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SOME STABLE PLASMA EQUILIBRIA IN COMBINED MIRROR-CUSP FIELDS

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SOME STABLE PLASMA EQUILIBRIA IN COMBINED MIRROR-CUSP FIELDS

by

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A B S T R A C T

The problem of equilibrium and stability of plasma confined in certain magnetic fields of combined mirror-cusp form is discussed. These fields have the properties that they are nowhere zero and everywhere increase toward the periphery. Attention is drawn to the importance of the existence of closed surfaces of constant $|B|$ - the magnetic isobars. The conditions for plasma equilibrium are derived and interpreted; then by exploiting the existence of closed magnetic isobars certain low - β confined equilibria are constructed. These equilibria are shown to be stable according to the fluid (double adiabatic) energy principle and according to the small Larmor radius limit theory. A direct proof of stability against motions which preserve the magnetic moment is given. These equilibria have the property that there is no current along lines of force so that they are also immune to several drift instabilities.

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CONTENTS

1. INTRODUCTION

It is well known[1] that the adiabatic invariance of the magnetic moment of a charged particle provides a mechanism whereby plasma may be contained within magnetic mirrors; however mirror systems are usually hydromagnetically unstable[2]. It is generally believed that a hydromagnetically stable situation is provided by fields which increase away from the centre[3], as in the spindle cusp; in these systems the adiabatic invariance is usually destroyed by a weak field region near the centre so that they are not genuine containment systems.

Recently there has been renewed interest[4] in magnetic field configurations which might provide both the inherent plasma stability attributed to fields whose strength increases towards the periphery, and the possibility of adiabatic containment.

One way of creating a field configuration of this type is by the addition of a multipole cusp field to the basic magnetic mirror (the stabilized mirror), as in the experiments of Ioffe[4]. Another method is by the insertion of a central, current-carrying, conductor along the axis of a spindle-cusp, thereby removing the weak field region which otherwise prevents adiabatic containment in the simple cusp.

General magnetic fields of the desired type can be identified by their two basic features, namely that there is a region in which

(a) the field is nowhere zero, so that adiabatic containment is possible, and

(b) the magnetic field strength $|B|$ 'increases outwards'.

By this second property of $|B|$ 'increasing outwards' one means that there exists a point, or in some cases a closed curve, which is a local minimum of B^2. In the neighbourhood of this point, or curve, the contours defined by B^2 = constant form a set of closed, nested, surfaces and a surface of larger B^2 encloses those of smaller B^2. Since these surfaces <u>are</u> closed one can unambiguously refer to inside and outside; then one can say that the magnetic pressure is lower inside any given surface than outside it. It is in a region such as this that one hopes for stable plasma confinement and in this paper we prove that there exists at least a class of stable equilibria in these 'non-vanishing outwardly increasing' fields.

It should be emphasized that these surfaces of B^2 = constant (which may be termed magnetic isobars) are not flux surfaces. A line of force will generally cut a magnetic isobar twice (or not at all) and the points of intersection could, for example, form the turning points of particles contained on that line by the mirror effect.

In section 2 a brief description is given of an example of a 'hybrid' mirror-cusp field configuration having the desired properties (a) and (b), while the main body of the paper, sections 3 - 5, is concerned with finding low - β equilibria in such fields. Using a fluid description of the plasma the necessary and sufficient conditions for equilibrium are derived and are then interpreted in terms of individual particle motions. By exploiting the concept of closed magnetic isobars a class of confined low - β equilibria are then constructed which satisfy these equilibrium conditions. These have the property that $p_\perp$ and p_{11} are themselves constant over a magnetic isobar.

In section 6 the stability of this class of equilibria is discussed and they are shown to be stable against interchange instability (which is the only form of magnetohydrodynamic instability possible at low - β) according to both the double adiabatic energy principle of Bernstein et al [5] and the small-Larmor-radius-limit energy principle of Kruskal and Oberman [6]. Finally this stability is demonstrated in a more direct manner.

As the equilibria have the additional property that there is no current along the lines of force they should also be immune to several of the non-magneto-hydrodynamic instabilities such as the drift instabilities.

2. MAGNETIC FIELD CONFIGURATION - AN EXAMPLE

As an example of the type of magnetic field under discussion we may consider the configuration employed by Ioffe [4]. Our object is merely to indicate some of the main features of this arrangement, particularly of the magnetic isobars.

