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MAGNETIC SURFACES IN A QUADRUPOLE

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MAGNETIC SURFACES IN A QUADRUPOLE

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ABSTRACT

Local imperfections in the magnetic field can alter the topology of the magnetic surfaces, thereby allowing plasma to escape. A modest longitudinal magnetic field will maintain the nested surfaces near the plasma boundary, so that the residual topological changes near the separatrix become unimportant.

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May, 1970.

1. INTRODUCTION

A toroidal multipole is able to confine plasma because the magnetic field has an average-minimum-B configuration. The stability is not much affected by the presence of an axisymmetric toroidal field B_φ as this merely opens out the closed lines of force to produce nested magnetic surfaces (Furth 1968). However, in any real device, local imperfections will destroy the perfect symmetry and so cause the magnetic surfaces to break up and possibly reach the walls. This is known to be a serious problem in the stellarator (Gibson 1967), but there is reason to believe (Morozov and Solov'ev 1966), that the presence of solid conductors threading the field lines will confer some stability on the magnetic configuration. This paper describes the topology of the perturbed magnetic surfaces in a quadrupole and estimates the size of imperfections that should be tolerable in practice.

On the whole, the toroidal curvature does not play an important role, so most of the results which follow were computed for a straight unshielded quadrupole (see figure 1). The closed lines of force in the unperturbed system were labelled by the coordinate z and the z-component of vector potential $A_z \equiv \Psi$. The origin of Ψ was chosen to be the separatrix, and the critical surface, at which $U = \oint d\ell/B$ has a minimum value, is $\Psi = 1$. Ideally, plasma can be contained between $\Psi = 1$ and $\Psi = -3.5$ which represents the surface of the conductors in CLIMAX (Allen et al. 1969). The differential equations describing each field line were integrated by the Runge-Kutta method using automatic step-length adjustment to keep within chosen error estimates, typically 10 μm in 30 cm. The magnetic potential, χ, was was taken as the independent variable so that

$$d\ell = d\chi/B,\ dx = B_x d\chi/B^2,\ dy = B_y d\chi/B^2,\ dz = B_z d\chi/B^2 \ .$$

The range of integration was from 0 to $\mu_0 I$, i.e. for one complete orbit around the conductors.

One important class of local imperfections arises from defects in the conductors within the plasma. For example, as in CLIMAX, the current does not always flow uniformly across the joints between the segments of the conductors. This does not necessarily lead to a change in the topology of the magnetic field, as the current density might retain reflection symmetry across the plane $y = 0$ (figure 1). In the absence of an externally applied longitudinal component B_z, the condition

$$j_x(-y) = j_x(y),\ j_y(-y) = -j_y(y),\ j_z(-y) = j_z(y)$$

means that

$$B_x(-y) = -B_x(y),\ B_y(-y) = B_y(y),\ B_z(-y) = -B_z(y)\ ,$$

hence the field lines must close on themselves, no matter how large the imperfections may be (Taylor 1967). This constitutes a very strong argument for trying to ensure that the apparatus is symmetric across the plane $y = 0$. The topology will be changed, however, if there is a component of current density which is not symmetric in y.

A convenient way of expressing the size of the perturbation is to quote the peak value of the z-component relative to the magnitude of the original field B_0 at the same place. This is the way that measurements were made on CLIMAX and B_1 was found to be greatest close to the conductor surface and at a distance of ± 2 cm from the joint. Before the joints were properly tightened, B_1 was as large as 30% of B_0, but it has now been reduced to less than 3% at all joints.

2. THE ADIABATIC APPROXIMATION

If the perturbation, B_1, is small enough, then the adiabatic approximation (the method of averaging) can be used. This assumes that lines of force follow their original closed orbits except for a slow drift in Ψ and z. As by definition no lines of force may cross a magnetic surface, the flux threading any cross-section intersecting the surface, $\int \underset{\sim}{B}.d\underset{\sim}{S}$, is invariant. By Stokes' theorem this can be rewritten as $\oint \underset{\sim}{A}.d\underset{\sim}{\ell}$ taken around the line of intersection which for convenience is chosen to be the unperturbed orbit. This integral is

$$\alpha(\Psi,z) = \oint \frac{\underset{\sim}{A}_1 \cdot \underset{\sim}{B}_0 d\chi}{B_0^2}$$

