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A CALCULATION OF THE INSTANTANEOUS POPULATION DENSITIES OF THE EXCITED LEVELS OF HYDROGEN-LIKE IONS IN A PLASMA

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A CALCULATION OF THE INSTANTANEOUS POPULATION DENSITIES OF THE EXCITED LEVELS OF HYDROGEN-LIKE IONS IN A PLASMA

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ABSTRACT

The instantaneous population densities for the excited levels of hydrogen-like ions in an optically thin plasma, which is not necessarily in equilibrium, have been calculated for a range of electron temperature of 4000 Z^2 °K to 256,000 Z^2 °K and electron density of $10^8 Z^7 \text{ cm}^{-3}$ to $10^{18} Z^7 \text{ cm}^{-3}$, where Z is the charge of the bare nucleus. The population densities depend linearly on the ground level population densities and tables are presented of two coefficients representing this relation for some of the lower excited levels.

The calculations include the processes of excitation, de-excitation, and ionization by electron collision, spontaneous radiative decay, three body recombination and radiative recombination. Processes involving the absorption of photons are neglected and it is assumed that the free electrons have a Maxwellian distribution.

The validity of the calculations in the extremes of the ranges is discussed.

The calculations illustrate the transition from high densities where all the excited levels have nearly a Saha-Boltzmann population to low densities where the radiative capture-cascade model is valid.

From the population density of the excited levels, the power lost by line radiation by hydrogen-like ions, radiative recombination of electrons onto the bare nuclei and bremsstrahlung of the free electrons in the field of the bare nuclei is calculated.

Since energy is lost by radiation, the total energy dissipated during the ionization of one hydrogen-like ion may be much greater than the simple ionization energy of the ion. This has been calculated for electron temperatures between 16,000 Z^2 °K and 256,000 Z^2 °K.

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C O N T E N T S

T A B L E S

APPENDICES

1. Introduction

The spectral line radiation emitted by a plasma is a powerful source of information about the conditions inside the plasma. In an optically thin plasma, the line intensities are proportional to the population densities of the excited levels and in this paper these population densities have been calculated for a plasma, composed of hydrogen-like ions, which is not necessarily in equilibrium. The calculations for hydrogen atoms have been done by Bates and Kingston (1963). Tables of two coefficients are presented which represent the linear relation between instantaneous excited level population densities and the ground level population density for a given electron temperature and density, for the levels of low quantum number.

Since the line intensities depend on the instantaneous population density of the ground level, the variation of the line intensities with time requires the calculation of the variation of the ground level population density. This involves the collisional-radiative recombination and ionization coefficients which have been calculated by Bates, Kingston and McWhirter (1962) using the same atomic model as that used here. This paper will be referred to as paper 1. The atomic model includes all inelastic electron collisions involving bound and free electrons and all radiative processes that do not involve the absorption of photons. It is assumed that the free electrons have a Maxwellian distribution.

The calculation of the population densities of each of an infinite number of excited levels is avoided because at a sufficiently high quantum number the excited level has a Saha-Boltzmann population density, and the contribution of these high levels to the lower ones is then easily calculated.

It is assumed that level 20 and above have a Saha-Boltzmann population density, and it is found that this assumption is justified for electron densities greater than $10^8 Z^7$ cm^{-3}.

From the instantaneous population densities, the power loss by radiation and the total energy required to ionise one hydrogen-like ion are calculated as a function of electron density and temperature.

2. The basic equations

The collisional and radiative processes which are included in these calculations are the following.

(1) Excitation of a hydrogen-like ion of charge Z-1 from a level p to a level q by electron impact, and the inverse process of de-excitation.

$$N^{(Z-1)+}(p) + e \rightleftharpoons N^{(Z-1)+}(q) + e \tag{1}$$

The rate coefficients for the forward and backward process are given the symbols $K(p,q)$ and $K(q,p)$ in units of $cm^3\ sec^{-1}$.

(2) Ionisation of the ion from a level p and its inverse of three body recombination.

$$N^{(Z-1)+}(p) + e \rightleftharpoons N^{Z+} + e + e \tag{2}$$

Their rate coefficients are given the symbols $K(p,c)$ and $K(c,p)$ respectively in units of $cm^3\ sec^{-1}$.

(3) Spontaneous radiative decay from a level p to a lower level q.

$$N^{(Z-1)+}(p) \rightarrow N^{(Z-1)+}(q) + h\nu \tag{3}$$

The transition probability is $A(p,q)\ sec^{-1}$. The inverse process of photo-excitation is not included, so it is assumed that the plasma is optically thin.

(4) Radiative recombination of an electron with a bare nucleus to form an ion in level p.

$$N^{Z+} + e \rightarrow N^{(Z-1)+}(p) + h\nu \tag{4}$$

The rate coefficient is $\beta(p)\ cm^3\ sec^{-1}$. The inverse of this process, photoionisation, is not included.

It is assumed that the bound levels are adequately specified by their principal quantum numbers and that the sub levels are populated according to their statistical weights. Generally the electron collision rate between the sub-levels is sufficiently large to ensure this.

With all these processes included, the rate at which the population density n(p) of level p changes is given by the differential equation

$$\frac{dn(p)}{dt} = -n(p)\left\{n(c)\left[K(p,c)+\sum_{q\neq p} K(p,q)\right]+\sum_{q<p} A(p,q)\right\}$$

$$+\,n(c)\sum_{q\neq p} n(q)K(q,p)+\sum_{q>p} n(q)A(q,p)$$

$$+\,\frac{n(c)^2}{X}\left\{K(c,p)+\beta(p)\right\} \qquad (5)$$

where n(c) is the number density of the free electrons and X is the ratio of n(c) to the number density of bare nuclei of charge Z. For plasma neutrality X must be equal to or greater than Z since ions of charge not equal to Z may also be present.

The description of the variation of the populations of the bound levels is represented by an infinite number of such equations, one for each bound level.

3. The atomic coefficients

The values of the coefficients K, A and β used in these calculations are the same as those described in paper 1 for Z greater than one. The coefficients K(1,c) and K(1,2) were evaluated by the numerical integration of the Coulomb-Born approximation for the cross section for Z of 2 over a Maxwellian electron distribution (Burgess 1961). The effect of 1s-2s transitions was included in the K(1,2) coefficient by increasing the values by 15%. This is the average ratio of the 1s-2s to 1s-2p cross-section according to the ordinary Born approximation for hydrogen (McCarroll 1957). K(1,3) and K(1,4) were obtained by adjusting Burgess' 1s-2s and 2p cross sections to the appropriate threshhold energy and fitting to the Born approximation of McCarroll at high electron energy.

The values used for these four coefficients are given in table 1. The rate coefficients of the inverse processes to these four were obtained from detailed balancing.

Table 1. Values of the ground level collision coefficients.

Θ	$Z^3K(1,2)$	$Z^3K(1,3)$	$Z^3K(1,4)$	$Z^3K(1,c)$
4,000	3.20^{-20}	2.81^{-23}	1.97^{-24}	9.07^{-26}
8,000	6.20^{-14}	8.41^{-16}	1.37^{-16}	4.90^{-17}
16,000	7.54^{-11}	4.00^{-12}	9.90^{-13}	1.32^{-12}
32,000	2.34^{-9}	2.44^{-10}	7.11^{-11}	2.35^{-10}
64,000	1.20^{-8}	1.75^{-9}	5.90^{-10}	3.38^{-9}
128,000	2.58^{-8}	4.45^{-9}	1.58^{-9}	1.34^{-8}
256,000	3.66^{-8}	6.98^{-9}	2.56^{-9}	2.63^{-8}

The indices give the power of ten by which the entries in the coefficient columns must be multiplied.

The oscillator strengths for transitions up to level 20 used to calculate the rate coefficients were taken from Green, Rush, and Chandler (1957). Beyond level 20 the asymptotic expression for the oscillator strength given by Unsöld (1955) was used.

$$f(p,q) = \frac{2^6}{3\sqrt{3}\,\pi}\left\{\frac{1}{p^2} - \frac{1}{q^2}\right\}^{-3} \frac{1}{p^3}\,\frac{1}{q^3}\,\frac{1}{2p^2} \tag{6}$$

The coefficients K, A and β all depend on the nuclear charge Z. With the values used for these coefficients the dependence of the equations on Z and X may be included by substituting the following reduced parameters.

electron temperature $\quad \Theta = T/Z^2 \quad$ (7)

electron density $\quad \eta(c) = n(c)/Z^7 \quad$ (8)

population density $\quad \eta(p) = X\,n(p)/Z^{11} \quad$ (9)

All the results will be given in terms of these reduced parameters. Since X, the ratio of the number density of free electrons to the bare nuclei is included, the calculations give the population densities of the excited levels for given electron and bare nuclei population densities and electron temperature. For convenience the values of the first nine integers raised to the various powers appearing in the Z-scaling laws are given in Appendix 1.

