi}{2\Omega} \left[ \varepsilon \frac{p_\theta}{r_o} - \frac{1}{2} \cdot r_o \cdot \Omega \cdot (1 - \varepsilon) \right]^2 + \mu_o \qquad \ldots (11)$$

In Eqs.(9)-(11)

$$\mu_o = 0 \qquad \text{for} \quad p_\theta > -\tfrac{1}{2} \cdot r_o^2\Omega$$

$$= \pi(p_\theta + \tfrac{1}{2}\ r_o^2\Omega) \qquad \text{for} \quad p_\theta < -\tfrac{1}{2} \cdot r_o^2\Omega \qquad \ldots (12)$$

For $eE_o < 0$, $E$ may be imaginary, in which case the particle spirals outwards continuously.

## A.2 Loss Cone

The loss cone is the accessible region of phase space at the cusp hole, and is simply given from Eqs.(8)-(12) by putting $r_o = 0$, $\Omega = \lambda\Omega_o$. In this case, (10) and (11) are automatically satisfied, whilst (8), (9) and (12) give the loss cone

$$0 < \mu < \mu^* = \frac{\pi}{2}\left[\varepsilon' \frac{2H}{\lambda\Omega_o} - p_\theta(1-\varepsilon')\right] \quad \text{for} \quad p_\theta > 0$$

$$= \frac{\pi}{2}\left[\varepsilon' \frac{2H}{\lambda\Omega_o} + p_\theta(\varepsilon' + 1)\right] \quad \text{for} \quad p_\theta < 0 \qquad \ldots (13)$$

where $$\varepsilon' = (1 + \frac{\alpha}{\lambda})^{-\frac{1}{2}}$$

This loss cone is sketched for $eE_o > 0$ in Fig.(1) as the cross-hatched area ADE; the upper limit $p_{max}$ may be greater or less than $R(2H)^{\frac{1}{2}}$. For $E_o < 0$, the loss cone is no longer truncated at large $p_\theta$.

## A.3 Distribution Functions

Firstly we consider a device in which all trapped orbits are adiabatic. The region of phase space for $r_o = R$, $\Omega = \Omega_o$, corresponding to the central plasma, which simultaneously satisfies all of Eqs.(8)-(11) is shown cross-hatched in Fig.2 for $eE_o > 0$, and is that region of phase space which is occupied by particles which nowhere penetrate the $\beta = 1$ plasma. For $H$ sufficiently large the second intersection of (10) and (11) occurs for $p_\theta > -R \cdot (2H)^{\frac{1}{2}}$, in which case an allowed region also appears for $p_\theta < 0$ corresponding to particles whose gyroradius is greater than the plasma radius. These particles are neglected.

As $E_0$ increases, the allowed region shrinks about the point $p_\theta = R \cdot (2H)^{1/2}$, $\mu = 0$, vanishing for

$$\varepsilon > 1 + 2 \cdot (2H)^{1/2}/R\Omega o$$

in which case no trapped particles can exist.

For $eE_0 < 0$, $E > 1$ and the parabola vertex occurs for $p_\theta < 0$, the allowed region the extending into $p_\theta < 0$, and not longer being truncated at large $p_\theta$.

The boundary between trapped and penetrating orbits in the region $p_\theta < R \cdot (2H)^{1/2}$ of interest is seen to be given by (11) which is readily shown to be a monotonically decreasing function of $r_0$, the criterion for a particle with the parameters $(H, p_\theta, \mu)$ at the cusp hole to be penetrating thus being given at $r_0 = R$ as

$$\mu > \mu_{min} = \frac{\pi}{2\Omega_0}\left[\varepsilon'' p_\theta - \frac{1}{2} \cdot R \cdot \Omega_0 \cdot (1 - \varepsilon'')\right]^2 \qquad \ldots (14)$$

$$= 0 \qquad \text{for} \qquad p_\theta \gtrless \tfrac{1}{2}R^2\Omega_0 \left[\frac{1-\varepsilon''}{\varepsilon''}\right] \text{ respectively}$$