Near the centre of a mirror machine the field strength increases as one moves along the axis toward either mirror, but decreases as one moves radially away from the axis. A method of creating a field having the property that B^2 increases both axially and radially would therefore appear to be to superimpose on the mirror a second field which increases as one moves from the axis but which is constant along the axis. Such a field is the 'multipole' field provided by 2ℓ straight rods parallel to the axis of the machine, adjacent rods carrying current in opposite directions. Near the axis the multipole field is approximately

$$\begin{aligned} B_r &= -\frac{\ell I^*}{R}\left(\frac{r}{R}\right)^{\ell-1}\cos \ell\theta \\ B_\theta &= +\frac{\ell I^*}{R}\left(\frac{r}{R}\right)^{\ell-1}\sin \ell\theta \end{aligned} \qquad ..(2.1)$$

where R is the distance of the rods from the axis and I^* is a measure of the current in each rod. (The relationship of I^* to the actual current I depends on the way that the current is distributed over the cross section of the rods and on the shape of this cross section; for thin rods $I^* = 2I$.) The original mirror field can be approximately represented by

$$\begin{aligned} B_z &= B_o\{1 - \alpha I_o(\tfrac{2\pi r}{L})\cos(\tfrac{2\pi z}{L})\} \\ B_r &= -\alpha B_o I_1(\tfrac{2\pi r}{L})\sin(\tfrac{2\pi z}{L}) \end{aligned} \qquad ..(2.2)$$

where I_o and I_1 are modified Bessel functions. The mirrors are situated at $z = \pm L/2$ and the mirror ratio is

$$R_m = \frac{1+\alpha}{1-\alpha} \qquad ..(2.3)$$

The formation of closed magnetic isobars of the required type can be illustrated easily when $\ell = 2$, for then near the centre of the machine, $z = 0$, $r = 0$, the field is given by

$$B^2 = B_o^2(1-\alpha)^2 + 4\pi^2 B_o^2\{\alpha(1-\alpha)\frac{z^2}{L^2} + \frac{r^2}{L^2}\left(\frac{I^{*2}L^2}{\pi^2 B_o^2 R^4} - \frac{\alpha(1-\alpha)}{2}\right)\}. \quad (2.4)$$

If the current in the multipole rods is small, so that

$$I^{*2} < \frac{\pi^2 R^4}{2L^2}\alpha(1-\alpha)B_o^2 \qquad ..(2.5)$$

then the isobars form a family of hyperboloids. However, as the current in the multipole rods is increased so that

$$I^{*2} > \frac{\pi^2 R^4}{2L^2}\alpha(1-\alpha)B_o^2 \qquad ..(2.6)$$

these magnetic isobars become closed (ellipsoidal) surfaces of the type we desire.

Before leaving this topic it is worth while noting that the situation is not so simple when $\ell > 2$. If $\ell > 2$ then sufficiently near the axis the multipole field is always too weak to compensate for the radial decrease in the basic mirror field. In this case closed magnetic isobars are still formed but instead of a single minimum at $r = 0$, $z = 0$, there are 2ℓ minima situated off the axis.

3. LOW - β EQUILIBRIA

We now consider the problem of plasma equilibrium in a magnetic field. For equilibrium the pressure tensor $\underline{\underline{P}}$ must satisfy

$$\underline{j} \times \underline{B} = \nabla \cdot \underline{\underline{P}} \qquad ..(3.1)$$

where $\underline{j}$ and $\underline{B}$ are connected by

$$\nabla \times \underline{B} = 4\pi \underline{j} \qquad ..(3.2)$$

$$\nabla \cdot \underline{B} = 0 \qquad ..(3.3)$$

A full solution to the problem of equilibrium would involve solving these equations subject to boundary conditions such as the given currents in the external conductors. However, apart from the impracticability of such a programme, it is our present aim to derive general results independent of the detailed arrangement of conductors, and so applicable to all fields possessing properties (a) and (b) of section 1. We therefore seek low-β solutions (where β is the ratio of plasma pressure to magnetic pressure).

At zero β the magnetic field is the vacuum field due to external currents; this is easily calculated and will be considered as given. The first order perturbation in the field, due to plasma pressure, is given by:

$$\underline{j}_1 \times \underline{B}_0 = \nabla \cdot \underline{\underline{P}} \qquad ..(3.4)$$

$$\nabla \times \underline{B}_1 = 4\pi \underline{j}_1 \qquad ..(3.5)$$

$$\nabla \cdot \underline{B}_1 = 0 \qquad ..(3.6)$$

where $\underline{j}_1$ is the plasma current density, $\underline{B}_0$ the original vacuum field and $\underline{B}_1$ the perturbation in this field due to the presence of plasma.

Now it might appear that these equilibrium equations should have solutions $\underline{j}_1$ and $\underline{B}_1$ for <u>any</u> given plasma pressure $\underline{\underline{P}}$ and that there is, therefore, no problem. Indeed in axi-symmetric configurations such as mirror or cusp this is true, but in general these equations will <u>not</u> possess a solution and our first task is to determine the conditions which $\underline{\underline{P}}$ must satisfy in order that a solution should exist.

This is perhaps most easily done as follows: Equations (3.5) and (3.6) are simply the magnetostatic equations which are known to have a solution if $\underline{j}_1$ exists and $\nabla \cdot \underline{j}_1 = 0$. Our procedure therefore will be to solve equation (3.4) for $\underline{j}_1$ and then to examine under what conditions $\nabla \cdot \underline{j}_1 = 0$. (As we shall be concerned only with $\underline{j}_1$ and $\underline{B}_0$ we may henceforth suppress all subscripts, provided we remember that $\underline{B}$ is always a vacuum field.)