4. The quasi steady state solution

If the system is in a completely steady state, then the population densities of the bound levels are determined by the solution of the equations (5) when $\frac{dn}{dt}$ is zero for every level. Under these conditions the rate at which electrons are transferred out of the level, which is represented by the term in n(p) in equation (5), equals the rate at which electrons are transferred in to the level, which is represented by all the other terms.

Suppose now that the population of level p is perturbed by a small amount. The return to its steady state value is given by

$$n(p) = A\left[1 - e^{-t/\tau(p)}\right] \qquad (10)$$

where

$$\tau(p) = n(c)\left[K(p,c) + \sum_{q \neq p} K(p,q)\right] + \sum_{q<p} A(p,q) \qquad (11)$$

Values of τ (p), the relaxation time constant for level p, have been calculated for a range of conditions. The relaxation time is inversely proportional to Z^4, and the values of $Z^4\tau(p)$ are given in seconds in table 2. It is apparent from these values that the relaxation time of the ground level is always much greater than that of any of the excited levels. The reason for this is that the electron collision rate coefficients between excited levels are much greater than the rate coefficients to the ground level and also that the ground level cannot decay by spontaneous radiative transitions. Inspection of the relaxation time constants for conditions where the plasma is not near its steady state supports the general conclusion that the ground level time constant is always much greater than that of the excited levels.

Table 2. Relaxation time constants $Z^4\tau(p)$ secs.

(H)	$\eta(c)$	p = 1	p = 2	p = 3	p = 15
4,000	10^{8}	3.1^{+11}	2.1^{-9}	1.0^{-8}	1.1^{-6}
4,000	10^{18}	3.1^{+1}	1.4^{-11}	2.8^{-13}	1.2^{-16}
16,000	10^{8}	1.2^{+2}	2.1^{-9}	1.0^{-8}	2.1^{-6}
16,000	10^{18}	1.2^{-8}	1.8^{-12}	1.7^{-13}	2.5^{-16}
64,000	10^{8}	5.4^{-1}	2.1^{-9}	1.0^{-8}	3.5^{-6}
64,000	10^{15}	5.4^{-8}	7.4^{-10}	2.3^{-10}	4.9^{-13}
64,000	10^{18}	5.4^{-11}	1.1^{-12}	2.3^{-13}	4.9^{-16}
256,000	10^{8}	1.3^{-1}	2.1^{-9}	1.0^{-8}	5.5^{-6}
256,000	10^{12}	1.3^{-5}	2.1^{-9}	9.8^{-9}	9.8^{-10}
256,000	10^{15}	1.3^{-8}	9.3^{-10}	4.1^{-10}	9.8^{-13}
256,000	10^{18}	1.3^{-11}	1.7^{-12}	4.3^{-13}	9.8^{-16}

The indices give the power of ten by which the entries in the time constant columns must be multiplied.

The complete description of the system would involve the variation with time of the population density of every bound level from a particular set of initial conditions. But in practice the time in which the population densities of the excited levels come into equilibrium with a particular population density of the ground level, free electrons and bare nuclei is so short that it is sufficient to give these equilibrium population densities as a function of the ground level population density. This is the quasi steady state solution, and it is obtained by setting $\frac{dn}{dt}$ for all levels except the ground level equal to zero. In general the ground level will not be in equilibrium, but its population density changes comparatively slowly and the population densities of the excited levels follow in a time that may be regarded as instantaneous. The rate at which the population density of the ground level changes is related to the collisional radiative recombination and ionisation coefficients discussed in paper 1.

It is assumed in these calculations that the population densities of the free electrons and bare nuclei remain constant during the time in which the quasi steady state is established. When the total population density of the discrete excited levels is greater than the population density of the bare nuclei this may not be true because a large proportion of the electrons may exchange between the continuum and the bound levels because of a change in the plasma conditions. Thus for the quasi steady state solution to be valid, the inequality

$$\sum_{p=2}^{\infty} n(p) < n(Z^+) \qquad \text{for } n(1) = 0 \tag{12}$$

must be satisfied. The evaluation of this sum requires the knowledge of the level which merges into the continuum. This is outside the scope of this paper, but the calculated values of n(p) for n(1) set at zero indicate that a fairly reliable estimate may be made of $\sum n(p)$ for this purpose. Table 3 is based on such an estimate and gives the values of Z that must not be exceeded for a range of reduced electron temperature and density if the inequality is to be satisfied.

Table 3. The greatest values of Z which still satisfy the inequality $\sum_{p=2}^{\infty} n(p) < n(Z^+)$ when $n(1) = 0$

$\eta(c) \leq$	10^{18}	10^{17}	10^{16}	10^{15}	10^{14}	10^{13}
H $\geq$ 4,000			$Z \leq (1)$	5	12	25
8,000	(1)	2	4	11	23	
16,000	3	5	10	17	30	
32,000	4	7	14	20	30	
64,000	5	10	16	30		
128,000	8	13	18	30		
256,000	10	14	22	36		

This may not be satisfied, for example, under conditions of low temperature and high density where recombination may be so rapid that the excited levels have insufficient time to reach their quasi steady state populations before the density of the bare nuclei changes by a large fraction.

5. The method of solution

The calculation of the quasi steady state solution requires the solution of an infinite set of equations, but fortunately the treatment of an infinite matrix may be avoided because at high quantum number levels the population density is sufficiently well described by the Saha-Boltzmann equation

$$n_E(p) = \frac{n(c)^2}{X} p^2 \left\{\frac{h^2}{2\pi m k T}\right\}^{3/2} \exp\left\{\frac{E(p,c)}{kT}\right\} \tag{13}$$

where $n_E(p)$ is the Saha-Boltzmann population density of level p, T is the electron temperature and E(p,c) the ionisation potential of level p.

The upper levels approach the Saha-Boltzmann population density because the spontaneous radiative decay rate decreases with increasing quantum number while the collisional rates increase. When the collisional processes between the bound levels themselves and the continuum dominate the radiative processes the population density has the Saha-Boltzmann value. It is assumed that level 20 and above have a Saha-Boltzmann population density. Level 20 was chosen because this is the limit of the tables of oscillator strength of Green, Rush and Chandler, but for many conditions this is much higher than is necessary.

The contribution of these high levels to the population density of levels 19 and below was included by using the asymptotic value of the oscillator strength to calculate the rate coefficients. The calculation assumed that the high levels all remained discrete and successive levels were included until the contribution of the last level was less than 0.1% of the total for the levels already calculated.

The microfields due to neighbouring electrons and ions cause the uppermost bound levels to lose their discrete nature and they merge into the continuum of free electrons. The level at which this takes place represents a possible cut off point for the calculations to avoid the difficulties associated with an infinite number of levels. However, it is difficult to calculate precisely where this occurs and fortunately it is not necessary since the contribution of the continuum which replaces the bound levels is exactly the same as the discrete bound levels. In the calculations the assumption that level 20 and above had a Saha-Boltzmann population density was checked by selected calculations assuming that level 16 and above had a Saha-Boltzmann population density. For the lowest reduced electron density 10^8 cm^{-3} the differences were never more than a few per cent. At all other densities the differences were quite negligible.

The quasi steady state solution of the population densities of the excited levels is a function of the ground level population density for a given electron density and temperature. But the equations (5) are linear in n(1) so that the solution for any excited level population density may be expressed in terms of two coefficients. The first is the population density of a given excited level when the ground level population density is zero, and this represents the contribution to the population density from the continuum of free electrons. The second coefficient is the increase in the excited level for a unit increase in the ground level, and this represents the contribution to the population density of the excited level by excitation from the ground level. It is convenient to express all the population densities as a fraction of the Saha-Boltzmann population density for the electron temperature and density involved. Thus the quasi steady state population density of an excited level is given by

$$\frac{n(p)}{n_E(p)} = r_0(p) + r_1(p)\,\frac{n(1)}{n_E(1)} \qquad (14)$$

where n (p) is the quasi steady state population density of level p, n_E(p) is the Saha-Boltzmann population density, and r_0(p) and r_1(p) are the two coefficients.

