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THE CRITERIA FOR THE OPTICALLY THIN APPROXIMATION IN SPECTRAL LINE INTENSITIES

A. G. HEARN

Culham Laboratory,

Culham, Abingdon, Berkshire

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THE CRITERIA FOR THE OPTICALLY THIN APPROXIMATION IN SPECTRAL LINE INTENSITIES

by

A. G. HEARN

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A B S T R A C T

In the interpretation of many spectroscopic measurements of plasmas, it is assumed that the plasma is optically thin for the radiation concerned. Criteria for the validity of this assumption are developed for the intensities of doppler broadened resonance lines using a model of the diffusion of resonance photons in frequency space. For a high electron density plasma where the excited level population is determined by collision processes and not radiative processes, the equation of transfer of radiation has a simple solution and the optical depth must be less than 0.3 if the total intensity of the line is to be proportional to the number of emitting atoms to within 10%. In a low electron density plasma where spontaneous transitions dominate electron de-excitation, the maximum optical depth can be much greater for a resonance line such as Lyman α where the resonance line is the only possible radiative transition. For a resonance line such as Lyman β, where there is a comparatively strong competing radiative transition Balmer α for which the plasma is optically thin, the effect of optical depth is much more severe.

U.K.A.E.A. Research Group,
Culham Laboratory,
Nr. Abingdon,
Berks.

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In the interpretation of many spectroscopic measurements of plasmas, it is assumed that the plasma is optically thin for the radiation concerned. This means that the intensity is assumed to be linearly proportional to the number of emitting atoms in the line of sight. This paper considers at what optical thickness this assumption breaks down in the interpretation of the intensity of a resonance line which is predominantly doppler broadened, and this is illustrated by reference to Lyman α and β resonance lines.

In high electron density plasmas, where the populations of the excited levels of the atom are determined by the collision processes and not by the radiative transfer of the resonance radiation, the intensity of the line may be calculated easily from the equation of transfer of radiation

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + \frac{J_\nu}{\chi_\nu} \qquad (1)$$

where I_ν is the intensity at the frequency ν as a function of the optical depth τ_ν, J_ν is the emission coefficient and χ_ν is the absorption coefficient. For a uniform plasma the ratio J_ν / χ_ν is independent of the optical depth and the equation of transfer has the simple solution

$$I_\nu = \frac{J_\nu}{\chi_\nu}\left(1 - e^{-\tau_\nu}\right) \qquad (2)$$

For a doppler broadened line

$$\tau_\nu = \tau e^{-(\nu-\nu_0)^2/\Delta\nu_D^2} \qquad (3)$$

where τ is the optical depth at the central frequency ν_0 of the absorption line and $\Delta\nu_D$ is the e^{-1} frequency of the gaussian profile. For a high density plasma the ratio J_ν / χ_ν is independent of frequency, so that expanding equation (2), substituting equation (3) and integrating gives the total intensity of the line as

$$\int_0^\infty I_\nu \, d\nu = \Delta\nu_D \sqrt{\pi} \, \frac{J_\nu}{\chi_\nu} \left(\tau - \frac{\tau^2}{2!\sqrt{2}} + \frac{\tau^3}{3!\sqrt{3}} - \cdots \right) \tag{4}$$

(Burton and Wilson, 1961). This shows that for very small optical depths, the intensity is linearly proportional to the optical depth, that is to the total number of atoms, but when the optical depth at the centre of the line is 0.3 the total intensity has already departed from this linear relation by 10%. So for a high electron density plasma the optical depth must be restricted to less than 0.3 if the optically thin approximation is not to be wrong by more than 10%.

In low density plasmas, when the collisional de-excitation rate is much less than the spontaneous emission rate, the population of an excited level is determined by both the collision processes and the radiative transfer processes and there is no longer a Boltzmann distribution. Attempts to treat the radiative transfer of a doppler broadened resonance line as simple diffusion have failed because of the difficulty of defining a mean free path. It has been suggested by Zanstra[(2)] and Osterbrock[(3)] that for a doppler broadened line, the change in frequency upon reëmission of the photon provides a more important escape mechanism than diffusion in space, since if a photon is emitted in the wings of the line where the optical depth of the plasma is small it will escape.

Assume that a photon will escape if it is emitted at a frequency greater than x_1 from the centre of the line, and it is absorbed if it is emitted at a frequency less than x_1 where the dimensionless frequency x is defined by

$$x = \frac{\nu - \nu_0}{\Delta\nu_D} \tag{5}$$

If the optical depth at the centre of the line from the edge to the centre of the plasma is τ_0 then at the frequency x_1 the optical depth τ_1 is

$$\tau_1 = \tau_0 e^{-x_1^2} \tag{6}$$

With this assumption, if the distribution of the photons upon reëmission is known as a function of frequency, the probability q that a photon will be emitted at a frequency greater than x_1 and hence escape, may be calculated.