To illustrate the argument consider the case of scalar pressure when equation (3.4) reduces to

$$\underline{j} \times \underline{B} = \nabla p \; . \qquad ..(3.7)$$

The first necessary condition on p is clearly

$$\underline{B} \cdot \nabla p = 0 \quad \text{or} \quad \frac{\partial p}{\partial s} = 0 \qquad ..(3.8)$$

i.e. p is constant along a field line. Given that (3.8) is satisfied we can then solve (3.7) for $\underline{j}_\perp$ (the component of $\underline{j}$ perpendicular to $\underline{B}$),

$$\underline{j}_\perp = \frac{-\nabla p \times \underline{B}}{B^2} \qquad ..(3.9)$$

and therefore

$$\underline{j} = \frac{-\nabla p \times \underline{B}}{B^2} + \lambda \underline{B} \qquad ..(3.10)$$

where λ is an arbitrary scalar.

The requirement div $\underline{j} = 0$ then gives

$$\underline{B} \cdot \nabla\lambda = \operatorname{div}\left\{\frac{\nabla p \times \underline{B}}{B^2}\right\} \qquad . (3.11)$$

or

$$\underline{B} \cdot \nabla\lambda = \frac{-2\nabla B \cdot (\nabla p \times \underline{B})}{B^3} \qquad ..(3.12)$$

Equation (3.12) can be written

$$\frac{d\lambda}{ds} = \frac{-2\nabla B \cdot (\nabla p \times \underline{B})}{B^4} \qquad ..(3.13)$$

where s is measured along the line of force. A necessary condition for this equation to possess a unique single-valued solution for λ is clearly

$$\oint \frac{\nabla B \cdot (\nabla p \times \underline{B})}{B^4} \, ds = 0 \qquad ..(3.14)$$

where the integral is taken along any closed line of force. Newcomb[7] has shown that this is also a sufficient condition.

In the case of scalar pressure, then, equations (3.8) and (3.14) are the necessary and sufficient conditions which the pressure must satisfy if the plasma is to be in equilibrium. We now turn to the situation of immediate interest, namely when the pressure is anisotropic, and seek the analogous conditions on the pressure tensor.

B Anisotropic Pressure

In a co-ordinate system with the principal axis along the magnetic field the pressure tensor can be written

$$\underline{\underline{P}} = p_{\perp} \Pi + (p_{\parallel} - p_{\perp})\underline{n}\,\underline{n} \qquad ..(3.15)$$

where $\underline{n}$ is a unit vector in direction of $\underline{B}$ and Π is the unit tensor.

The momentum balance equation is now

$$\underline{j} \times \underline{B} = \nabla \cdot \underline{\underline{P}} \qquad ..(3.16)$$

and from the parallel component of this equation the first condition on $p_{\perp}$ and $p_{\parallel}$ is obtained,

$$\underline{n} \cdot \nabla p_{\perp} + \underline{n} \cdot \mathrm{div}\{(p_{\parallel} - p_{\perp})\underline{n}\,\underline{n}\} = 0 \qquad ..(3.17)$$

or

$$\frac{\partial p_{\parallel}}{\partial s} + \frac{(p_{\perp} - p_{\parallel})}{B}\frac{\partial B}{\partial s} = 0 \qquad ..(3.18)$$

where s is measured along the magnetic field. This condition specifies a relation between $p_{\perp}$ and $p_{\parallel}$ along a field line, replacing the simpler condition $\partial p/\partial s = 0$ of the scalar pressure theory. However, if (3.18) is satisfied then equation (3.16) can be solved for $\underline{j}_{\perp}$ as before,

$$\underline{j}_{\perp} = \frac{-\nabla p_{\perp} \times \underline{B}}{B^2} + \frac{\underline{B} \times \mathrm{div}[(p_{\parallel} - p_{\perp})\underline{n}\,\underline{n}]}{B^2} \qquad ..(3.19)$$

and so

$$\nabla \cdot \underline{j}_{\perp} = \frac{2\nabla p_{\perp} \cdot (\underline{B} \times \nabla B)}{B^3} + \mathrm{div}\left\{\frac{\underline{B} \times \mathrm{div}[(p_{\parallel} - p_{\perp})\underline{n}\,\underline{n}]}{B^2}\right\} \qquad ..(3.20)$$

It can be shown that because $\underline{B}$ is a vacuum magnetic field the last term can be transformed to give

$$\mathrm{div}\left\{\frac{\underline{B} \times \mathrm{div}[(p_{\parallel} - p_{\perp})\underline{n}\,\underline{n}]}{B^2}\right\} = \frac{\nabla(p_{\parallel} - p_{\perp}) \cdot (\underline{B} \times \nabla B)}{B^3} \qquad ..(3.21)$$