6. The results

The quasi steady state population densities have been calculated using an IBM 7090 electronic computer for a wide range of conditions. The results are given in terms of the reduced parameters which correspond to a range of $4000Z^2$

to $256000Z^2$ °K in electron temperature and of $10^8 Z^7$ to $10^{18} Z^7$ cm^{-3} in electron density. For each condition, the quasi steady state population densities were calculated for two values of the ground level population density. The first was zero which yields the coefficients $r_0(p)$, and the second was an arbitrary large value to give the coefficients $r_1(p)$.

Table 4 gives the quantum numbers of the lowest levels having $r_0(p)$ within 1%, 3%, 10% and 30% of unity. Since a value of unity corresponds to the Saha-Boltzmann population density, this table illustrates the way in which the Saha-Boltzmann population density extends further down the bound levels from the continuum as the density rises, until at the highest temperature and density quoted all the excited levels are within 3% of the Saha-Boltzmann population density.

Tables 5 and 6 give the coefficients $r_0(p)$ and $r_1(p)$ for six of the lower quantum numbers. $r_1(15)$ is also given since it is a guide to the departure of the upper levels from the Saha-Boltzmann population density.

The values of $r_0(p)$ show clearly the transition from the low density situation typified by the radiative capture-cascade model of Seaton (1959b) to the high density situation where the population densities approach the Saha-Boltzmann value. The coefficients $r_0(p)$ are less than unity for reduced temperatures less than 256,000 °K, but above this they sometimes exceed unity, increasing with increasing temperature. At the reduced temperature of 512,000 °K and a reduced density of 10^8 cm^{-3}, $r_0(2)$ is 1.7 and $r_0(10)$ is 1.2. These figures illustrate that the population densities can be considerably greater than the Saha-Boltzmann value, particularly during the period of ionisation when the contribution to the population density from the ground level can be large. Departures of this nature have been observed experimentally (McWhirter, Griffin and Jones, 1959).

The fall in the calculated values of $r_0(2)$ at reduced densities greater than 10^{16} cm^{-3} for reduced temperatures of 128,000 and 256,000 °K is caused by the adoption of the special values of the rate coefficients K(1,c), K(1,2), K(1,3), K(1,4). This effect is too small to cause concern.

The values of $\gamma_1(p)$ are always less than unity. The limiting values of $\gamma_1(p)$ as the density tends to zero could not be calculated because of the limited range of the computer. For quantum numbers less than 7 extrapolation to a lower density from the values given is satisfactory.

The calculations show that the populations of the upper levels are determined predominantly by a balance between collisional excitation and de-excitation whereas for the lower levels radiative decay and recombination are more important. By defining the collision limit as the lowest level from which an electron has a greater probability of making a transition to an upper level rather than to a lower level, a measure of the level of change-over is obtained. The values of the collision limit are given in Appendix 2.

The population densities of the excited levels for a true steady state, that is when the ground level is in equilibrium, may be calculated from the steady state population density of the ground level $n_s(1)$ using equation (14) and these are given expressed as a ratio of the corresponding Saha-Boltzmann population density in table 7. The steady state population density of an excited level for specified electron and bare nuclei number densities and electron temperature always exceeds the Saha-Boltzmann population density. The ratio between these two is usually closer to unity than the corresponding value of $\gamma_0(p)$, and it falls outside the limits of the ranges in table 4 in the top right hand region marked off by the boundary line.

The accuracy of these calculations depends mainly on the accuracy of the cross sections for the collisional processes, since the radiative rate coefficients are well established. Seaton (1962) estimates that the positive ion cross sections near threshhold are accurate to a factor of two. The slightly modified version used agrees with the Coulomb-Born approximation of Burgess (1961) for 1s-2p within a factor of two, so that the greatest error in the cross section should be less than a factor of three.

Because the equations (5) are linear in the rate coefficients, changing one of them by three cannot alter the calculated population density by more than three. At high levels where the population density is collision dominated the effect on the population density may be very small, and since detailed balancing is used to calculate the rate coefficient of the inverse process the departure from the Saha-Boltzmann population density will not be altered by more than a factor of three.

Table 4. Values of the smallest quantum number p for which $b_0(p)$ is within 1%, 3%, 10% and 30% of unity.

$\Theta \rightarrow$	4,000				8,000				16,000				32,000			
$\eta(c) \downarrow$	1%	3%	10%	30%	1%	3%	10%	30%	1%	3%	10%	30%	1%	3%	10%	30%
10^8	All >15				All >15				All >15				>15,	>15,	>15,	14
10^9	>15,	>15,	>15,	13	>15,	>15,	>15,	13	>15,	>15,	>15,	12	>15,	>15,	15,	11
10^{10}	>15,	15,	12,	10	>15,	15,	12,	10	>15,	15,	12,	9	>15,	15,	12,	9
10^{11}	14,	12,	10,	8	14,	12,	9,	7	14,	11,	9,	7	13,	11,	9,	7
10^{12}	12,	10,	8,	6	11,	9,	7,	6	10,	9,	7,	5	10,	9,	7,	5
10^{13}	10,	8,	7,	5	9,	7,	6,	5	8,	7,	5,	4	8,	6,	5,	4
10^{14}	9,	7,	6,	5	7,	6,	5,	4	7,	5,	4,	3	6,	5,	4,	3
10^{15}	9,	7,	6,	4	6,	5,	4,	3	5,	4,	3,	3	5,	4,	3,	3
10^{16}	9,	7,	6,	4	6,	5,	4,	3	4,	3,	3,	2	4,	3,	2,	2
10^{17}	9,	7,	5,	4	5,	4,	3,	2	4,	3,	2,	2	3,	3,	2,	2
10^{18}	9,	7,	5,	4	5,	4,	3,	2	4,	3,	2,	2	3,	3,	2,	2

Table 4 continued. Values of the smallest quantum number p for which $\gamma_0(p)$ is within 1%, 3%, 10% and 30% of unity.

$\gamma_0(p) \leqslant 1 \leftrightarrow \gamma_0(p) > 0.97$

Ⓗ →	64,000				128,000				256,000			
$\eta(c)$ ↓	1%	3%	10%	30%	1%	3%	10%	30%	1%	3%	10%	30%
10^{8}	>15,	>15,	>15,	9	>15,	>15,	>15,	3	6,	4,	3,	2
10^{9}	>15,	>15,	15,	9	>15,	>15,	12,	3	6,	4,	3,	2
10^{10}	>15,	14,	11,	7	15,	13,	9,	3	6,	4,	3,	2
10^{11}	13,	11,	8,	6	12,	10,	7,	3	6,	4,	3,	2
10^{12}	10,	8,	6,	5	9,	7,	6,	3	5,	4,	3,	2
10^{13}	7,	6,	5,	4	7,	6,	4,	2	4,	3,	3,	2
10^{14}	6,	5,	4,	3	5,	4,	3,	2	4,	3,	2,	2
10^{15}	4,	4,	3,	2	4,	3,	2,	2	3,	2,	2,	2
10^{16}	3,	3,	2,	2	3,	3,	2,	2	3,	2,	2,	2
10^{17}	3,	2,	2,	2	3,	2,	2,	2	3,	2,	2,	2
10^{18}	3,	2,	2,	2	3,	2,	2,	2	3,	2,	2,	2

Table 5. The coefficient $\gamma_0(p)$ $\eta(c) \to 0$

Ⓗ \ p	2	3	4	5	7	10
4,000	4.7^{-6}	1.3^{-3}	1.0^{-2}	2.9^{-2}	7.9^{-2}	1.5^{-1}
8,000	1.1^{-3}	1.9^{-2}	5.9^{-2}	1.0^{-1}	1.8^{-1}	2.5^{-1}
16,000	2.2^{-2}	9.3^{-2}	1.7^{-1}	2.3^{-1}	3.1^{-1}	3.7^{-1}
32,000	1.2^{-1}	2.5^{-1}	3.3^{-1}	3.9^{-1}	4.6^{-1}	5.1^{-1}
64,000	3.1^{-1}	4.6^{-1}	5.3^{-1}	5.7^{-1}	6.2^{-1}	6.5^{-1}
128,000	6.8^{-1}	7.3^{-1}	7.6^{-1}	7.8^{-1}	7.9^{-1}	7.9^{-1}
256,000	1.1	1.0	1.0	9.9^{-1}	9.7^{-1}	9.4^{-1}