together with

$$p_\theta < R \cdot (2H)^{1/2} \qquad \ldots (15)$$

where

$$\varepsilon'' = (1 + \alpha)^{-1/2}$$

For simplicity, we assume a monoenergetic isotropic distribution $g$ in the contained plasma to give the adiabatic cusp distribution function

$$f(H, p_\theta, \mu) = \frac{n}{4\pi(2H_0)^{1/2}} \delta(H - H_0) \cdot U(R(2H_0)^{1/2} - |p_\theta|) \cdot U(\mu - \mu_{min}) \quad \ldots (16)$$

which includes only penetrating particles, U being the Unit Function.

In a non-adiabatic cusp, the sole criterion for penetration is (15) the relevant distribution function then being given by deleting the final factor of Eq.(16). It should be recalled[5,1] however, that this in general gives an upper limit to non-adiabatic losses, as an orbit $(H, p_\theta, \mu)$ at the cusp hole is in general to be included only if it penetrates the plasma in one transit through the device, a limitation not included in (15).

A.4 Particle Loss Rate

The region from Fig.2 occupied by trapped particles is also shown superimposed on the loss cone in Fig.1, where, in addition, various limits required below are defined on the $p_\theta$ axis. We introduce the dimensionless parameters

$$H = \frac{4H_o}{\lambda R^2\Omega_o^2} \quad , \quad P = \frac{2p_\theta}{R^2\Omega_o} \quad , \quad \mu' = \frac{4\mu}{\pi R^2\Omega_o}$$

in terms of which the quantities defined in Fig.1 become

$$P_{max} = \frac{H\varepsilon'}{1-\varepsilon'} \quad , \quad P_o = \frac{1-\varepsilon''}{\varepsilon''} \quad , \quad P_{min} = -\frac{H\varepsilon'}{1+\varepsilon'}$$

$$P* = \frac{1}{\varepsilon''}\left\{\varepsilon' - \varepsilon'' + \left[(\varepsilon'-1)(\varepsilon'-2\varepsilon''+1) + 2\varepsilon'\varepsilon''H\right]^{\frac{1}{2}}\right\}$$

whilst

$$\left.\begin{aligned}\mu*' &= \varepsilon' H + P(\varepsilon'-1)\\ &= \varepsilon' H + P(\varepsilon'+1)\end{aligned}\right\} \quad \text{for} \quad p_\theta \gtrless 0$$

and

$$\left.\begin{aligned}\mu'_{min} &= \frac{1}{2\varepsilon''}(\varepsilon'' P + \varepsilon'' - 1)^2\\ &= 0\end{aligned}\right\} \quad \text{for} \quad p_\theta \gtrless 0$$

For the adiabatic cusp with $eE_o > 0$ the particle loss rate is given from (1) by integrating f over the area ABCD, Fig.1. It can be shown that $P^* < R(2H)^{1/2}$ for all $E_o$, leaving two regimes. In the first, $P_o < P_{max}$, i.e.

$$H > \frac{(1-\varepsilon')(1-\varepsilon'')}{\varepsilon'\varepsilon''} \qquad \ldots (18)$$

and the loss rate becomes

$$F = K\left[\int_{P_{min}}^{P^*} \mu^{*\prime}\, dP \int_{P_o}^{P^*} \mu'_{min}\, dP \right. \qquad \ldots (19)$$

where

$$K = \frac{n \cdot \pi \cdot R^4 \Omega_o^2}{8(2H_o)^{1/2}}$$

which reduces after considerable algebra to

$$F = K \cdot \left[\frac{1}{2} \cdot H^2 \cdot \frac{\varepsilon'^2}{1+\varepsilon'} + \frac{\varepsilon'(\varepsilon'-\varepsilon'')}{\varepsilon''} H + \frac{\varepsilon'-1}{6\varepsilon''^2}\left[3(\varepsilon'-\varepsilon'')^2 - (\varepsilon'-1)^2\right] \right.$$

$$\left. + \frac{\varepsilon}{3}\left(\frac{2\varepsilon'}{\varepsilon''} H - \frac{(\varepsilon'-1)(\varepsilon'-2\varepsilon''+1)}{\varepsilon''^2}\right)^{3/2}\right] \qquad \ldots (20)$$