Therefore we finally obtain

$$\nabla \cdot \underline{j}_{\perp} = \frac{\nabla(p_{\perp} + p_{\parallel}) \cdot (\underline{B} \times \nabla B)}{B^3} \qquad ..(3.22)$$

Then, just as in the case of scalar pressure, the vanishing of $\nabla \cdot \underline{j}$ requires

$$\nabla \cdot \underline{j}_{\parallel} = \underline{B} \cdot \nabla\lambda = - \nabla \cdot \underline{j}_{\perp} \qquad ..(3.23)$$

so that

$$\underline{B} \cdot \nabla\lambda = - \nabla(p_{\perp} + p_{\parallel}) \cdot \frac{(\underline{B} \times \nabla B)}{B^3} \qquad ..(3.24)$$

As before this can be written

$$\frac{d\lambda}{ds} = - \nabla(p_{\perp} + p_{\parallel}) \cdot \frac{(\underline{B} \times \nabla B)}{B^4} \qquad ..(3.25)$$

and if the lines of force were closed this would lead to the condition

$$\oint \nabla(p_{\perp} + p_{\parallel}) . \frac{(\underline{B} \times \nabla B)}{B^4} \, ds = 0 \qquad ..(3.26)$$

In the system we are considering the lines of force are not closed within the plasma volume but leave the region of interest. In this case, provided the plasma is surrounded by a region in which no current flows, we must have

$$\int \nabla(p_{\perp} + p_{\parallel}) \cdot \frac{(\underline{B} \times \nabla B)}{B^4} \, ds = 0 \qquad ..(3.27)$$

where the integral is taken from the point where the line of force first enters the plasma to the point where it first leaves it. (If this condition were not satisfied λ would not be zero when the line of force left the plasma and there would be currents flowing in the plasma free region.) Furthermore it is clear that if this condition (3.27) is satisfied a unique λ can always be constructed from (3.25). The condition (3.27) is therefore both necessary and sufficient.

With anisotropic pressure, then, the necessary and sufficient conditions for equilibrium are (3.18) and (3.27). Before discussing some distributions satisfying these conditions we will first interpret these equilibrium constraints from the point of view of individual particle motions.

4. PARTICLE MOTION

The first constraint (3.18) is simply the requirement that the particles be in equilibrium along each field line considered individually. This is entirely consistent with the basic idea of adiabatic mirror containment; for if the magnetic moment of a particle

$$\mu = \frac{v_{\perp}^2}{2B} \qquad ..(4.1)$$

is constant as it moves along a field line then

$$p_{\perp} \propto \int \rho(\mu,\varepsilon) \, \frac{\mu B}{2} \, d\mu d\varepsilon \qquad ..(4.2)$$

$$p_{\parallel} \propto \int \rho(\mu,\varepsilon)(\varepsilon - \mu B) \, d\mu d\varepsilon \qquad ..(4.3)$$

where ρ is the local density of particles of specified magnetic moment μ and energy ε. This is proportional to (i) the number of such particles on the line = $f(\mu,\varepsilon,L)$, (ii) to the density of lines = B, (iii) to the fraction of the time each particle spends near the point of interest.

$$dt \propto \frac{d\ell}{(\varepsilon - \mu B)^{\frac{1}{2}}} \qquad ..(4.4)$$

Therefore, for particles contained by the mirror effect,

$$p_{\perp} = \int f(\mu,\varepsilon,L) \frac{\mu B^2}{2(\varepsilon - \mu B)^{\frac{1}{2}}} d\mu d\varepsilon \qquad ..(4.5)$$

$$p_{\parallel} = \int f(\mu,\varepsilon,L) B (\varepsilon - \mu B)^{\frac{1}{2}} d\mu d\varepsilon \qquad ..(4.6)$$

It can be verified by direct substitution that these expressions satisfy (3.18).

The second constraint (3.27) may be interpreted in terms of the guiding centre drifts of the particles on a field line. As is well known[(2)] the first order guiding centre drift of a particle in an inhomogeneous magnetic field is

$$\underline{v}_D = \frac{mc}{e} \frac{(\underline{B} \times \nabla B)}{B^3}\left(\frac{v_{\perp}^2}{2} + v_{\parallel}^2\right) \qquad ..(4.7)$$

where $v_{\perp}$ is the velocity perpendicular to the field and $v_{\parallel}$ that along it.