Table 5. The coefficient $\gamma_0(p)$ $\eta(c) = 10^8$ cm^{-3}

H \ p	2	3	4	5	7	10
4,000	5.9^{-6}	1.6^{-3}	1.4^{-2}	3.9^{-2}	1.1^{-1}	2.4^{-1}
8,000	1.3^{-3}	2.2^{-2}	6.7^{-2}	1.2^{-1}	2.2^{-1}	3.4^{-1}
16,000	2.4^{-2}	1.0^{-1}	1.8^{-1}	2.5^{-1}	3.5^{-1}	4.6^{-1}
32,000	1.3^{-1}	2.6^{-1}	3.5^{-1}	4.1^{-1}	5.0^{-1}	5.8^{-1}
64,000	3.5^{-1}	4.8^{-1}	5.5^{-1}	6.0^{-1}	6.6^{-1}	7.2^{-1}
128,000	6.9^{-1}	7.4^{-1}	7.8^{-1}	8.0^{-1}	8.3^{-1}	8.6^{-1}
256,000	1.1	1.0	1.0	1.0	1.0	1.0

$\eta(c) = 10^9$ cm^{-3}

H \ p	2	3	4	5	7	10
4,000	6.7^{-6}	1.9^{-3}	1.6^{-2}	4.6^{-2}	1.4^{-1}	4.4^{-1}
8,000	1.3^{-3}	2.4^{-2}	7.3^{-2}	1.3^{-1}	2.5^{-1}	4.9^{-1}
16,000	2.4^{-2}	1.0^{-1}	1.9^{-1}	2.6^{-1}	3.7^{-1}	5.6^{-1}
32,000	1.3^{-1}	2.6^{-1}	3.5^{-1}	4.2^{-1}	5.2^{-1}	6.5^{-1}
64,000	3.6^{-1}	4.8^{-1}	5.6^{-1}	6.1^{-1}	6.7^{-1}	7.5^{-1}
128,000	6.9^{-1}	7.5^{-1}	7.8^{-1}	8.1^{-1}	8.4^{-1}	8.7^{-1}
256,000	1.1	1.0	1.0	1.0	1.0	1.0

Table 5. The coefficient $\gamma_0(p)$ $\eta(c) = 10^{10}\ cm^{-3}$

Θ \ p	2	3	4	5	7	10
4,000	8.8^{-6}	2.5^{-3}	2.2^{-2}	7.0^{-2}	3.2^{-1}	7.7^{-1}
8,000	1.5^{-3}	2.7^{-2}	8.5^{-2}	1.6^{-1}	4.0^{-1}	7.9^{-1}
16,000	2.6^{-2}	1.1^{-1}	2.0^{-1}	2.9^{-1}	4.8^{-1}	8.1^{-1}
32,000	1.3^{-1}	2.7^{-1}	3.7^{-1}	4.4^{-1}	5.9^{-1}	8.3^{-1}
64,000	3.6^{-1}	4.9^{-1}	5.7^{-1}	6.2^{-1}	7.1^{-1}	8.7^{-1}
128,000	6.9^{-1}	7.5^{-1}	7.9^{-1}	8.1^{-1}	8.5^{-1}	9.3^{-1}
256,000	1.1	1.0	1.0	1.0	1.0	1.0

$\eta(c) = 10^{11}\ cm^{-3}$

Θ \ p	2	3	4	5	7	10
4,000	1.5^{-5}	4.5^{-3}	4.8^{-2}	2.1^{-1}	6.8^{-1}	9.3^{-1}
8,000	2.0^{-3}	3.6^{-2}	1.3^{-1}	3.0^{-1}	7.2^{-1}	9.5^{-1}
16,000	2.9^{-2}	1.3^{-1}	2.5^{-1}	4.0^{-1}	7.5^{-1}	9.5^{-1}
32,000	1.4^{-1}	2.9^{-1}	4.0^{-1}	5.2^{-1}	7.9^{-1}	9.6^{-1}
64,000	3.7^{-1}	5.0^{-1}	5.9^{-1}	6.7^{-1}	8.4^{-1}	9.7^{-1}
128,000	6.9^{-1}	7.5^{-1}	8.0^{-1}	8.3^{-1}	9.1^{-1}	9.8^{-1}
256,000	1.1	1.0	1.0	1.0	1.0	1.0

Table 5. Coefficient $r_0(p)$

$\eta(c) = 10^{12}\ cm^{-3}$

Θ \ p	2	3	4	5	6	7
4,000	3.6^{-5}	1.5^{-2}	2.1^{-1}	5.5^{-1}	7.7^{-1}	8.8^{-1}
8,000	3.1^{-3}	6.9^{-2}	3.3^{-1}	6.5^{-1}	8.3^{-1}	9.2^{-1}
16,000	3.6^{-2}	1.8^{-1}	4.3^{-1}	7.1^{-1}	8.6^{-1}	9.3^{-1}
32,000	1.5^{-1}	3.3^{-1}	5.4^{-1}	7.5^{-1}	8.8^{-1}	9.4^{-1}
64,000	3.8^{-1}	5.3^{-1}	6.7^{-1}	8.1^{-1}	9.1^{-1}	9.6^{-1}
128,000	7.1^{-1}	7.7^{-1}	8.3^{-1}	9.0^{-1}	9.5^{-1}	9.7^{-1}
256,000	1.1	1.0	1.0	1.0	1.0	1.0

$\eta(c) = 10^{13}\ cm^{-3}$

Θ \ p	2	3	4	5	6	7
4,000	1.6^{-4}	9.6^{-2}	5.2^{-1}	7.8^{-1}	9.0^{-1}	9.5^{-1}
8,000	7.4^{-3}	2.5^{-1}	6.9^{-1}	8.8^{-1}	9.5^{-1}	9.8^{-1}
16,000	5.7^{-2}	3.8^{-1}	7.6^{-1}	9.1^{-1}	9.6^{-1}	9.8^{-1}
32,000	1.9^{-1}	5.1^{-1}	8.1^{-1}	9.3^{-1}	9.7^{-1}	9.9^{-1}
64,000	4.2^{-1}	6.5^{-1}	8.6^{-1}	9.4^{-1}	9.8^{-1}	9.9^{-1}
128,000	7.3^{-1}	8.1^{-1}	9.2^{-1}	9.7^{-1}	9.9^{-1}	9.9^{-1}
256,000	1.1	1.0	1.0	1.0	1.0	1.0

Table 5. The coefficient $\gamma_0(p)$

$\eta(c) = 10^{14}\ cm^{-3}$

Θ \ p	2	3	4	5	6	7
4,000	1.4^{-3}	3.2^{-1}	6.9^{-1}	8.7^{-1}	9.4^{-1}	9.7^{-1}
8,000	3.4^{-2}	6.1^{-1}	8.7^{-1}	9.5^{-1}	9.8^{-1}	9.9^{-1}
16,000	1.5^{-1}	7.4^{-1}	9.3^{-1}	9.8^{-1}	9.9^{-1}	1.0
32,000	3.1^{-1}	8.1^{-1}	9.5^{-1}	9.8^{-1}	9.9^{-1}	1.0
64,000	5.2^{-1}	8.6^{-1}	9.6^{-1}	9.9^{-1}	1.0	1.0
128,000	7.7^{-1}	9.3^{-1}	9.8^{-1}	9.9^{-1}	1.0	1.0
256,000	1.1	1.0	1.0	1.0	1.0	1.0

$\eta(c) = 10^{15}\ cm^{-3}$

Θ \ p	2	3	4	5	6	7
4,000	1.3^{-2}	4.3^{-1}	7.5^{-1}	8.9^{-1}	9.5^{-1}	9.8^{-1}
8,000	2.1^{-1}	7.7^{-1}	9.3^{-1}	9.8^{-1}	9.9^{-1}	1.0
16,000	5.2^{-1}	9.1^{-1}	9.8^{-1}	9.9^{-1}	1.0	1.0
32,000	6.8^{-1}	9.5^{-1}	9.9^{-1}	1.0	1.0	1.0
64,000	7.8^{-1}	9.7^{-1}	9.9^{-1}	1.0	1.0	1.0
128,000	8.9^{-1}	9.8^{-1}	1.0	1.0	1.0	1.0
256,000	1.0	1.0	1.0	1.0	1.0	1.0