Now expanding $\varepsilon'$ and $\varepsilon''$ in powers of $\alpha$, Eq.(18) becomes to highest order in $\alpha$

$$\alpha < 8^{1/2} \cdot (a/R)$$

where

$$a = (2H_o)^{1/2}/\Omega_o$$

is the gyroradius in the vacuum field. This ratio $a/R$ is assumed small giving $\alpha \ll 1$ over the range of validity of (2). Taking $\alpha \sim a/R$ and retaining only terms of highest order in $a/R$, (20) gives on expansion of $\varepsilon'$ and $\varepsilon''$

$$\frac{F}{F_o} = \frac{2}{3\lambda^{1/2}}\left[1 - 2E^2\left(1 - \frac{1}{2\lambda}\right)\right]^{3/2} + E\left(1 - \frac{1}{\lambda}\right) - \frac{1}{3}E^3\left[3\left(1-\frac{1}{\lambda}\right)^2 - \frac{1}{\lambda^2}\right]$$

$$\ldots (21)$$

for $0 < E < 2^{-\frac{1}{2}}$ where

$$F_o = n \cdot \pi \cdot R \cdot H_o / \lambda \Omega_o$$

is the zero electric field, non-adiabatic loss rate to this order[5,1], and

$$E = \frac{R}{4a} \cdot \alpha = \frac{a}{L_E}$$

where $L_E$ is the distance in the sheath at maximum radius $R$ over which the potential changes by $2H_o/e$, and as such, is a measure of the sheath width for $L_E < a$.

In the second regime, $P^* < P_o$, giving

$$\frac{F}{F_o} = \frac{1}{4E} \quad \text{for} \quad E > 2^{-\frac{1}{2}} \qquad \ldots (22)$$

without the need for expansion in $\alpha$ as above. In this case, there is no allowed region for trapped particles which no longer exist.

If $eE_o < 0$, $P_o < 0$, and the lower limit of integration in (19) lies in the region $0 < P < P_{min}$. However, as this case is primarily of relevance to electrons in a narrow sheath, the greatest negative value of $E$ expected (corresponding to an electron gyroradius sheath width[7]) is of the order $-1$, for which $P_{min} \sim a/R \cdot P_{max}$. The lower limit is therefore taken as zero to order $a/R$, giving for $P^* > 0$, i.e., for $0 > E > -\lambda^{-\frac{1}{2}}$,

$$\frac{F}{F_o} = F' + \frac{1}{3} \cdot E^3 \cdot \lambda \qquad \ldots (23)$$

where $F'$ is $\frac{F}{F_o}$ given by (21).

For $E < -\lambda^{-\frac{1}{2}}$ the loss cone is filled entirely by trapped particle orbits and the end loss falls to zero.

The non-adiabatic cusp loss rate is obtained by integrating f over the area ADE (Fig.1), noting the region $p_\theta > R(2H)^{\frac{1}{2}}$ to be empty. We now have $P_{max} \gtrless R(2H)^{1/2}$ giving respectively

$$\frac{F}{F_0} = 1 - E + O\left(\frac{a}{R}\right) \quad \text{for} \quad E < \frac{1}{2} \qquad \ldots (24)$$

$$= \frac{1}{4E} \quad \text{for} \quad E > \frac{1}{2} \qquad \ldots (25)$$

which is just (22) for large E as expected from the disappearance of trapped particles from the loss cone.