An electron in an excited level of an atom may leave it by a spontaneous transition to the ground level emitting a resonance photon, but there are other competing processes such as collisional de-excitation to lower levels, collisional excitation to upper levels, spontaneous transitions to lower levels other than the ground level, and ionisation. Let the probability that the excited atom will emit a resonance photon be b. If an excited atom emits resonance photons, a fraction q of these will escape and 1 - q will be absorbed producing excited atoms. Of these a fraction b will emit resonance photons of which a fraction q will escape. Thus the total fraction W of photons emitted as the result of an original excitation which escapes is therefore

$$\begin{aligned} W &= q + b(1-q)q + b^2(1-q)^2 q + \cdots \\ &= \frac{q}{1-b(1-q)} \end{aligned} \tag{7}$$

If the processes competing with the emission of a resonance photon are small so that b is close to unity, then at moderate optical depths W is also close to unity and all the resonance photons eventually escape. As the optical depth of the plasma increases, the probability q of escape at each emission becomes smaller and so does the fraction W of photons that eventually leave the plasma. While W is effectively unity, the intensity of the resonance radiation is proportional to the total number of atoms and the optically thin approximation is valid. The limiting optical depth for this approximation may be calculated from the value of q necessary to reduce W by say 10% for a given branching factor b. Thus if W is 0.90 equation (7) gives the optically thin criterion to be

$$q \geqslant q_{min} = \frac{0.90(1-b)}{1-0.90b} \tag{8}$$

This is the criterion that must be satisfied if the total intensity of the line is to be given by the optically thin approximation. The line profile may be significantly different from gaussian and at high optical depths the line profile becomes self-reversed(4). To ensure that the line profile is gaussian to within 10% the optical depth may not exceed 0.2.

It is often assumed in radiative transfer calculations that the source function J_ν / χ_ν is independent of frequency. Since for a doppler broadened line, the absorption coefficient χ_ν has a gaussian profile with frequency, this assumption means that the emission coefficient must have a gaussian profile also. This assumption is valid when the plasma is optically thin or in local thermodynamic equilibrium, but in general it is only justified when the collision time between atoms is small compared with the life-time of the excited level, so that the excited atoms are distributed into a Maxwellian distribution. If the source function is assumed to be independent of frequency the probability q that a photon is emitted at a frequency greater than x_1 is

$$q = \frac{2}{\sqrt{\pi}} \int_{x_1}^{\infty} e^{-x^2} dx = 1 - \operatorname{erf} x_1 = \operatorname{erfc} x_1 \tag{9}$$

Assuming that the frequency x_1 is that frequency at which the optical depth

τ_1 is unity gives

$$q = \text{erfc}\left[\left(\log_e \tau_0\right)^{1/2}\right] \quad \text{for } \tau_0 > 1 \qquad (10)$$

The probability q is plotted in figure 1 against $2\tau_0$ the total optical thickness of the plasma measured at the centre of the line for the assumption of a source function constant in frequency.

In a recent paper(4) the integral equations defining the population of the excited level, when it is determined by collisional and radiative processes including photo-excitation derived from the equation of transfer for resonance radiation, were solved numerically for a uniform plane parallel plasma composed of model atoms of two levels assuming that the source function is constant in frequency. In a two level atom where stimulated emission is neglected, the excited level is depopulated by only two processes, spontaneous emission of resonance radiation and collisional de-excitation, so that the branching factor b is given by

$$b = \frac{A_{21}}{n_e Y_{21} + A_{21}} \qquad (11)$$

where A_{21} is the Einstein spontaneous transition probability, Y_{21} is the rate coefficient for electron de-excitation collisions and n_e is the electron density. For a low electron density plasma this is very close to unity. For example using the rate coefficients for the excited level of Lyman α radiation in conditions which occur in the early stages of the ZETA discharge of an electron density of 10^{14} cm^{-3} and an electron temperature of 10^4 ^oK, the branching factor b is 0.99. A comparison of the loss of energy calculation from the solution of the integral equations and the photon diffusion in frequency space shows that for such a low electron density plasma the diffusion model is accurate to a factor of two even for such large optical depths as 10^4. Under these conditions the value of q_{min} derived from equation (8) is 0.0625 which corresponds to an optical depth of 9. So the maximum optical depth for which the optically thin approximation is valid to 10% assuming a constant source function in frequency is 30 times greater than that given by the simple treatment for a high density plasma.

For a line such as Lyman β the calculation is radically affected by the Balmer α radiation for which the plasma may be optically thin. For a three level atom corresponding to the emission of Lyman β and Balmer α radiation, the branching factor b for Lyman β radiation is

$$b = \frac{A_{31}}{n_e Y_{31} + n_e Y_{32} + A_{31} + A_{32}} \tag{14}$$

In a low electron density plasma, this is dominated by the two spontaneous transition probabilities giving a ratio of 0.44. The escape probability corresponding to a 10% deviation from the optically thin approximation is 0.89 from equation (8) which corresponds to an optical depth of less than 2, but because of the way the probability q is defined it has no meaning for a total optical depth of less than 2. The population of the upper excited level is determined predominantly by the collision processes and the loss of the optically thin Balmer α radiation and not by the radiative transfer of the Lyman β radiation, so that the situation resembles the simple high electron density plasma where the optical depth should be restricted to 0.3 to be sure that the optically thin approximation is valid.