Table 5. The coefficient $\hat{\gamma}_0(p)$

$\eta(c) = 10^{16}\ \mathrm{cm}^{-3}$

Θ \ p	2	3	4	5	6	7
4,000	6.1^{-2}	4.7^{-1}	7.7^{-1}	9.0^{-1}	9.5^{-1}	9.8^{-1}
8,000	6.0^{-1}	8.9^{-1}	9.7^{-1}	9.9^{-1}	1.0	1.0
16,000	8.6^{-1}	9.8^{-1}	9.9^{-1}	1.0	1.0	1.0
32,000	9.3^{-1}	9.9^{-1}	1.0	1.0	1.0	1.0
64,000	9.5^{-1}	9.9^{-1}	1.0	1.0	1.0	1.0
128,000	9.6^{-1}	1.0	1.0	1.0	1.0	1.0
256,000	9.8^{-1}	1.0	1.0	1.0	1.0	1.0

$\eta(c) = 10^{17}\ \mathrm{cm}^{-3}$

Θ \ p	2	3	4	5	6	7
4,000	9.8^{-2}	4.9^{-1}	7.8^{-1}	9.0^{-1}	9.6^{-1}	9.8^{-1}
8,000	7.4^{-1}	9.3^{-1}	9.8^{-1}	9.9^{-1}	1.0	1.0
16,000	9.3^{-1}	9.9^{-1}	1.0	1.0	1.0	1.0
32,000	9.6^{-1}	1.0	1.0	1.0	1.0	1.0
64,000	9.7^{-1}	1.0	1.0	1.0	1.0	1.0
128,000	9.8^{-1}	1.0	1.0	1.0	1.0	1.0
256,000	9.7^{-1}	1.0	1.0	1.0	1.0	1.0

Table 5. The coefficient $\gamma_0(p)$

$\eta(c) = 10^{18}$

H \ p	2	3	4	5	6	7
4,000	1.0^{-1}	5.0^{-1}	7.8^{-1}	9.0^{-1}	9.6^{-1}	9.8^{-1}
8,000	7.6^{-1}	9.4^{-1}	9.8^{-1}	9.9^{-1}	1.0	1.0
16,000	9.4^{-1}	9.9^{-1}	1.0	1.0	1.0	1.0
32,000	9.7^{-1}	1.0	1.0	1.0	1.0	1.0
64,000	9.8^{-1}	1.0	1.0	1.0	1.0	1.0
128,000	9.8^{-1}	1.0	1.0	1.0	1.0	1.0
256,000	9.7^{-1}	1.0	1.0	1.0	1.0	1.0

$\eta(c) \rightarrow \infty$

H \ p	2	3	4	5	6	7
4,000	1.1^{-1}	5.0^{-1}	7.8^{-1}	9.0^{-1}	9.6^{-1}	9.8^{-1}
8,000	7.6^{-1}	9.4^{-1}	9.8^{-1}	9.9^{-1}	1.0	1.0
16,000	9.4^{-1}	9.9^{-1}	1.0	1.0	1.0	1.0
32,000	9.7^{-1}	1.0	1.0	1.0	1.0	1.0
64,000	9.8^{-1}	1.0	1.0	1.0	1.0	1.0
128,000	9.8^{-1}	1.0	1.0	1.0	1.0	1.0
256,000	9.7^{-1}	1.0	1.0	1.0	1.0	1.0

The indices give the power of ten by which the entries in the coefficient columns must be multiplied.

Table 6. The coefficient γ_1 (p)

$\eta(c) = 10^8$ cm^{-3}

H \ p	2	3	4	5	7	10	15
4,000	1.2^{-8}	5.5^{-9}	5.1^{-9}	4.3^{-9}	3.4^{-9}	2.7^{-9}	1.2^{-9}
8,000	8.9^{-9}	4.2^{-9}	3.5^{-9}	3.2^{-9}	2.5^{-9}	2.0^{-9}	9.8^{-10}
16,000	6.8^{-9}	3.2^{-9}	2.6^{-9}	2.3^{-9}	1.8^{-9}	1.4^{-9}	7.6^{-10}
32,000	5.4^{-9}	2.6^{-9}	1.9^{-9}	1.7^{-9}	1.3^{-9}	1.0^{-9}	5.9^{-10}
64,000	4.5^{-9}	2.1^{-9}	1.5^{-9}	1.2^{-9}	9.2^{-10}	7.2^{-10}	4.5^{-10}
128,000	3.9^{-9}	1.8^{-9}	1.2^{-9}	8.5^{-10}	6.5^{-10}	5.1^{-10}	3.3^{-10}
256,000	3.5^{-9}	1.6^{-9}	1.1^{-9}	6.0^{-10}	4.6^{-10}	3.6^{-10}	2.4^{-10}

$\eta(c) = 10^9$ cm^{-3}

H \ p	2	3	4	5	7	10	15
4,000	1.2^{-7}	5.5^{-8}	5.1^{-8}	4.2^{-8}	3.3^{-8}	2.0^{-8}	4.1^{-9}
8,000	8.9^{-8}	4.2^{-8}	3.5^{-8}	3.1^{-8}	2.4^{-8}	1.5^{-8}	3.3^{-9}
16,000	6.8^{-8}	3.2^{-8}	2.5^{-8}	2.3^{-8}	1.7^{-8}	1.2^{-8}	2.8^{-9}
32,000	5.4^{-8}	2.6^{-8}	1.9^{-8}	1.6^{-8}	1.2^{-8}	8.6^{-9}	2.2^{-9}
64,000	4.5^{-8}	2.1^{-8}	1.5^{-8}	1.2^{-8}	8.9^{-9}	6.4^{-9}	1.8^{-9}
128,000	3.9^{-8}	1.8^{-8}	1.2^{-8}	8.4^{-9}	6.3^{-9}	4.6^{-9}	1.4^{-9}
256,000	3.5^{-8}	1.6^{-8}	1.1^{-8}	5.9^{-9}	4.5^{-9}	3.3^{-9}	9.6^{-10}

Table 6. The coefficient $\nu_1(p)$

$\eta(c) = 10^{10}\ \mathrm{cm}^{-3}$

(H) \ p	2	3	4	5	7	10	15
4,000	1.2^{-6}	5.5^{-7}	5.0^{-7}	4.1^{-7}	2.7^{-7}	8.5^{-8}	9.8^{-9}
8,000	8.9^{-7}	4.1^{-7}	3.4^{-7}	3.0^{-7}	2.0^{-7}	6.4^{-8}	7.2^{-9}
16,000	6.8^{-7}	3.2^{-7}	2.5^{-7}	2.2^{-7}	1.5^{-7}	5.1^{-8}	5.6^{-9}
32,000	5.4^{-7}	2.5^{-7}	1.9^{-7}	1.6^{-7}	1.1^{-7}	4.2^{-8}	5.0^{-9}
64,000	4.5^{-7}	2.1^{-7}	1.5^{-7}	1.1^{-7}	7.9^{-8}	3.3^{-8}	3.9^{-9}
128,000	3.9^{-7}	1.8^{-7}	1.2^{-7}	8.1^{-8}	5.8^{-8}	2.5^{-8}	2.8^{-9}
256,000	3.5^{-7}	1.6^{-7}	1.1^{-7}	5.8^{-8}	4.1^{-8}	1.9^{-8}	1.9^{-9}

$\eta(c) = 10^{11}\ \mathrm{cm}^{-3}$

(H) \ p	2	3	4	5	7	10	15
4,000	1.2^{-5}	5.5^{-6}	4.9^{-6}	3.6^{-6}	1.3^{-6}	2.6^{-7}	2.1^{-8}
8,000	8.8^{-6}	4.1^{-6}	3.3^{-6}	2.5^{-6}	9.3^{-7}	1.7^{-7}	1.3^{-8}
16,000	6.8^{-6}	3.2^{-6}	2.4^{-6}	1.9^{-6}	7.1^{-7}	1.4^{-7}	1.0^{-8}
32,000	5.4^{-6}	2.5^{-6}	1.8^{-6}	1.4^{-6}	5.6^{-7}	1.1^{-7}	7.4^{-8}
64,000	4.5^{-6}	2.0^{-6}	1.4^{-6}	1.0^{-6}	4.5^{-7}	9.3^{-8}	6.7^{-9}
128,000	3.9^{-6}	1.7^{-6}	1.2^{-6}	7.4^{-7}	3.5^{-7}	7.2^{-8}	4.6^{-9}
256,000	3.5^{-6}	1.6^{-6}	1.0^{-6}	5.4^{-7}	2.6^{-7}	5.1^{-8}	2.8^{-9}