These results (21)-(25) are plotted in Fig.3 as a function of E and $\lambda$. The reduction of the loss rate at $E = 0$ by a factor $2/3\lambda^{1/2}$ is in agreement with previous results[1]. However, it is apparent that the enhanced mirror scaling rapidly decreases with increasing transverse field, vanishing for a potential about equivalent to the ion energy across an ion gyroradius in the sheath at maximum plasma radius. This reversion to normal $\lambda^{-1}$ scaling is a consequence of the disappearance of trapped particle orbits from the sheath. All particles must then enter the $\beta = 1$ region where the distribution is assumed isotropic and hence independent of $\mu$. The critical electric field is observed to be too weak to decrease significantly the sheath width, and a factor $(m_I/m_e)^{1/2}$ less than that required in an electron gyroradius sheath[7]. This disappearance of the enhanced scaling will not therefore be much dependent on the assumed field profile, whilst similar behaviour would also be expected at a line cusp.

As a result of the absence of trapped particle orbits for $E \gtrsim 1$ we now require only that

$$\ell \gg L_N$$

when $\ell$ is the mean-free-path and L the nozzle length, as distinct from plasma length L. In this case collisions may consistently provide the assumed non-adiabaticity in the $\beta = 1$ regions of the theta-pinch remote from the hole if $\ell \ll L$. The $\beta = 1$ asumption is, however, rather more stringent, it being required that $P_{max}$ be large relative to the total trapped magnetic flux, $\psi(R)$, giving, for $L_E \lesssim a_I$,

$$(1-\beta)^{1/2} \ll \frac{a_I}{R}\left(\frac{L_E}{R}\right)^{1/2}$$

B. SPINDLE-CUSP

It is assumed that, as in the theta-pinch geometry above, the limiting criterion for a particle to penetrate the $\beta = 1$ plasma is given at the maximum plasma radius, R, a result demonstrated in paper I for $E = 0$. At the maximum radius, $\underline{B} \approx B_r$ and

$$\psi(R, z) = \Omega_o \cdot R \cdot z$$

giving from (3)

$$\Phi(R, z) = E_o \cdot |z|$$

For a trapped particle

$$p_z^2 = C_1 + C_2 \cdot z + \Omega_o^2 \cdot z^2$$

where

$$C_1 = 2H - p_r^2 - \left(\frac{p_\theta}{R}\right)^2$$

and

$$C_2 = 2\Omega_o \left(\frac{p_\theta}{R} - \frac{1}{4} R \cdot \Omega_o \cdot \alpha\right)$$

giving

$$\mu = \frac{1}{2} \cdot \oint p_z \, dz = \frac{\pi}{8\Omega_o^3}\left[C_2^2 + 4\Omega_o^2 C_1\right]$$

subject as before to the conditions $p_r^2 \geqslant 0$, $p_z^2 \geqslant 0$ and $z > 0$, leading respectively to

$$\mu \geqslant 0$$

$$\mu \leqslant \frac{\pi}{2}\left[\frac{2H}{\Omega_o} - p_\theta \cdot \frac{a}{2} + \frac{1}{16} \cdot R^2 \cdot \Omega_o a^2\right]$$

and

$$\begin{cases} \mu \leqslant \frac{\pi}{2\Omega_o R^2} \left(p_\theta - \frac{1}{4} \cdot R^2 \Omega_o a\right)^2 \\ p_\theta > \frac{1}{4} \cdot R^2 \cdot \Omega_o \cdot a \end{cases}$$

which to order $a/R$ give the same allowed region of phase space for trapped orbits as (8)-(12), except that in this case the neglected class of particles with $p_\theta < 0$ and $a > R$ are no longer allowed, as is obvious from the geometry. The results (21)-(25) are therefore equally applicable to a spindle-cusp point cusp. Calculations using the same sheath structure in plane geometry also give very much the same results for a line cusp, except that marginally stronger electric fields are now required to eliminate the enhanced mirror scaling.