because the zero-th order term $\underset{\sim}{A}_0 \cdot \underset{\sim}{B}_0$ vanishes everywhere. Figure 2 shows the trajectories in Ψ,z space of the magnetic surfaces when B_1 is the field of a magnetic dipole centred on one conductor at $z = 0$ with its polar axis in the x direction. The trajectories do not depend on the magnitude of B_1: all that happens as B_1 is increased is that the surfaces are traced out more quickly. The full lines represent the surfaces which encircle only the defective conductor or both, while the broken lines represent those which encircle only the other conductor. Thus the crossing of trajectories when $\Psi < 0$ does not mean that the magnetic surfaces intersect as they are centred on different conductors.

Because the separatrix $\Psi = 0$ has a field zero, the adiabatic approximation must break down in its vicinity. However, the magnetic surfaces are well defined for both positive and negative Ψ, and the region of invalidity shrinks as B_1 is reduced. We may therefore join up the surfaces across the separatrix noting that the surfaces coming in from the shared flux region ($\Psi > 0$) must bifurcate so as to enter each private flux region. This will be correct for most lines of force, as the majority cross $\Psi = 0$ when B_0 is finite. A few lines cross near the field zero and some of these can shoot off along the axis of the machine and emerge later on a different surface.

In the stippled regions in figure 2, the magnetic surfaces are nested tori surrounding the two magnetic axes. The rest of the Ψ,z plane is occupied by a single surface with a rather complicated configuration. It can be thought of as being made up of a series of gloves having a thumb and only one finger, figure 3, one inside the other. The digits of each glove are pulled inside out and the finger sticks out through its own wrist, and ends by merging onto the wrist of a larger sized surrounding glove. The thumb ends while still inside the hand, so it merges onto the wrist of a glove which is smaller than its own. Each wrist is therefore connected to one finger from a smaller glove and one thumb from a larger. Lines of force then progress according to the rules of Snakes and Ladders and, unless they chance upon the right mixture of thumbs and fingers, quickly reach a wall or conductor. Thus, no matter how small B_1 is, only the volume represented by the stippled regions can be used to confine plasma for an indefinite period. In fact, even some of this volume is useless as, unless B_1 is very large, the circulation time around the magnetic surface is longer than the growth time for flutes, so that the plasma whose surface goes outside $\Psi = 1$ is also lost. The time scales will be considered further in Section 5.

3. THE ADDITION OF B_z

One way to trying to suppress the effects of local imperfections is apply a uniform longitudinal field B_z of the same order as B_1. The vector potential can be taken to be $A_x = -yB_z$ and hence the invariant (which is the flux the long way round) is

$$\Phi = \oint \underset{\sim}{A}_1 \cdot \underset{\sim}{B}_0 \, d\chi/B_0^2 - B_z \oint y \, dx$$

$$= \alpha(\Psi,z) - B_z v(\Psi) \; ,$$

where v is volume enclosed per unit length within the orbit Ψ. (It is this simple dependence on B_z which makes it worthwhile to use the adiabatic approximation rather than compute exact trajectories). Figure 4 plots the surfaces in Ψ,z space for $B_z = 2\% \, B_1$. The steady B_z dominates the numerically larger but sign-reversing B_1, but the bifurcation of the surfaces still means that plasma will eventually escape. With $B_z = 6.5\% \, B_1$, (figure 5), closed magnetic surfaces exist both inside and outside the separatrix so that the topology of the surfaces in between becomes irrelevant. The fingers of the glove have now been pulled the right way out again, and, in principle, plasma can be contained along the whole length of the machine.