Table 6. The coefficient $\gamma_1(p)$

$\eta(c) = 10^{12}\ cm^{-3}$

(H) \ p	2	3	4	5	6	7	15
4,000	1.2^{-4}	5.5^{-5}	4.2^{-5}	2.3^{-5}	1.2^{-5}	6.3^{-6}	9.4^{-8}
8,000	8.8^{-5}	4.0^{-5}	2.6^{-5}	1.3^{-5}	6.5^{-6}	3.3^{-6}	4.0^{-8}
16,000	6.7^{-5}	3.0^{-5}	1.8^{-5}	9.3^{-6}	4.4^{-6}	2.1^{-6}	1.5^{-8}
32,000	5.4^{-5}	2.4^{-5}	1.4^{-5}	7.6^{-6}	3.8^{-6}	2.0^{-6}	2.5^{-8}
64,000	4.4^{-5}	1.9^{-5}	1.2^{-5}	5.9^{-6}	2.8^{-6}	1.3^{-6}	7.9^{-9}
128,000	3.8^{-5}	1.7^{-5}	1.0^{-5}	5.1^{-6}	2.6^{-6}	1.4^{-6}	1.4^{-8}
256,000	3.5^{-5}	1.5^{-5}	9.2^{-6}	3.9^{-6}	1.9^{-6}	9.0^{-7}	4.0^{-9}

$\eta(c) = 10^{13}\ cm^{-3}$

(H) \ p	2	3	4	5	6	7	15
4,000	1.2^{-3}	5.7^{-4}	2.9^{-4}	1.3^{-4}	6.2^{-5}	3.0^{-5}	2.6^{-7}
8,000	8.8^{-4}	3.6^{-4}	1.4^{-4}	5.4^{-5}	2.2^{-5}	9.9^{-6}	5.2^{-8}
16,000	6.6^{-4}	2.4^{-4}	8.6^{-5}	3.2^{-5}	1.3^{-5}	5.7^{-6}	3.1^{-8}
32,000	5.2^{-4}	1.8^{-4}	6.4^{-5}	2.3^{-5}	9.4^{-6}	4.1^{-6}	1.9^{-8}
64,000	4.3^{-4}	1.5^{-4}	5.5^{-5}	2.0^{-5}	7.7^{-6}	3.3^{-6}	1.4^{-8}
128,000	3.7^{-4}	1.4^{-4}	5.2^{-5}	1.8^{-5}	7.1^{-6}	3.0^{-6}	1.3^{-8}
256,000	3.4^{-4}	1.3^{-4}	5.2^{-5}	1.8^{-5}	6.9^{-6}	3.0^{-6}	1.0^{-8}

Table 6. The coefficient $\gamma_1(p)$

$\eta(c) = 10^{14}\ \mathrm{cm}^{-3}$

H \ p	2	3	4	5	6	7	15
4,000	1.2^{-2}	6.4^{-3}	2.9^{-3}	1.3^{-3}	6.0^{-4}	3.0^{-4}	3.8^{-6}
8,000	8.5^{-3}	2.6^{-3}	8.4^{-4}	3.1^{-4}	1.3^{-4}	5.9^{-5}	5.3^{-7}
16,000	6.0^{-3}	1.3^{-3}	3.6^{-4}	1.2^{-4}	4.8^{-5}	2.1^{-5}	1.2^{-7}
32,000	4.5^{-3}	8.8^{-4}	2.3^{-4}	7.4^{-5}	2.9^{-5}	1.2^{-5}	7.4^{-8}
64,000	3.6^{-3}	7.1^{-4}	1.8^{-4}	5.8^{-5}	2.2^{-5}	9.7^{-6}	5.9^{-8}
128,000	3.2^{-3}	6.6^{-4}	1.7^{-4}	5.3^{-5}	2.0^{-5}	8.4^{-6}	3.7^{-8}
256,000	2.9^{-3}	6.7^{-4}	1.8^{-4}	5.4^{-5}	2.0^{-5}	8.5^{-6}	3.8^{-8}

$\eta(c) = 10^{15}\ \mathrm{cm}^{-3}$

H \ p	2	3	4	5	6	7	15
4,000	1.1^{-1}	6.0^{-2}	2.7^{-2}	1.2^{-2}	5.6^{-3}	2.8^{-3}	4.6^{-5}
8,000	6.4^{-2}	1.8^{-2}	5.5^{-3}	2.0^{-3}	8.3^{-4}	3.8^{-4}	4.4^{-6}
16,000	3.2^{-2}	5.8^{-3}	1.5^{-3}	4.9^{-4}	1.9^{-4}	8.5^{-5}	7.4^{-7}
32,000	2.0^{-2}	2.9^{-3}	6.8^{-4}	2.2^{-4}	8.3^{-5}	3.6^{-5}	2.6^{-7}
64,000	1.5^{-2}	2.1^{-3}	4.8^{-4}	1.5^{-4}	5.6^{-5}	2.4^{-5}	1.4^{-7}
128,000	1.4^{-2}	1.9^{-3}	4.4^{-4}	1.3^{-4}	5.0^{-5}	2.1^{-5}	1.2^{-7}
256,000	1.4^{-2}	2.0^{-3}	4.5^{-4}	1.4^{-4}	5.2^{-5}	2.2^{-5}	1.3^{-7}

Table 6. The coefficient $\gamma_1(p)$

$\eta(c) = 10^{16}$ cm^{-3}

Θ \ p	2	3	4	5	6	7	15
4,000	5.1^{-1}	2.9^{-1}	1.3^{-1}	5.8^{-2}	2.7^{-2}	1.4^{-2}	2.4^{-4}
8,000	1.9^{-1}	5.1^{-2}	1.6^{-2}	5.7^{-3}	2.4^{-3}	1.1^{-3}	1.5^{-5}
16,000	5.7^{-2}	9.9^{-3}	2.5^{-3}	8.4^{-4}	3.3^{-4}	1.5^{-4}	1.4^{-6}
32,000	3.0^{-2}	4.2^{-3}	9.8^{-4}	3.1^{-4}	1.2^{-4}	5.1^{-5}	3.7^{-7}
64,000	2.3^{-2}	2.9^{-3}	6.7^{-4}	2.1^{-4}	7.8^{-5}	3.4^{-5}	2.4^{-7}
128,000	2.2^{-2}	2.7^{-3}	6.1^{-4}	1.9^{-4}	7.1^{-5}	3.1^{-5}	2.2^{-7}
256,000	2.4^{-2}	3.0^{-3}	6.8^{-4}	2.1^{-4}	7.7^{-5}	3.3^{-5}	2.4^{-7}

$\eta(c) = 10^{17}$ cm^{-3}

Θ \ p	2	3	4	5	6	7	15
4,000	8.3^{-1}	4.7^{-1}	2.1^{-1}	9.4^{-2}	4.4^{-2}	2.2^{-2}	4.0^{-4}
8,000	2.3^{-1}	6.3^{-2}	1.9^{-2}	7.1^{-3}	2.9^{-3}	1.4^{-3}	1.8^{-5}
16,000	6.2^{-2}	1.1^{-2}	2.7^{-3}	9.0^{-4}	3.6^{-4}	1.6^{-4}	1.6^{-6}
32,000	3.1^{-2}	4.4^{-3}	1.0^{-3}	3.2^{-4}	1.2^{-4}	5.4^{-5}	3.9^{-7}
64,000	2.4^{-2}	3.1^{-3}	6.9^{-4}	2.2^{-4}	8.1^{-5}	3.5^{-5}	2.5^{-7}
128,000	2.3^{-2}	2.9^{-3}	6.4^{-4}	2.0^{-4}	7.4^{-5}	3.2^{-5}	2.3^{-7}
256,000	2.6^{-2}	3.2^{-3}	7.1^{-4}	2.2^{-4}	8.1^{-5}	3.5^{-5}	2.5^{-7}