The total current associated with this drift is then

$$\underline{j}_D = \frac{(\underline{B} \times \nabla B)}{B^3} (p_{\perp} + p_{\parallel}) \qquad ..(4.8)$$

and the divergence of this expression is

$$\nabla \cdot \underline{j}_D = \nabla(p_{\perp} + p_{\parallel}) \cdot \frac{(\underline{B} \times \nabla B)}{B^3} \qquad ..(4.9)$$

so that the second condition for equilibrium can be written

$$\int (\nabla \cdot \underline{j}_D)\frac{ds}{B} = 0 \qquad ..(4.10)$$

The meaning of this is made clear if we consider not the integral along a field line but the integral over an infinitesimal flux tube. This can be obtained by multiplying (4.10) by B d A when we have

$$\int_{\text{Flux tube}} (\nabla \cdot \underline{j}_D) d\tau = 0 \qquad ..(4.11)$$

So that the condition found for the existence of a solution to the magnetostatic fluid equations is equivalent to the statement that the divergence of the current associated with the guiding centre drifts should vanish when averaged over any flux tube. Of course, the current due to the guiding centre drifts is <u>not</u> the same as the total current but the difference between them can be expressed as the Curl of the magnetisation per unit volume, whose divergence vanishes identically. The constraint might therefore equally well be applied to the total current or to the drift current.

5. A CLASS OF EQUILIBRIA

Now let us consider some particular solutions of the equilibrium constraints (3.18) and (3.27), appropriate to the type of magnetic field under discussion. It should first be noted that the second constraint (3.27) is not serious in systems of axial symmetry such as the mirror or the spindle cusp. For in these systems the symmetry ensures that ∇p, ∇B and $\underline{B}$ are coplanar vectors (lying in the r,z, plane) so that the expression

$$\nabla(p_{\perp} + p_{\parallel}) \cdot \nabla B \times \underline{B} \qquad ..(5.1)$$

vanishes identically. Similarly in any cylindrically symmetric system ∇p and ∇B are both radial and (5.1) again vanishes.

In other field configurations the constraint (3.27) can be a severe restriction; for example, the condition (3.27)(or rather (3.26) which is then the appropriate form) can never be satisfied by any <u>confined</u> plasma distribution within a circular torus. For in such a configuration, symmetry ensures that the integral (3.26) can only vanish if the integrand vanishes. As $(\nabla B \times \underline{B})$ is in the direction parallel to the symmetry axis of the torus this means that p must be constant in this direction, thus the plasma is not confined. This, of course, is the well known lack of equilibrium in a simple toroidal field.

If we leave aside for the moment the question of whether it represents contained plasma or not, a restricted class of solutions to the equilibrium constraints can <u>always</u> be found by demanding that (5.1) should vanish. This is certainly achieved if $(p_{\perp} + p_{\parallel})$ is a function only of B, then, since the 'parallel' equilibrium equation (3.18) gives $p_{\perp}$ in terms of $p_{\parallel}$ this will make $p_{\perp}$ and $p_{\parallel}$ individually functions of B alone. Making $p_{\perp}$ and $p_{\parallel}$ functions of B alone means that the surfaces of constant B, the magnetic isobars, are also surfaces of constant $p_{\perp}$ and $p_{\parallel}$.

The significance of magnetic field configurations which possess closed magnetic isobars now becomes apparent. Equilibria in which $p_\perp$ and $p_{\shortparallel}$ are functions only of B exist in all field configurations, but only in those which possess closed magnetic isobars do these equilibria correspond to <u>confined</u> plasma configurations.

This class of low - β equilibria, which have

$$p_\perp = p_\perp(B), \quad p_{\shortparallel} = p_{\shortparallel}(B) \qquad ..(5.2)$$

and from (3.18)

$$Bp'_{\shortparallel} = p_{\shortparallel} - p_\perp \qquad ..(5.3)$$

where the prime denotes differentiation with respect to B, is one whose stability will be proved in the next section.

An example of this class of equilibrium distribution is

$$\left.\begin{array}{ll} p_{\shortparallel} = C\,B(B_o - B)^n & \\ & \text{if } B < B_o \\ p_\perp = nC\,B^2(B_o - B)^{n-1} & \\ p_\perp = p_{\shortparallel} = 0 & \text{if } B > B_o \end{array}\right. \qquad ..(5.4)$$

where n, B_o are arbitrary parameters. These equilibria correspond to plasma confined within the contour $B = B_o$ which by the basic property of our fields can be a closed contour.

Particle distribution functions corresponding to the equilibria (5.4) can also be written down in terms of the distribution in μ,ε space (see section IV). A particle distribution function which leads to the pressure distributions (5.4) is

$$\left.\begin{array}{ll} f(\mu,\varepsilon) = (\mu B_o - \varepsilon)^{n-\frac{3}{2}} g(\mu) & \varepsilon < \mu B_o \\ f(\mu,\varepsilon) = 0 & \varepsilon > \mu B_o \end{array}\right. \qquad ..(5.5)$$

where $g(\mu)$ is an arbitrary function of the magnetic moment.

6. STABILITY OF THE SPECIAL EQUILIBRIA

To examine the stability of the equilibria described in the previous section let us first continue with a fluid description and consider the double adiabatic hydromagnetic energy principle derived by Bernstein et al[5].