## C. ROTATIONAL STABILITY

In paper I a new mechanism for producing rotation in collisionless theta-pinches was discussed. It was shown that such a device becomes $M = 2$ unstable after a time

$$\tau_{inst} \approx 0{\cdot}5 \cdot \frac{a_I}{\bar{p}_{\theta z}} \cdot \left(\frac{2H_o m_I}{3}\right)^{\frac{1}{2}} \cdot \tau_c \qquad \ldots (26)$$

where $\tau_c = \frac{n \cdot V}{F}$ is the containment time, $V$ the plasma volume and $\bar{p}_{\theta z}$ the average angular momentum per ion-electron pair lost through the end holes. We consider now the effect of a transverse electric field on this mechanism in a non-adiabatic cusp. The flux of angular momentum per particle of the ith species is found by inserting an additional factor $p_\theta$ in the integrals leading to (24) and (25) and using these results to be

$$\bar{p}_{\theta zi} = \frac{1}{2} \cdot R \cdot (2m_i H_o)^{\frac{1}{2}} \left[\frac{1 - \frac{4}{3} E_i}{1 - E_i}\right] + 0\left(\frac{a_i}{R}\right) \quad \text{for } E_i \lessgtr \tfrac{1}{2}$$

$$= \frac{1}{6E_i} \cdot R \cdot (2m_i H_o)^{\frac{1}{2}}$$

where $E_i = \frac{a_i}{L_E}$ is the field seen by the ith species. For $E_I \ll (m_I/m_e)^{\frac{1}{2}}$ the electron contribution is negligible giving $\bar{p}_{\theta z} \approx p_{\theta zI}$. The stable time therefore scales as

$$\tau_{inst} \propto (1 - \tfrac{4}{3} E_I)^{-1} \quad \text{for } E_I \gtrless \tfrac{1}{2}$$

$$\propto 6E_I$$

An inward electric field is thus stabilizing to this mechanism. For stability for the order of the containment time however we require $E_I \sim R/3a_I$. That is, a sheath width much less than the ion gyroradius is required to stabilize this mechanism in the absence of other factors.

## III. LOSSES FROM A $\beta < 1$ CUSP

The results of paper I were shown to be valid if

$$(1 - \beta)^{\frac{1}{2}} \ll a/R \ll \frac{L}{\ell} \ll 1$$

where $a = \frac{(3kT)^{\frac{1}{2}}}{\Omega_o}$ is now that R.M.S. gyroradius. The most demanding assumption was thus that of $\beta = 1$, the effect of reducing $\beta$ by the introduction of a weak trapped parallel magnetic field being the subject of this Section. Electric fields are assumed to be zero.

Losses from $\beta < 1$ cusps have previously been considered qualitatively by Grad[2], and with regard to theta-pinches, by Roberts[8] and by Taylor et al.[9] using particle and MHD theory respectively. Both these authors neglect the sheath, however, their results therefore breaking down as $\beta$ approaches unity. Morse[10,11] has given expressions for the loss from a near-Gaussian radial density profile theta-pinch. This profile is, however, suitable only for an average $\beta$ less than about 80%.

Assuming the inside of the sheath to be still sharply defined at radius $r_o$, the total trapped magnetic flux is

$$\psi_o = \int_0^{r_o} r \cdot B_z \cdot dr = \frac{1}{2} \bar{B}_t \cdot R^2 \qquad \ldots (28)$$

where $\bar{B}_t = B_o\sqrt{1-\bar{\beta}}$ is an average trapped magnetic field. For a non-adiabatic cusp, the criterion for a particle to penetrate the high $\beta$ plasma inside the critical flux line $\psi_o$ is now, using (13),

$$R(2H)^{\frac{1}{2}} + \psi_o > p_\theta \gtrsim \begin{cases} \psi_o - R(2H)^{\frac{1}{2}} \\ -H/\Omega_t \end{cases} \quad \text{for } R \lessgtr \frac{(2H)^{\frac{1}{2}}}{\Omega_t} \qquad \ldots (29)$$

The lower limit assumes uniform trapped field. However, it will be seen that in the cases of most interest, this limit does not enter into the loss calculations and is therefore not accurately required.

