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THE SPECTRA AND CORRELATION FUNCTIONS FOR ION SOUND TURBULENCE

by

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A B S T R A C T

A theory is given for the spectra and correlation functions for ion sound turbulence. It is shown that the correlation time can be short enough to affect appreciably the non-linear interactions. The actual value of the anomalous resistivity for plasma in an external electric field is calculated. The ion sound spectra resulting from the non-linear conversion is discussed.

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C O N T E N T S

1. PHYSICAL DESCRIPTION OF THE PROBLEM

The theory of weak turbulence was first applied by KADOMTSEV (1965) (see also KADOMTSEV and PETVIASHVILI, 1963) to find the spectrum of ion-sound turbulence. However, he did not consider the effects of the finite correlation time which actually occurs in most experimental investigations of ion sound spectra and of the structure of collisionless shock waves (HAMBERGER and JANCARIK, 1970; PAUL, et al, 1969, 1970; SAGDEEV, 1967). The theory of correlation functions in plasma has only recently been developed (MAKHANKOV and TSYTOVICH, 1969*). The purpose of the present paper is to investigate the influence of finite correlation effects on the effectiveness of those non-linear interactions of ion sound waves which determine both the spectra and the correlation function in plasma. The experimentally observed shock width seems to indicate that the non-linear interactions of ion sound waves must be at least one order of magnitude stronger than was predicted by KADOMTSEV (1965). Further, in experiments on ion sound turbulence (HAMBERGER and JANCARIK, 1970) not only the spectrum but also the correlation width and anomalous resistivity are measured. These are not independent, and it will be shown that in this theory they are closely connected. Hence an adequate theory must account for all these experimentally measured quantities just referred to. Such a detailed comparison of theory and experiment is of course desirable both to further the development of weak turbulence theory and also for the experimenter. Until now this has not been possible because theory has been able to give only a rough estimate of the anomalous resistivity (see SAGDEEV, 1967) with a numerical** factor given only to order of magnitude. We wish to point out that the correlation broadening makes it necessary to consider a new non-linear interaction for which it is possible to construct a more precise theory***.

As shown in MT and RUDAKOV and TSYTOVICH (1970)**** the method of expanding in terms of the turbulent energy is inconsistent near resonance, i.e. we must include the effect of the correlation broadening. The broadening of the resonance $\omega = \bar{k}.\bar{v}$ between ion sound waves and electrons enormously reduces the non-linear interaction between them compared to the quasi-linear interaction so that non-linear electron - ion-sound interaction never can be important (RT). The broadening of the decay resonance, as shown in MT means that the frequency and wave number of the turbulent pulsation are not single valued. This can be expressed mathematically in terms of the correlation function for the electric fields E of the turbulent pulsation:

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* Hereafter referred to as MT.

** Fitted parameter.

*** There will remain only some slowly varying logarithmic functions defined by non-linear integral equations. These may be solved numerically.

**** Hereafter referred to as RT.

$$<E(\bar{r}_1 t_1)\ E(\bar{r}_2, t_2)> = \int \frac{4\pi\omega^2}{\omega_{pi}^2}\ d\bar{k}d\omega\ W_{\bar{k}\omega}\ e^{ik(r_1 - r_2)\ -\ i\omega(t_1 - t_2)} \qquad (1.1)$$

where

$$W = \int W_{\bar{k}\omega} d\bar{k}d\omega \qquad (1.2)$$

is the energy density of the ion sound turbulence, and $\omega_{pi}^2 = \frac{4\pi ne^2}{m_i}$. The existence of correlation broadening means that the dependence of $W_{\vec{k}\omega}$ on ω cannot be expressed in terms of a δ-function, $W_{\bar{k}}\ \delta(\omega - \omega(\bar{k}))$, but by some function with a finite width $\Delta\omega$. To remain within the assumption of weak turbulence we must have $\Delta\omega \ll \omega(\bar{k})$. This usually also means that

$$\frac{W}{nT} \ll 1 \qquad (1.3)$$

Even if the correlation width is finite it is possible to introduce the spectrum of turbulence as an integral of $W_{\vec{k}\omega}$ over frequency