According to this, the stability of a plasma configuration with anisotropic pressure is determined by the sign of the minimum of the energy integral.

$$\delta W_D = \int d\tau \{ |\underline{Q}|^2 - \underline{j} \cdot \underline{Q} \times \underline{\xi} + \tfrac{5}{3} p_\perp (\nabla \cdot \underline{\xi})^2 + (\nabla \cdot \underline{\xi})(\underline{\xi} \cdot \nabla p_\perp)$$

$$+ \tfrac{1}{3}[\nabla \cdot \underline{\xi} - 3q]^2 + q \nabla \cdot [\underline{\xi}(p_{\parallel} - p_\perp)]$$

$$- (p_{\parallel} - p_\perp)[\underline{n} \cdot (\underline{a} \cdot \nabla)\underline{\xi} + \underline{a} \cdot (\underline{n} \cdot \nabla)\underline{\xi} - 4q^2] \quad ..(6.1)$$

where

$$\underline{Q} = \nabla \times (\underline{\xi} \times \underline{B})$$
$$q = \underline{n} \cdot (\underline{n} \cdot \nabla)\underline{\xi} \qquad ..(6.2)$$
$$\underline{a} = (\underline{n} \cdot \nabla)\underline{\xi} - (\underline{\xi} \cdot \nabla)\underline{n}$$

and $\underline{\xi}$ is an arbitrary displacement vector. δW_{min} should be positive for stability.

Examination of the energy integral shows that only the first term $|Q^2|$ is independent of β so that at low-β it must dominate (and so make δW positive) except for those displacements which themselves make Q zero. Physically these displacements are those which do not change the vacuum magnetic field - the so called interchange modes.

Hence, at sufficiently low β we can determine stability by examining δW for displacements which satisfy

$$\underline{Q} = \nabla \times (\underline{\xi} \times \underline{B}) = 0 \qquad ..(6.3)$$

and for these displacements

$$q = \nabla \cdot \underline{\xi} + \frac{\underline{\xi} \cdot \nabla B}{B}$$

$$\underline{a} = (\nabla \cdot \underline{\xi} + \frac{\underline{\xi} \cdot \nabla B}{B})\underline{n} \qquad ..(6.4)$$

With the aid of (6.3) and (6.4) the energy integral can be greatly simplified. In fact

$$\delta W_D = \int d\tau \{ 3p_{\parallel} d^2 + ds(5p_{\parallel} - p_\perp) + s^2(p_\perp + 2p_{\parallel})$$

$$+ d(\underline{\xi} \cdot \nabla p_{\parallel}) + s(\underline{\xi} \cdot \nabla(p_{\parallel} - p_\perp)) \} \qquad ..(6.5)$$

where, for brevity, we have written

$$\nabla \cdot \underline{\xi} \equiv d \; ; \quad \frac{\underline{\xi} \cdot \nabla B}{B} \equiv s$$

So far this is quite general. For the equilibria found in section 5 namely those which have the properties.

$$p_\perp = p_\perp(B), \quad p_{\parallel} = p_{\parallel}(B), \quad Bp_{\parallel}' = p_{\parallel} - p_\perp \qquad ..(6.6)$$

δW_D reduces to

$$\delta W_D = \int d\tau \left\{ \frac{1}{3p_{\parallel}} \left[3p_{\parallel}(d+s) - p_{\perp}s \right]^2 + s^2 \left[2p_{\perp} - \frac{p_{\perp}^2}{3p_{\parallel}} - Bp_{\perp}' \right] \right\} \qquad ..(6.7)$$

The first term is clearly non-negative so a sufficient criterion for stability according to the double adiabatic principle is

$$2p_{\perp} - \frac{p_{\perp}^2}{3p_{\parallel}} - Bp_{\perp}' > 0 \qquad ..(6.8)$$

Some explicit examples of equilibria were given by equations (5.4). For these examples

$$Bp_{\perp}' = 2p_{\perp} - \frac{(n-1)}{n} \frac{p_{\perp}^2}{p_{\parallel}} \qquad ..(6.9)$$

and a sufficient stability condition is $n > \frac{3}{2}$. (Note that this is also the condition for $f(\mu, \varepsilon)$ in equation (5.5) to be continuous at $\varepsilon = \mu B_0$).

There is, however, one reservation to be made about the argument above. The last term in the energy integral contains the expression $p_{\perp}^2/p_{\parallel}$ and for some of the equilibria of the form (5.4) this quantity tends to infinity at the plasma boundary. This will make possible, even at low - β, some instabilities in which the magnetic field is perturbed. These are the 'mirror' instabilities. As the plasma density falls to zero at the surface it is not clear whether this particular instability is to be taken seriously, but in any case it can be avoided by demanding that $p_{\perp}^2/p_{\parallel}$ be finite at the surface. This requirement is satisfied by the equilibria given in (5.4) if $n > 2$.