A. THETA-PINCH LOSSES

In this case, the distribution functions, assuming a Maxwellian plasma in the central trapped field, is given for a non-adiabatic cusp as

$$F(H, p_\theta) = \frac{n}{(2\pi kT)^{3/2}} e^{-H/kT} \, U(R(2H)^{\frac{1}{2}} + \psi_o - p_\theta) \cdot U(p_\theta - p_\theta^*) \qquad \ldots (30)$$

where $p_\theta^*$ is the R.H.S. of (29). In the region $\psi < \psi_o$ this gives a complete Maxwellian and therefore uniform density and pressure. Thus for equilibrium, we require that either

$$\left(\frac{B_t}{B_o}\right)^2 = (1-\bar{\beta}) \ll 1 \qquad \ldots (31)$$

or

$$B = \text{constant} \quad \text{for} \quad r < r_o \qquad \ldots (32)$$

It is noted that (31) is a considerable relaxation of (27), whilst if (32) is satisfied, (30) is valid for all $\beta$ although no longer physically justifiable, trapped and penetrating particles now being lost on the same time scale.

Again assuming uniform magnetic field at the cusp hole, the loss cone is given by (13) with $E = 0$. Thus, for $\lambda \geqslant 1$, the lower limit of (29) is seen always to lie outside the loss cone. Using this the loss rate at a point cusp becomes

$$\frac{F}{F_o} = 1 + \left(\frac{8}{27\pi}\right)^{\frac{1}{2}} \cdot \frac{a}{\lambda R} + \left(\frac{8}{3\pi}\right)^{\frac{1}{2}} \cdot \frac{\psi_o}{\Omega_c R_a} \qquad \ldots (33)$$

differing from the $\beta = 1$ result only by the final term, which is observed to be the rate of free effusion across the area occupied by the trapped flux at the cusp hole. This simple addition of an extra term is a consequence of the unchanged loss cone and will not occur for $E \neq 0$. Using (28), (33) can be written

$$\frac{F}{F_o} = 1 + \left(\frac{8}{27\pi}\right)^{\frac{1}{2}} \frac{a}{\lambda R} + \left[\frac{2}{3\pi}(1 - \bar{\beta})\right]^{\frac{1}{2}} \cdot R/a \qquad \ldots (34)$$

the final term now being just the result of Roberts[8], which, for $\lambda \gtrsim 2$, is a constant factor $(4\pi/3\Upsilon)^{\frac{1}{2}}$ less than the MHD result of Taylor et al.[9]

For large $R/a$, it is apparent that the particle loss rate rapidly increases as $\beta$ is reduced. However, for $(1 - \beta)^{\frac{1}{2}} \gg a/R$ such that the final term dominates, the containment time is of the order

$$\frac{\tau_c}{\tau_t} \sim \frac{\lambda}{2(1 - \beta)^{\frac{1}{2}}} \gg 1$$

for steady state theory to be valid, $\tau_t$ being the ion thermal transit time through the device. This provides a lower limit on $\beta$ for the validity of these results. In this intermediate $\beta$ regime, however, the mean-free-path is only required to be long relative to the nozzle length.

If particles trapped in the sheath are adiabatic, then the penetration criterion is readily found, as in Paper I, to be

$$\mu \geqslant \frac{\pi}{2\Omega_0} \left(\frac{p_\theta - \psi_0}{R}\right)^2 \quad \text{for} \quad p_\theta > 0$$

together with (29).

Appropriate modification of the distribution function is then found to modify the particle loss rate (34) by a factor $\frac{2}{3\lambda^{\frac{1}{2}}}$ in the first term only. The enhanced mirror effect is therefore seen to be a very high $\beta$ phenomena, being significant if the ion gyroradius in the trapped magnetic field is comparable with the plasma radius. The enhancement thus appears only in a narrow regime of weak electric fields and high $\beta$, which is just that in which the mean-free-path was required to be long relative to the plasma length. This, however, leads to equilibrium[12] and rotational stability[1] problems, it being concluded that the mirror enhancement will not be seen in theta-pinches.