$$W_{\bar{k}} = \int W_{\bar{k}\omega} d\omega \qquad (1.4)$$

As is shown in MT the correlation function near resonance becomes

$$W_{\bar{k}\omega} = \frac{W_{\bar{k}}\ \Delta\omega_{\bar{k}}}{\pi\left[(\omega - \omega(\bar{k}))^2 + \Delta\omega_{\bar{k}}^2\right]} \qquad (1.5)$$

where $\Delta\omega_{\bar{k}}$ (the broadening) is of the order of the characteristic non-linear growth rate of the process which produces the spectrum. Although in principle the correlation function $W_{\bar{k}\omega}$ can be found only by summing the series in W (since $\Delta\omega_{\bar{k}}$ is proportional to $W_{\bar{k}}$) the expansion in W can be used to derive an approximate equation for $W_{\bar{k}}$. This is because the value of $W_{\bar{k}}$ obtained by integrating over ω does not appreciably appect broadening, equation (1.5). The equation for $W_{\bar{k}}$ is called the "balance equation" and can easily be found by using the concept of induced processes described in detail in the author's book (TSYTOVICH, 1967).

The three types of process that one needs consider to formulate a balance equation for ion sound turbulence are

$$e + s \rightleftharpoons e' + s' \qquad (1.6)$$

$$i + s \rightleftharpoons i' + s' \qquad (1.7)$$

$$s \rightleftharpoons s' + s'' \qquad (1.8)$$

Here the symbol s represents an ion sound wave, i a plasma ion, and e a plasma electron. As mentioned above, process (1.6) can be neglected (RT). Process (1.7), considered by KADOMTSEV (1963), leads to the equation

$$\frac{dW_{\bar{k}}}{dt} + \gamma_{\bar{k}} W_{\bar{k}} = W_{\bar{k}}\, k^2 \frac{\partial}{\partial k} \int dx_1 \frac{T_i}{T_e} \frac{(2\pi)^2 k^3 v_s}{n\, Te} W_{k,x_1}\, x_1^2\, (1 - x_1^2)\ ; \quad \frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial \omega_{\bar{k}}}{\partial \bar{k}} \frac{\partial}{\partial \bar{r}} \tag{1.9}$$

where $x_1 = \cos(\bar{\bar{k}}.\bar{k})$, $\gamma_{\bar{k}}$ is the linear (or quasi-linear) damping ($\gamma_{\bar{k}} > 0$) or growth rate ($\gamma_{\bar{k}} < 0$). Equation (1.9) is presented here to illustrate two statements.

The first is that the interaction (1.9) leads to a considerable change in direction as the waves decrease in frequency. Thus, in the example of current-driven ion-sound turbulence, waves must exist which move in a direction opposite to that of the current, which appears to disagree with observation. Indeed, the damping of such waves is only of order u/v_s times smaller than the growth rate in the unstable region (here u is the drift velocity, $v_s = \sqrt{T_e/m_i}$ the ion sound speed). It can also be easily seen that in the region of damping a spectrum is formed similar to that found by KADOMTSEV (1963). For example, for the fully isotropic case where there is only Landau damping we have

$$\sqrt{\frac{\pi}{2} \frac{m_e}{m_i}}\ \Omega = \frac{4\pi}{15} \frac{T_i}{T_e}\ \Omega^2 \frac{\partial}{\partial \Omega}\ \Omega\, W_\Omega \frac{1}{n\, Te}, \tag{1.10}$$

Here

$$W_\Omega = \frac{4\pi k^2}{v_s} W_{\bar{k}}; \qquad \Omega = k v_s; \qquad W = \int W_\Omega d\Omega \tag{1.11}$$

The energy originates from some $\Omega > \Omega^*$, that is outside the region for which (1.10) is assumed to be valid*. For each Ω there is a balance between the frequency transfer of the turbulent energy and the Landau damping. The solution of (1.10) is

$$\frac{W_\Omega}{nT_e} = \frac{1}{\Omega} \ln \frac{\Omega}{\Omega min} \cdot \frac{15}{2\pi} \cdot \sqrt{\frac{\pi}{8} \frac{m_e}{m_i}} \cdot \frac{T_e}{T_i} \tag{1.11}$$

where Ω_{min} is the lowest frequency to which the waves can be transferred without being completely damped.