Table 6. The coefficient $\gamma_1(p)$

$\eta(c) = 10^{18}\ cm^{-3}$

H \ p	2	3	4	5	6	7	15
4,000	8.9^{-1}	5.0^{-1}	2.2^{-1}	1.0^{-1}	4.7^{-2}	2.4^{-2}	4.3^{-4}
8,000	2.4^{-1}	6.5^{-2}	2.0^{-2}	7.2^{-3}	3.0^{-3}	1.4^{-3}	1.8^{-5}
16,000	6.2^{-2}	1.1^{-2}	2.8^{-3}	9.1^{-4}	3.6^{-4}	1.6^{-4}	1.6^{-6}
32,000	3.1^{-2}	4.4^{-3}	1.0^{-3}	3.3^{-4}	1.2^{-4}	5.4^{-5}	3.9^{-7}
64,000	2.4^{-2}	3.1^{-3}	7.0^{-4}	2.2^{-4}	8.2^{-5}	3.5^{-5}	2.5^{-7}
128,000	2.3^{-2}	2.9^{-3}	6.5^{-4}	2.0^{-4}	7.5^{-5}	3.2^{-5}	2.3^{-7}
256,000	2.6^{-2}	3.2^{-3}	7.2^{-4}	2.2^{-4}	8.2^{-5}	3.5^{-5}	2.5^{-7}

$\eta(c) \rightarrow \infty$

H \ p	2	3	4	5	6	7	15
4,000	8.9^{-1}	5.1^{-1}	2.2^{-1}	1.0^{-1}	4.7^{-2}	2.4^{-2}	4.3^{-4}
8,000	2.4^{-1}	6.5^{-2}	2.0^{-2}	7.3^{-3}	3.0^{-3}	1.4^{-3}	1.8^{-5}
16,000	6.3^{-2}	1.1^{-2}	2.8^{-3}	9.1^{-4}	3.6^{-4}	1.6^{-4}	1.6^{-6}
32,000	3.1^{-2}	4.4^{-3}	1.0^{-3}	3.3^{-4}	1.2^{-4}	5.4^{-5}	3.9^{-7}
64,000	2.4^{-2}	3.1^{-3}	7.0^{-4}	2.2^{-4}	8.2^{-5}	3.5^{-5}	2.5^{-7}
128,000	2.3^{-2}	2.9^{-3}	6.5^{-4}	2.0^{-4}	7.5^{-5}	3.2^{-5}	2.3^{-7}
256,000	2.6^{-2}	3.2^{-3}	7.2^{-4}	2.2^{-4}	8.2^{-5}	3.5^{-5}	2.5^{-7}

The indices give the power of ten by which the entries in the coefficient columns must be multiplied.

Table 7. $n_S(1)/n_E(1)$

$\eta(c)$ \ H	4,000	8,000	16,000	32,000	64,000	128,000	256,000
$\eta(c)\to 0$	$3.6\ ^{16}/\eta(c)$	$4.4\ ^{16}/\eta(c)$	$5.5\ ^{16}/\eta(c)$	$7.3\ ^{16}/\eta(c)$	$9.7\ ^{16}/\eta(c)$	$1.3\ ^{17}/\eta(c)$	$1.9\ ^{17}/\eta(c)$
10^{8}	3.6^{8}	4.4^{8}	5.5^{8}	7.3^{8}	9.7^{8}	1.3^{9}	1.9^{9}
10^{9}	3.6^{7}	4.4^{7}	5.5^{7}	7.2^{7}	9.7^{7}	1.3^{8}	1.9^{8}
10^{10}	3.3^{6}	4.2^{6}	5.4^{6}	7.1^{6}	9.6^{6}	1.3^{7}	1.9^{7}
10^{11}	2.9^{5}	3.9^{5}	5.1^{5}	6.8^{5}	9.3^{5}	1.3^{6}	1.9^{6}
10^{12}	2.4^{4}	3.4^{4}	4.6^{4}	6.4^{4}	8.8^{4}	1.2^{5}	1.8^{5}
10^{13}	1.8^{3}	2.6^{3}	3.8^{3}	5.4^{3}	7.7^{3}	1.1^{4}	1.6^{4}
10^{14}	1.1^{2}	1.7^{2}	2.5^{2}	3.8^{2}	5.7^{2}	8.7^{2}	1.3^{3}
10^{15}	9.8	1.3^{1}	1.8^{1}	2.6^{1}	3.8^{1}	5.8^{1}	8.9^{1}
10^{16}	1.8	2.2	2.6	3.2	4.2	5.7	7.9
10^{17}	1.1	1.1	1.2	1.2	1.3	1.5	1.7
10^{18}	1.0	1.0	1.0	1.0	1.0	1.0	1.1
$\eta(c)\to\infty$	1.0	1.0	1.0	1.0	1.0	1.0	1.0

7. The radiated power

In the calculation of the population densities, two radiative processes were included, spontaneous radiative decay giving line radiation and radiative recombination giving a continuum. To these processes which cause a radiative power loss a third, bremsstrahlung is added. This radiation is produced by electrons making a free-free transition in the field of the bare nuclei of charge Z, and this process does not affect the population densities of the bound levels. The radiation produced by the transitions of the second electron in the field of a hydrogen-like ion is not included.

The power lost by line radiation is

$$P_{line} = \sum_{p,q} n(p)\, A(p,q)\, E(p,q) \text{ ergs cm}^{-3}\text{ sec}^{-1} \qquad (15)$$

where $E(p,q)$ is the energy difference between levels p and q in ergs.

The power lost by radiative recombination is given by

$$P_{recomb} = \frac{n(c)^2}{X} 2.853 \times 10^{-27} Z^2 T^{1/2} \left\{ -0.0713 + 0.5 \log_e \lambda + 0.640 \lambda^{-1/3} \right\}$$

$$+ \sum_{p} \left[\frac{n(c)^2}{X} 2.177 \times 10^{-11} Z^2 p^{-2} \beta(p) \right] \text{ergs cm}^{-3} \text{sec}^{-1} \qquad (16)$$

where $\lambda = \frac{1.579 \times 10^5 Z^2}{T}$ when T is in $^\circ$K

The first term in this expression is the kinetic energy lost by the free electrons (Seaton 1959a), and the second term is the potential energy lost by the electrons.

The power lost by bremsstrahlung is given by

$$P_{brem} = \frac{n(c)^2}{X} 1.420 \times 10^{-27} Z^2 T^{1/2} \text{ ergs cm}^{-3} \text{sec}^{-1} \qquad (17)$$

when T is in $^\circ$K. In the range of interest, this is a comparatively small contribution and the Kramers-Gaunt correction is neglected.

The power radiated by radiative recombination and bremsstrahlung depend on the electron density, while the line radiation depends on the excited level population densities. But since these are involved linearly, the total power radiated may be expressed in the form

$$P = P_o + P_1 n(1) \qquad (18)$$

where $n(1)$ is the population density of the ground level. The radiative recombination and bremsstrahlung contribute only to P_o.

The two coefficients have scaling laws for Z, and table 8 gives $X P_o Z^{-17}$ ergs/cc/sec and table 9 $P_1 Z^{-6}$ ergs/sec for the range of conditions studied.

Table 8. The coefficient XP_0/Z^{17} in ergs cm^{-3} sec^{-1}

$\eta(c)$ \ $\Theta(H)$	4,000	8,000	16,000	32,000	64,000	128,000	256,000
10^8	2.0^{-7}	1.2^{-7}	7.2^{-8}	4.6^{-8}	3.2^{-8}	3.4^{-8}	2.0^{-8}
10^9	2.2^{-5}	1.2^{-5}	7.3^{-6}	4.6^{-6}	3.2^{-6}	2.4^{-6}	2.0^{-6}
10^{10}	2.9^{-3}	1.4^{-3}	7.7^{-4}	4.7^{-4}	3.2^{-4}	2.4^{-4}	2.0^{-4}
10^{11}	4.7^{-1}	1.7^{-1}	8.4^{-2}	4.9^{-2}	3.2^{-2}	2.4^{-2}	2.0^{-2}
10^{12}	1.1^{+2}	2.5^{+1}	1.0^{+1}	5.2^{0}	3.3^{0}	2.4^{0}	2.0^{0}
10^{13}	3.7^{4}	5.1^{3}	1.4^{3}	5.9^{2}	3.4^{2}	2.4^{2}	2.0^{2}
10^{14}	1.9^{7}	1.4^{6}	2.3^{5}	7.4^{4}	3.7^{4}	2.4^{4}	2.0^{4}
10^{15}	1.3^{10}	5.9^{8}	5.2^{7}	1.0^{7}	4.1^{6}	2.5^{6}	2.0^{6}
10^{16}	5.9^{12}	1.6^{11}	7.8^{9}	1.2^{9}	4.3^{8}	2.5^{8}	2.0^{8}
10^{17}	9.6^{14}	1.9^{13}	8.3^{11}	1.3^{11}	4.4^{10}	2.5^{10}	2.0^{10}
10^{18}	1.0^{17}	1.9^{15}	8.3^{13}	1.3^{13}	4.4^{12}	2.5^{12}	2.0^{12}

The indices give the power of ten by which the entries in the coefficient columns must be multiplied.