B. SPINDLE-CUSP LOSSES

These devices are mirror symmetric about the line cusp, the magnetic field being oppositely directed at the two point cusps. This is allowed for by the transformation $p_\theta \rightarrow - p_\theta$ in the relevant half, giving the region of phase space occupied by penetrating particles in a non-adiabatic cusp as

$$|p_\theta| < R(2H)^{\frac{1}{2}} + \psi_o$$

This results in the same loss rate (33) at a point cusp. The line cusp loss rate is also readily found, using the loss cone of paper I, as

$$\frac{F}{F_o} = 2\left[1 + \left(\frac{8}{3\pi}\right)^{\frac{1}{2}} \frac{\psi_o}{\Omega_o R_a}\right]$$

which is just twice (33) if the small second term there is neglected.

If the trapped orbits are adiabatic, then the first term only is again reduced by a factor $2/3\lambda^{\frac{1}{2}}$. In this geometry, there is no simple relationship between $\psi_o$ and the trapped magnetic field. However, it is apparent that for enhanced mirror scaling to be observed, the total magnetic flux inside the separatrix between adiabatic and non-adiabatic orbits must satisfy.

$$\psi_o \lesssim \Omega_o \cdot R \cdot a$$

The further problems of long mean-free-path rotation and equilibrium do not arise in this geometry due respectively to mirror symmetry and to the inherent non-adiabaticity of the central region. Higher order devices in the hierarchy, such as the cusp-ended theta-pinch[13], may be subject to the former, but not the latter. Their end losses are readily calculated as above; general results are not conveniently formulated however.

## IV. CONCLUSION

Previous calculations of steady-state, collisionless, $\beta = 1$ cusp losses have been extended to include separately a transverse electric field of a simple assumed profile, and a trapped parallel magnetic field giving a reduced $\beta$. Particular reference has been paid to the effect of these on the $\lambda^{-3/2}$ enhanced mirror scaling with applied mirror ratio recently found in an adiabatic cusp[1].

Expressions have been given for the point cusp particle loss rate from a monoenergetic isotropic $\beta = 1$ plasma for arbitrary electric field E. An inward field reduces the enhanced scaling, which vanishes completely with the disappearance of trapped particle orbits when a potential greater than about kT/e appears across an ion gyroradius in the sheath at maximum plasma radius, a field which is too weak to significantly decrease the sheath width. This electric field has a stabilizing effect on the production of rotation in theta-pinches by the ion loss being asymmetric in $p_\theta$ when the sheath is broadened by trapped electrons, as discussed in Paper I. However, a sheath width much less than the ion gyroradius is required for stability in the absence of other factors.

Useful analytic expressions have also been given for the particle losses through $\beta \leqslant$ line and point cusps, connecting the previously available results for $\beta = 1$[4] and $1 - \beta \sim 1$[8], and extending them to an adiabatic cusp. Reduction of $\beta$ leads to a rapid increase in the particle loss rate with a corresponding decrease in the mirror enhancement. This enhancement is then shown not to be expected in theta-pinches due to associated equilibrium[12] and rotational stability[1] problems. The effect is valid in spindle-cusp and other devices, however, if one simultaneously satisfies $E < 1$, $\ell \gg L$, $R \gg a$, $\psi_0 \lesssim \Omega_0 Ra$, and has mirror symmetric point cusps, adiabatic trapped orbits and non-adiabatic penetrating orbits. The first two are perhaps the most demanding, the former being difficult to control experimentally[13], whilst the latter is inconsistent with Spalding's[14] recent estimate of $\ell \sim 0{\cdot}1$ L for an economic cusp-eneded reactor.

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## REFERENCES


1 A.S. Kaye. To be published.

2 H. Grad. Prog. in Nucl. Energy, Series XI. Plasma Physics, 2, 189, (1963).

3 J. Berkowitz, K.O. Friedrichs, H. Goertzel, H. Grad, J. Killeen, and E. Rubin. Proc. 2nd. U.N. International Conference on the Peaceful Uses of Atomic Energy, 31, 171, (1958).