In the infinite conductivity approximation the plasma is shear stabilised, but because there are buried conductors, we must use the V** criterion (Johnson and Greene 1967) to see whether it is stable against resistive modes. In the expression

$$V^{**} = V'' - V'W''/W'$$

the prime denotes differentiation but it is easily shown that the sign of V** does not depend on the choice of independent variable. Rather than using the flux Φ, it is convenient here to label the surfaces by Ψ_0 which is the value of Ψ at $z = 0$ and $z = \infty$ (where the perturbation $\alpha(\Psi,z) = 0$). The volume enclosed by the surface is

$$V(\Psi_0) = \int v(\Psi(z))dz$$

but, as the effects of the perturbation are felt only locally and because the contributions from positive and negative z tend to cancel, V is dominated by the original $v(\Psi_0)$. In the expansion of

$$W(\Psi_0) = \int B^2 d^3x \ ,$$

the leading term is $\int \mu_0 I d\Psi dz$ which is a linear function of Ψ_0, the second term vanishes as the integrand $\underset{\sim}{B}_0 \cdot (\underset{\sim}{B}_1 + \underset{\sim}{B}_z)$ is antisymmetric in y and the term $\int (\underset{\sim}{B}_1 + \underset{\sim}{B}_z)^2 d^3x$ is negligible in the adiabatic approximation. Hence W''/W' is zero in this approximation, $V^{**} = V''$ and the usual $\int d\ell/B$ criterion is valid.

Thus as long as the surface $\Psi_0 = 1$ does not cross the separatrix, the plasma has an outer boundary. Similarly an inner boundary exists if there is a surface which neither crosses the separatrix nor touches a conductor. These boundaries were shown as the two darker lines in figure 5. Figure 6 shows these limits plotted as a function of B_1/B_z, the minimum value of B_z for confinement being 6.0% B_1. It is still, of course, preferable for B_z/B_1 to be much greater than this limit as the break-down of nesting near $\Psi = 0$ will allow anomalously fast transport of plasma there. Combined with the ordinary diffusion mechanism in the outer regions, this will lead to an enhanced plasma loss rate.

When the perturbation is very small, MHD instabilities will let the plasma fall back to the separatrix before it can drift to the conductors. In this case the limiting value is that for which lines

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When the perturbation is very small, MHD instabilities will let the plasma fall back to the separatrix before it can drift to the conductors. In this case the limiting value is that for which lines of force in the critical surface just graze the separatrix, i.e., $B_z = 2.7\% \; B_1$.

4. HIGHER ORDER TERMS AND TOROIDICITY

The discussion has been restricted so far to a magnetic dipole perturbation as this is the predominant term measured in CLIMAX. The magnitude originally observed is also reasonable, for, assuming the local currents to flow on the surface of a sphere embedded within the conductor, $B_1 = 30\% \; B_0$ is produced when the net current just falls to zero at one point on the conductor's surface. Higher order modes do make small contributions, but in general their effect is small because their range is short compared with the r^{-3} of the dipole. For instance, the hexapole term can be neutralised by $B_z/B_1 = 1.5\%$ or 0.32% depending upon which of the criteria mentioned in the previous section is adopted.

The only exception is the quadrupole perturbation which, despite its r^{-4} dependence, has a large effect because the integrand $\underset{\sim}{A} \cdot d\underset{\sim}{\ell}$ does not change sign while travelling around the defective conductor. The drift trajectories are shown in figure 7, and it can be seen that no absolute confinement regions exist when $B_z = 0$. The motion along the broken lines (orbits encircling only the perfect conductor) is very slow in this case. The minimum value of B_z/B_1 for confinement is now 7.3% or 1.2%, which is comparable with that for the dipole term. Figure 8 shows the effect of increasing B_z to $2\% \; B_1$, by which time the plasma has a proper outer boundary but the separatrix and the conductor surfaces are still confounded.

The toroidal curvature of the machine does not change the general features of the magnetic surfaces as it does little more than alter the weighting of different parts of the orbit by the R^{-1}

factor. Calculations were made for an unshielded toroidal quadrupole with major radii of 65 and 95 cm with the currents adjusted so that the field zero fell midway between the conductors. The effect of a dipole perturbation at the outer conductor is very much like figure 2 with, for example, a pair of magnetic axes in the positive Ψ region at $z = \pm\ 6$ cm. Plasma confinement can be obtained right round the torus using the field from a wire on the major axis, giving a very similar ratio for B_z to B_1, viz. 5.9%.