Table 9. The coefficient P_1/Z^6 in ergs sec^{-1}

$\eta(c)$ \ $\Theta(H)$	4,000	8,000	16,000	32,000	64,000	128,000	256,000
10^8	5.2^{-23}	1.0^{-16}	1.3^{-13}	4.6^{-12}	2.6^{-11}	5.7^{-11}	8.3^{-11}
10^9	5.2^{-22}	1.0^{-15}	1.3^{-12}	4.6^{-11}	2.6^{-10}	5.7^{-10}	8.3^{-10}
10^{10}	5.2^{-21}	1.0^{-14}	1.3^{-11}	4.6^{-10}	2.5^{-9}	5.7^{-9}	8.2^{-9}
10^{11}	5.2^{-20}	1.0^{-13}	1.3^{-10}	4.6^{-9}	2.5^{-8}	5.6^{-8}	8.1^{-8}
10^{12}	5.2^{-19}	1.0^{-12}	1.3^{-9}	4.5^{-8}	2.4^{-7}	5.5^{-7}	8.0^{-7}
10^{13}	5.2^{-18}	1.0^{-11}	1.3^{-8}	4.2^{-7}	2.3^{-6}	5.1^{-6}	7.5^{-6}
10^{14}	5.2^{-17}	9.9^{-11}	1.2^{-7}	3.5^{-6}	1.8^{-5}	4.1^{-5}	6.1^{-5}
10^{15}	4.6^{-16}	7.5^{-10}	6.2^{-7}	1.5^{-5}	7.6^{-5}	1.8^{-4}	2.9^{-4}
10^{16}	2.2^{-15}	2.2^{-9}	1.1^{-6}	2.3^{-5}	1.1^{-4}	2.8^{-4}	4.9^{-4}
10^{17}	3.6^{-15}	2.7^{-9}	1.2^{-6}	2.4^{-5}	1.2^{-4}	2.9^{-4}	5.3^{-4}
10^{18}	3.9^{-15}	2.8^{-9}	1.2^{-6}	2.4^{-5}	1.2^{-4}	2.9^{-4}	5.3^{-4}

The indices give the power of ten by which the entries in the coefficient columns must be multiplied.

In the low density limit P_0 shows a quadratic dependence on the electron density. This is because the radiative recombination and bremsstrahlung are both proportional to the product of the densities of electrons and bare nuclei, and since the coefficient $\gamma_0(p)$ tends to a constant value at low densities and the Saha-Boltzmann population density is proportional to $n(c)^2$, the contribution of the line radiation has the same variation. In the high density limit the coefficient again shows a quadratic dependence on the electron density, since the excited levels have a Saha-Boltzmann population density. Except by chance, the constants of proportionality are not the same.

The dependence of P_o at low densities on the electron temperature is small because the radiative recombination rate coefficient is not sensitive to the temperature. The main contribution to the variation with temperature is that on average each photon produced during recombination at a high temperature carries a correspondingly larger energy. The large power loss at high density and low temperature reflects the large recombination rates for these conditions (Paper 1 table 6).

The P_1 coefficients depend on the excitation of line radiation by electron collisions with the ground level. At low densities, they have a linear dependence on the electron density but as stepwise excitation becomes important, they become independent of electron density. The steep temperature dependence is governed by the dependence of the excitation rate coefficients $K(1,p)$ on the temperature.

8. The energy of ionisation

In a plasma where ionisation is occurring, energy has to be provided by the free electrons not only for the actual ionisation but also for the power lost by radiation. So the total energy needed for each ionisation is given by

$$\frac{\text{Power absorbed from the free electrons}}{\text{net ionisation rate}} =$$

$$\frac{Z^2\left[I_H\,\eta(c)\left\{\eta(1)Z^3S-\eta(c)\alpha/Z\right\}+X\,P_0/Z^{17}+\eta(1)P_1/Z^6\right]}{\eta(1)\,\eta(c)Z^3S-\eta(c)\alpha/Z} \qquad (19)$$

where I_H is the hydrogen ionisation potential in ergs and S and α are the collisional radiative ionisation and recombination coefficients described in paper 1. The energy transferred in elastic collisions between the electrons and ions, and the energy of 3/2 kT required to give the newly released electron the mean kinetic energy of the others are neglected.

When the reduced electron temperature is greater than 16,000°K the ionisation rate in most practical cases greatly exceeds the recombination rate. The terms $\eta(c)\frac{\alpha}{Z}$ and $\frac{XP_0}{Z^{17}}$ are then negligible compared with the others and the energy of ionisation may be written

$$\text{Energy/ionisation} = \frac{Z^2\left[I_H Z^3 S + P_1 Z^{-6}\eta(c)\right]}{Z^3 S} \qquad (20)$$

so that the energy per ionisation divided by Z^2 is a function of the reduced electron density $\eta(c)$ and the reduced electron temperature Θ only, and the results are shown in figure 1.

At low temperatures and densities excitation followed by radiative decay is much more probable than stepwise excitation leading to ionisation, so the energy required for each ionisation is large. At high temperatures the energy of ionisation rises again because the excitation rate coefficients particularly from level 2 to the upper excited levels fall with increasing temperature. This is illustrated in the curve for $\eta(c)$ of 10^{16} cm^{-3}. At high densities, radiative processes are dominated by collision processes and the curves tend towards the simple ionisation energy of 13.6 ev. At low densities, stepwise excitation is negligible and all excited ions decay radiatively. In the low density limit, the energy required for each ionisation is

$$\frac{\eta(1)\eta(c)Z^2 I_H\left\{Z^3K(1,c) + \sum_p Z^3K(1,p)\left[1-1/p^2\right]\right\}}{\eta(1)\eta(c)Z^3K(1,c)} \qquad (21)$$

This is independent of $\eta(c)$ and $\eta(1)$ and the approach to this low density limit is shown in the results.

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Appendix 1.

The first nine integers raised to the various powers appearing in the Z scaling laws.

Z	Z^6	Z^7	Z^{11}	Z^{17}
2	6.40^{1}	1.28^{2}	2.05^{3}	1.31^{5}
3	7.29^{2}	2.19^{3}	1.77^{5}	1.29^{8}
4	4.10^{3}	1.64^{4}	4.19^{6}	1.72^{10}
5	1.56^{4}	7.81^{4}	4.88^{7}	7.63^{11}
6	4.67^{4}	2.80^{5}	3.63^{8}	1.69^{13}
7	1.18^{5}	8.23^{5}	1.98^{9}	2.33^{14}
8	2.62^{5}	2.10^{6}	8.59^{9}	2.25^{15}
9	5.31^{5}	4.78^{6}	3.14^{10}	1.67^{16}

Appendix 2. The collision limits.

$\eta(C)$ \ (H)	4,000	8,000	16,000	32,000	64,000	128,000	256,000
10^8	14	15	15	16	17	17	18
10^9	11	11	11	12	12	13	14
10^{10}	8	8	9	9	9	10	10
10^{11}	6	6	6	7	7	7	8
10^{12}	5	5	5	5	5	5	6
10^{13}	4	4	4	4	4	4	4
10^{14}	4	3	3	3	3	3	3
10^{15}	3	3	2	2	2	2	2
10^{16}	3	2	2	2	2	2	2
10^{17}	3	2	2	2	2	2	2
10^{18}	3	2	2	2	2	2	2

CLM- P23 Fig. 1 The energy in electron volts per ionisation divided by Z^2 plotted against the reduced electron temperature Θ for various values of the reduced electron density η(c).