4 J.B. Taylor. Culham Laboratory Report CLM-R 58, (1966).

5 O.B. Firsov, in: M.A. Leontonvich, ed., Plasma Physics and the Problem of Controlled Thermonuclear Reactions, Pergamon Press, London, 3, 386, (1959).

6 W.B. Grossman. Phys. Fluids, 9, 12, 2478, (1966)

7 R.L. Morse. Phys. Fluids, 8, 2, 308, (1965).

8 K.V. Roberts. J. Nuclear Energy Pt. C, 1, 243, (1960).

9 J.B. Taylor and J.A. Wesson. Nuclear Fusion, 5, 159, (1965).

10 R.L. Morse. Phys. Fluids, 9, 12, 2536, (1966).

11 R.L. Morse, and J.P. Friedburg. Proc. A.P.S. Topical Conference on Pulsed High Density Plasmas, Los Alamos report LA-3770, Paper No.B7, (1967).

12 R.W. Kilb. Ref.11, Paper No.B5

13 I.J. Spalding, M.J. Eden, A.D.R. Phelps and T.K. Allen. Third I.A.E.A. Conference on Plasma Physics and Controlled Nuclear Fusion Research, Novosibirsk, Paper No.CN-24/K9, (1968).

14 I.J. Spalding, Nuclear Fusion, 8, 3, 161, (1968).

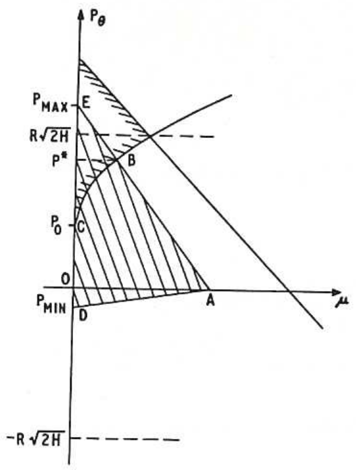


Fig. 1 The loss cone, ADE, and the region occupied by trapped particles in the $\mu - p_\theta$ phase plane.

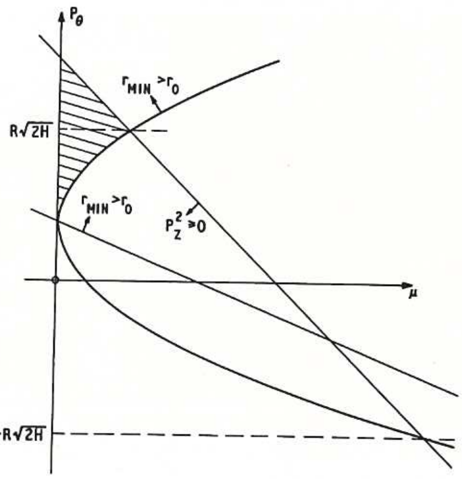


Fig. 2 The allowed region of phase space for particles trapped in the sheath for plasma radius $r_0$

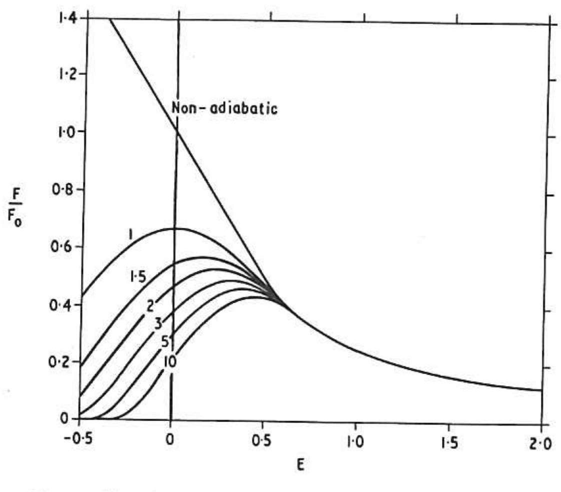


Fig. 3 Normalised particle loss rate from adiabatic and non-adiabatic cusps as a function of the transverse electric field E and the parameter $\lambda$.