6. LARGE PERTURBATIONS

When B_1 is no longer small compared with B_0, the drift per orbit becomes significant and the adiabatic approximation breaks down. The trajectories must now be computed directly, but, as was shown in Section 3, the important feature is whether closed magnetic surfaces exist between the separatrix and the surface of the conductors. The computations show that plasma can always be contained if the longitudinal field is large enough (see figure 9 for the dipole perturbation). The slope of the curve starts off at B_z/B_1, as given by figure 6, but then the necessary B_z rises more steeply. This is because, as B_1 is increased, the lines of force thrash about wildly from their adiabatic trajectories. However, the worst perturbation now present in CLIMAX, $B_1 = 3\%\ B_0$, lies well within the scope of the adiabatic approximation and its effect can be suppressed using $B_z = 0.17\%\ B_0$, typically 15 G.

The time scales for the drift motion in figure 2 can now be calculated. When B_1 is 3% B_0, 68 orbits are needed to trace out the magnetic surface which just reaches $\Psi = 1$. For 5 eV electrons this corresponds to 50 μs compared with the observed plasma lifetime of 2 ms. The drift velocities on the open surfaces are larger than this, e.g. electrons on the critical surface at $z = 0$ can crash down onto the conductor in only 4 μs.

In Section 1 it was mentioned that symmetric current perturbations leave the lines closed on themselves. When an external B_z is applied, this symmetry is destroyed but, as long as the adiabatic approximation is still valid, the magnetic surfaces are not altered by the perturbation. However, when B_1 is large, the surfaces will be affected, although B_z can always be made large enough to dominate. Thus in the case of a large perturbation having both symmetrical and antisymmetrical components, applying an inadequate B_z might conceivably enhance the loss rate.

Even when the field lines close on themselves, the perturbation can influence the stability of the plasma. For an equilibrium to exist at all

$$\underset{\sim}{\nabla} p \times \underset{\sim}{\nabla} U = 0,$$

i.e. the plasma pressure is constant along each contour of U. The integral U is now a function of z as well as of Ψ. Figure 10 shows these isobars for a magnetic dipole perturbation $B_1 = \pm 10\% \ B_0$ having its axis in the y direction. When the fields oppose in the region between the conductors (figure 10a) the wall of the magnetic well is breached and plasma can drift out along the contours from as far in as $\Psi = 0.65$. When the fields reinforce each other (figure 10b) the rim of the wall is raised locally. This does not in fact help to confine the plasma but all that happens is that the contours bulge inwards near the perturbation. The critical surface remains at $\Psi = 1$ along the length of the machine apart from this small bulge. Thus it is safer for the local current perturbation to reinforce the uniform flow on the side furthest from the other conductor.

6. CONCLUSIONS

It has been shown that the opening of magnetic surfaces by a dipole perturbation can be suppressed by a modest longitudinal magnetic field. It is desirable that the perturbations remain within the scope of the adiabatic approximation ($B_1 \leqslant 10\% \ B_0$) so as to minimise the necessary B_z and to avoid any side effects from symmetrical perturbations. Even though these symmetrical perturbations do not change the topology of the surfaces, they can let plasma escape by making a hole in the side of the magnetic well, but this can always be averted by adding B_z to remove the degeneracy.

7. ACKNOWLEDGEMENTS

I wish to thank my many colleagues in Experimental Division A and Theory Division for their helpful comments and suggestions. In particular, I am grateful to Dr R.J. Bickerton for his critical appraisal of this work and to Dr A.N. Dellis and his group, who, in addition, measured the perturbed field in CLIMAX.

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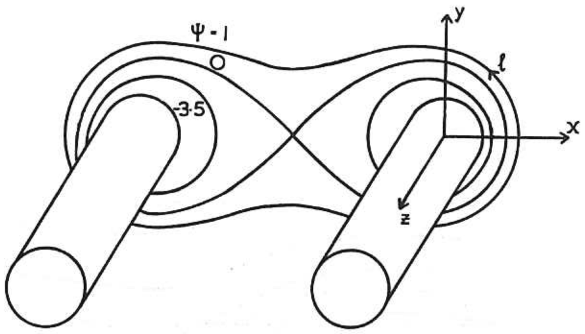

Fig.1 The quadrupole.