B The small Larmor radius theory

The double adiabatic energy principle is open to at least two objections; firstly that it is based on the assumption that in the plasma motion there is no heat flow along the lines of force and secondly that although a component of the displacement $\underline{\xi}$ along the lines of force is formally allowed, it is hard to see what is the real significance of this parallel displacement (since in collisionless-plasma motion arises from $\underline{E} \times \underline{B}$ drifts).

An energy principle which is sufficient, though not necessary, for stability and which overcomes these objections was given by Kruskal and Oberman(6). This is based on the use of the Boltzman equation in the limit of small Larmor radius. In this case the appropriate energy integral can be written

$$\delta W_{ko} = \delta W_D - \int d\tau \{ 2p_{\perp} q (\nabla \cdot \underline{\xi}) + (3p_{\parallel} - 2p_{\perp}) q^2 \} + I \qquad ..(6.10)$$

where

$$I = \int d\tau \left\{ \Sigma m_i \iint \frac{B}{v_{\parallel}} \, d\mu d\varepsilon \left[\mu^2 B^2 \left(\frac{\partial f_o}{\partial \varepsilon} \right) (\nabla \cdot \underline{\xi} - q)^2 - \frac{f^{*2}}{\frac{\partial f_o}{\partial \varepsilon}} \right] \right\} \qquad ..(6.11)$$

and

$$\frac{v_{\parallel}^2}{2} = \varepsilon - \mu B \qquad ..(6.12)$$

In these expressions ε and μ are again the energy and magnetic moment as in section 5, and f^* is the perturbation in the particle distribution function. The quantity $f_o(\mu, \varepsilon, L)$ is the unperturbed particle distribution and in their derivation of the energy principle Kruskal and Oberman require that

$$\frac{\partial f_o}{\partial \varepsilon} < 0 \qquad ..(6.13)$$

The minimisation of δW_{ko} has to be carried out over $\underline{\xi}$ and also over f^* subject to certain constraints. The minimisation over f^* is carried out in the Kruskal and Oberman paper but we will have no need of this in the present discussion.

It can be shown that the minimum of δW_{ko} is independent of $\xi_{\parallel}$ as it should be, so that $\xi_{\parallel}$ can be taken to be zero.

As before, at sufficiently low-β we need only consider displacements which satisfy

$$\nabla \times (\underline{\xi} \times \underline{B}) = 0 \qquad ..(6.14)$$

so that equations (6.4) are again valid. However as $\underline{\xi}$ is now perpendicular to $\underline{B}$ a further simplification can also be obtained. For (6.14) implies that

$$\underline{\xi} \times \underline{B} = \nabla \phi \qquad ..(6.15)$$

and so $\underline{\xi}$ can now be written

$$\underline{\xi} \equiv \underline{\xi}_{\perp} = \frac{\underline{B} \times \nabla \phi}{B^2} \qquad ..(6.16)$$

whence

$$\nabla \cdot \underline{\xi} = -2 \frac{\underline{\xi} \cdot \nabla B}{B} \qquad ..(6.17)$$

With the aid of equations (6.4) and (6.17) the energy integral may be reduced to

$$\delta W_{ko} = \int d\tau \left\{ (2p_{\perp} - Bp_{\perp}') s^2 \right\} + \int d\tau \Sigma m_i \iint \frac{B}{v_{\parallel}} \, d\mu d\varepsilon \left[\mu^2 B^2 \left(\frac{\partial f_o}{\partial \varepsilon} \right) s^2 - \frac{f^{*2}}{\frac{\partial f_o}{\partial \varepsilon}} \right] \qquad ..(6.18)$$

This can be further simplified, for

$$p_{\perp} = \Sigma\, m_i \iint \frac{\mu B^2}{v_{\parallel}} f_o \, d\mu d\varepsilon \qquad ..(6.19)$$

and since

$$\frac{1}{v_{\parallel}} = \frac{\partial v_{\parallel}}{\partial \varepsilon} \qquad ..(6.20)$$

a partial integration leads to

$$p_{\perp} = -\Sigma\, m_i \iint \mu B^2 v_{\parallel} \left(\frac{\partial f_o}{\partial \varepsilon}\right) d\varepsilon d\mu \qquad ..(6.21)$$

Then if $p \equiv p_{\perp}(B)$, differentiation with respect to B gives

$$Bp_{\perp}' = 2p_{\perp} + \Sigma\, m_i \iint \frac{\mu^2 B^3}{v_{\parallel}} \left(\frac{\partial f_o}{\partial \varepsilon}\right) d\varepsilon d\mu \qquad ..(6.22)$$

and using this result the energy integral is finally reduced to

$$\delta W_{ko} = -\int d\tau\, \Sigma\, m_i \iint \frac{B}{v_{\parallel}} \, d\mu d\varepsilon \left\{ \frac{f^{*2}}{\frac{\partial f_o}{\partial \varepsilon}} \right\} \qquad ..(6.23)$$

which is certainly positive if $\frac{\partial f_o}{\partial \varepsilon} < 0$, a condition which is in any case required for the present energy principle to be valid.