The more severe limitation on Lyman β comes because of the alternative paths of radiative decay. At low electron densities the radiation energy still leaves the plasma, but many of the Lyman β photons are degraded to Lyman α and Balmer α photons. The Lyman α photons can only be prevented from leaving the plasma by collisional de-excitation and the photon energy is converted into electron kinetic energy.

At large optical depths the plasma will become optically thick to the

Balmer α radiation, but the optical depth of the Balmer α radiation is determined by the radiative transfer of the Lyman α radiation and this would require the full solution of the radiative transfer of the three level atom.

These calculations show that for resonance radiation such as Lyman α where there is no alternative radiative transition the optically thin intensity approximation can be valid for much higher optical depths than the simple solution of the equation of transfer would suggest. But for resonance radiation such as Lyman β radiation where there is an alternative radiative transition for which the plasma is optically thin, the restriction on the maximum optical depth is much more severe.

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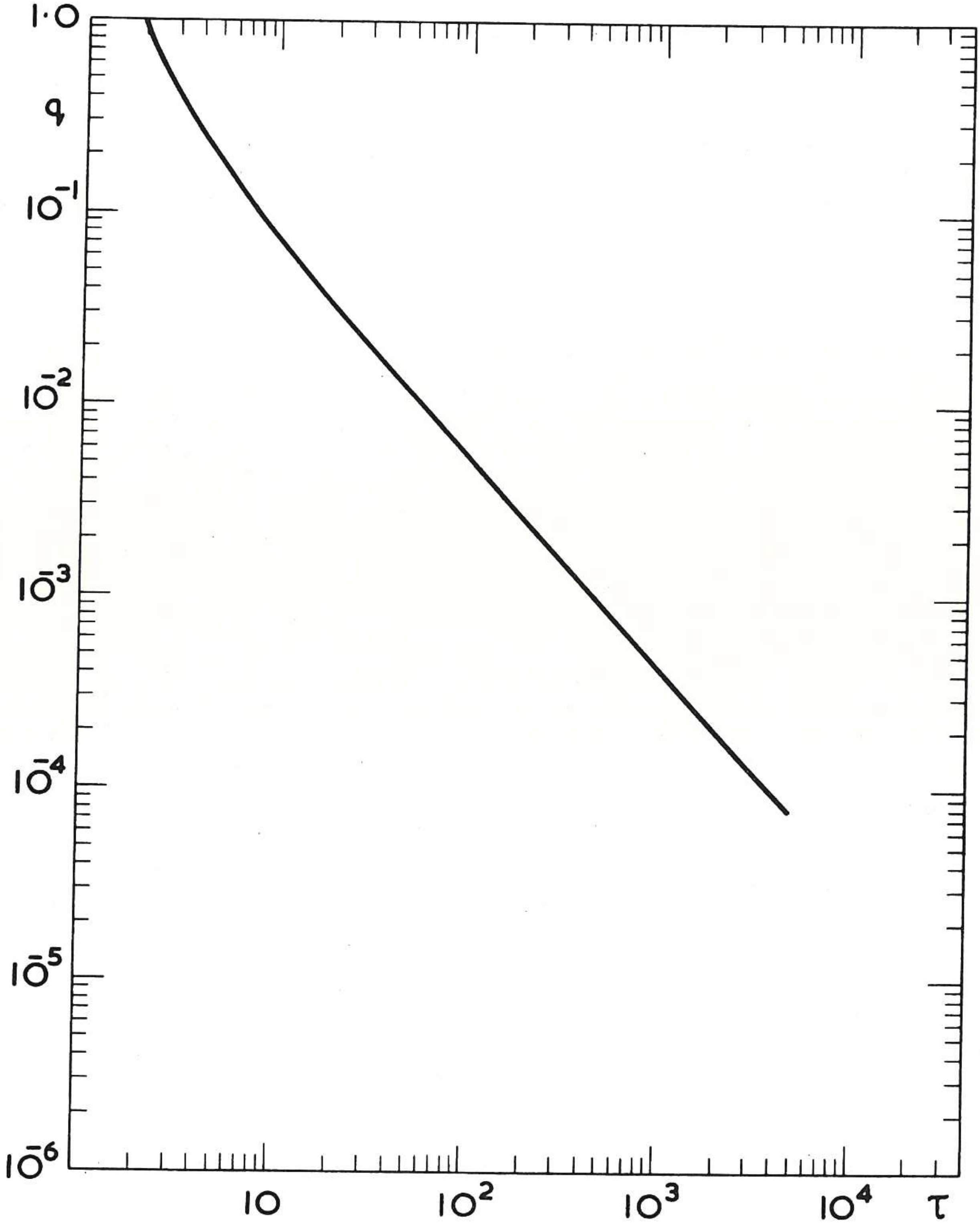

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