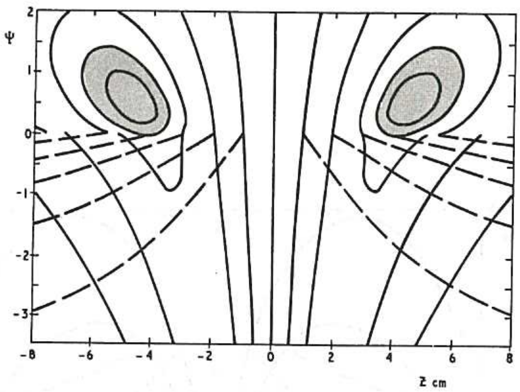

Fig.2 Projection of the magnetic surfaces for a dipole perturbation.

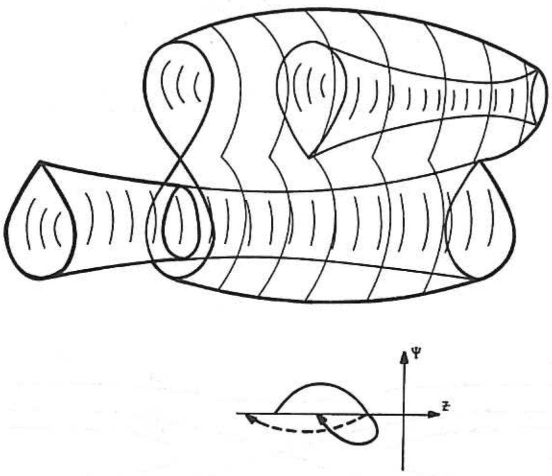

Fig.3 Topology of the magnetic surfaces.

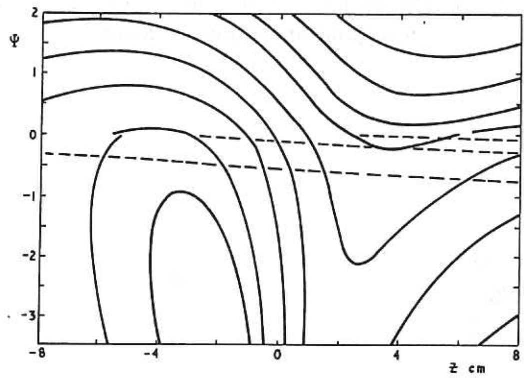

Fig.4 As Fig.2 with the addition of $B_Z = 2\% B_1$.

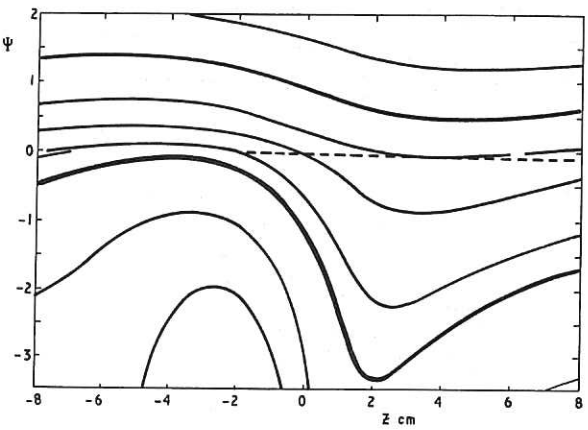

Fig.5 As Fig.2 with the addition of $B_Z = 6.5\% B_1$.

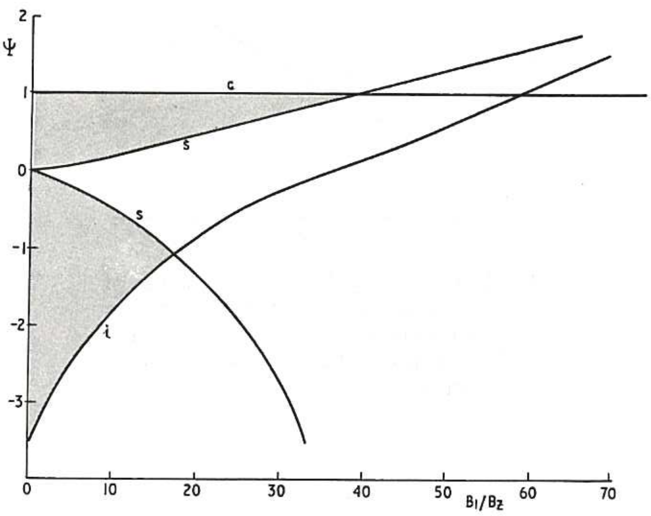

Fig.6 Position of the limiting magnetic surfaces in the presence of a dipole perturbation as a function of B_1/B_Z. The critical surface is marked c, the surface which grazes the conductor is i and those which graze the separatrix are s.