According to the small Larmor radius theory of Kruskal and Oberman, then, equilibria of the class (6.6) are stable if their corresponding particle distributions satisfy

$$\frac{\partial f_o}{\partial \varepsilon} < 0 \qquad ..(6.24)$$

Thus the specific examples (5.4) correspond to the particle distributions (5.5) and so are stable if

$$\frac{\partial}{\partial \varepsilon}(\mu B_o - \varepsilon)^{n-\frac{3}{2}} < 0 \qquad ..(6.25)$$

that is if $n > \frac{3}{2}$. In this case, therefore, the two energy principles lead to the same criterion.

7. DIRECT PROOF OF STABILITY

The simplicity of the form of the final expression for δW_{ko} suggests that a more direct demonstration of the stability of our equilibria should be possible which did not make use of the full Kruskal - Oberman theory. Such a proof of stability can be developed by extension of the argument given by Newcomb[8] in discussing stability of infinite maxwellian plasma.

Let us consider a general particle motion in which the magnetic moment of a particle is invariant, (as in small Larmor radius theory), then a general constant of the motion constructed from individual particle constants is

$$S = \int \frac{B}{v_{\parallel}} \, d\mu d\varepsilon d\tau \; G(f,\mu) \qquad ..(7.1)$$

Now consider a distribution function $f = f_o + \delta f$ where f_o is the initial equilibrium distribution whose stability we want to discuss. Then we can write

$$\delta S = 0 = \int \frac{B}{v_{\parallel}} \, d\mu d\varepsilon d\tau \left\{ G'(f_o,\mu) \cdot \delta f + G''(f_o,\mu) \cdot \frac{(\delta f)^2}{2} + \dots \right\} \qquad ..(7.2)$$

where

$$G'(f,\mu) \equiv \frac{\partial G}{\partial f} \quad .$$

Now the equilibria we are considering have the property that $p_{\perp}$ and $p_{\parallel}$ are functions of B only and satisfy the parallel equilibrium equation. Such equilibria correspond to particle distribution functions which depend only on μ and ε (i.e. $f_o(\varepsilon,\mu,L)$ is independent of the particular flux line considered). For these equilibria, therefore, the function G can be chosen so that

$$G'(f_o,\mu) \equiv \varepsilon \qquad ..(7.3)$$

(at least if $\partial f_o/\partial\varepsilon$ is monotonic) and with this choice for G equation (7.2) becomes

$$\int \frac{B}{v_{\parallel}} \, d\mu d\varepsilon d\tau \cdot (\varepsilon \delta f) = - \int \frac{B}{v_{\parallel}} \, d\mu d\varepsilon d\tau \cdot \frac{(\delta f)^2}{2 \frac{\partial f_o}{\partial \varepsilon}} + \dots \qquad ..(7.4)$$

which may be written

$$\delta K = - \int \frac{B}{v_{\parallel}} \, d\mu d\varepsilon d\tau \, \frac{(\delta f)^2}{2 \frac{\partial f_o}{\partial \varepsilon}} + \dots \qquad ..(7.5)$$

where K is the total kinetic energy of the particles,

$$K = \int \frac{B}{v_{\parallel}} \, d\mu d\varepsilon d\tau \cdot (\varepsilon f) \quad .$$

If now

$$\frac{\partial f_o}{\partial \varepsilon} < 0$$

it is clear that to second order in δf, $\delta K > 0$ so that any change δf in f around f_o will <u>increase</u> the kinetic energy. Furthermore if the equilibrium has no electric fields and is of such low-β that the magnetic field is a vacuum field, then any perturbations can also only increase the field energies. As the total energy is constant it is clear that δf cannot grow indefinitely and in particular cannot grow exponentially. Therefore the system is stable.

Thus it has been shown that any low-β equilibrium with f_o a function only of μ, ε is stable against all perturbations in which the magnetic moment is an invariant. This is certainly sufficient to demonstrate stability against hydromagnetic motions.

8. CONCLUSIONS

Attention has been drawn to the importance of the existence of closed magnetic isobars in certain hybrid mirror-cusp magnetic fields. The existence of these closed isobars enables one to construct a class of confined plasma distributions, those with $p_\perp$ and $p_\parallel$ functions of B alone, which satisfy the conditions for equilibrium. These equilibria are stable against interchanges according to both the double-adiabatic energy principle and the more complete small-Larmor-radius theory. A direct proof of stability against all motions in which the magnetic moment of a particle is an invariant has also been given.

It is easily shown that these equilibria have the property that $j_\parallel = 0$, which ensures that they are also stable against several forms of 'drift' instability; the large amount of 'shear' and the high curvature of some of these hybrid fields may also inhibit some other micro-instabilities. One concludes, therefore, that these non-vanishing outwardly-increasing fields do indeed offer the possibility of stable plasma confinement.

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