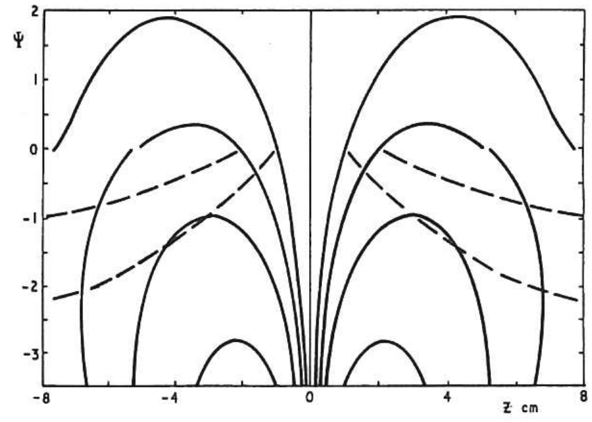

Fig.7 Projection of the magnetic surfaces for a quadrupole perturbation.

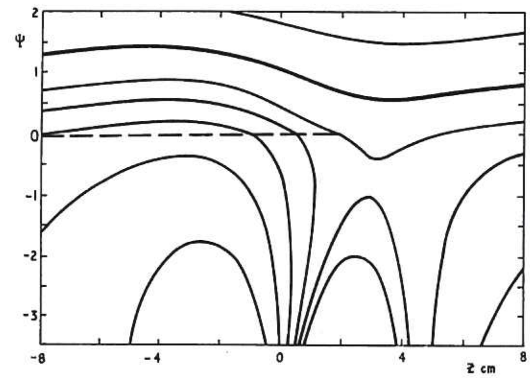

Fig.8 As Fig.7 with the addition of $B_Z = 2\%B_1$.

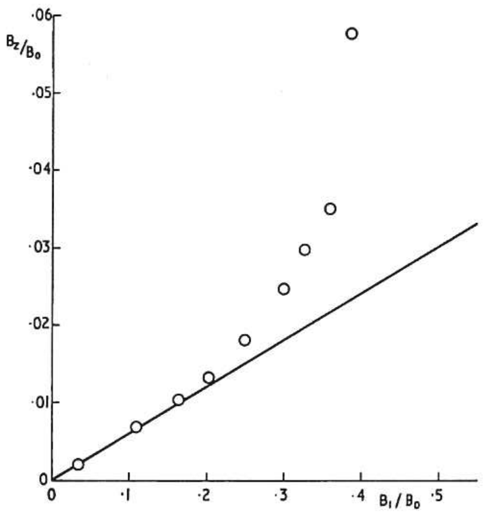

Fig.9 Minimum B_Z for confinement of plasma as a function of the dipole perturbation field B_1. The solid line denotes the adiabatic approximation.

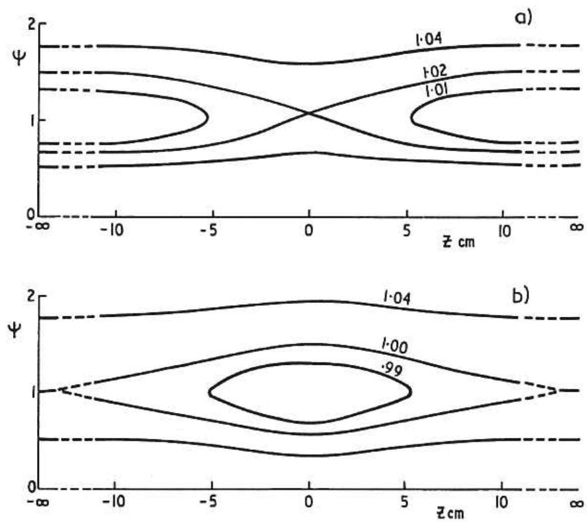

Fig.10 Plasma isobars in the presence of a symmetrical dipole perturbation $B_1 = 10\%B_O$. The value of U is shown on each contour. (a) B_1 opposes B_O between the conductors. (b) B_1 reinforces B_O between the conductors.

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