The second statement is that the non-linear interaction described by (1.10) is rather weak, so that to balance the growth or damping rate (which are quite large for ion sound waves, whose growth rate is bigger than the Landau damping) we need a high level of turbulent energy. Indeed, even from (1.11) one finds

$$\frac{W}{nT_e} \approx \frac{15}{4\pi} \frac{T_e}{T_i} \sqrt{\frac{\pi}{8} \frac{m_e}{m_i}} \cdot \ln^2 \frac{\Omega_*}{\Omega_{min}}. \tag{1.12}$$

For the case of hydrogen on the threshold of instability, even if we take $T_e \approx 5T_i$, W can be small compared to nT_e only if Ω_{min} is close to Ω_*. This means that the spectrum must have a narrow frequency spread, in contradiction with the observations. The only other possibility of reducing the ratio (1.12) occurs if γ_k is very small, i.e. to say that the regions near to the threshold of instability are most important, or that the system, as a result of quasi-linear relaxation, is brought automatically to the condition

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* This situation occurs precisely in the case in which the ion-sound waves arise from the non-linear decay of Langmuir waves.

near to the threshold (KORABLEV and RUDAKOV, 1966; KOVRIZHNIK, 1968). But in this case if we include the quasi-linear interaction with the resonant ions the frequency spectrum of the turbulence must also be very narrow. The observation of electron drift velocities much bigger than the mean ion sound velocity (if the ion sound velocity and the drift velocity are indeed measured precisely) also contradicts the statement that the turbulence is maintained by quasi-linear effects close to threshold conditions.

Let us now come to the last of the possible interactions (1.8) which has not been considered before. Using well defined values of $\omega \equiv \omega(\bar{k})$ it is not possible to satisfy the conservation laws for such a process:

$$\bar{k} = \bar{k}_1 + \bar{k}_2 \qquad (1.13)$$

$$\omega = \omega_1 + \omega_2 \qquad (1.14)$$

However, as can be seen from (1.5), the correlation broadening $\Delta\omega$ tends towards the order of ω as $W \to nT$. On the other hand, as was mentioned above, W tends to be high if the process (1.8) is neglected. It is easy to show that process (1.8) is allowed if the correlation broadening is sufficient that

$$\omega < \omega_{pi} \sqrt{\frac{\Delta\omega}{\omega}} \qquad (1.15)$$

It is known (see TSYTOVICH, 1967) that the process of induced scattering corresponds to the wings of the decay process, where the resonance conditions (1.13) and (1.14) are not satisfied. This means firstly that the decay process is much more effective when it is allowed by the conservation laws, which from the viewpoint of observation is desirable since it enhances the non-linear interaction. Secondly, if one includes both the resonant interaction and its wings, then both the decay processes and ion scattering are taken into account which seems to offer a more general theoretical picture of ion sound turbulence.

Thus it is necessary to re-examine the problem of the formation of ion sound turbulence taking into account the correlation broadening. That is the purpose of the present paper.

Before coming to the theoretical calculations it is desirable to give some physical picture of the process together with some rough estimates. First, let us mention that the process (1.8) is allowed only for small angular difference $\Delta\theta$ between the directions of the interacting waves;

$$\Delta\theta \leq \Delta\omega/\omega \qquad (1.16)$$

and if $\Delta\omega/\omega \ll 1$ the waves mainly interact without significantly changing their angular distribution. This is quite the opposite to the induced scattering on ions. This implies that if process (1.8) predominates, waves excited in some definite direction do not change

$$\frac{dW_{\bar{k}}}{dt} + \gamma_{\bar{k}} W_{\bar{k}} = W_{\bar{k}} k^2 \frac{\partial}{\partial k} \int dx_1 \frac{T_i}{T_e} \frac{(2\pi)^2 k^3 v_s}{n\, T_e} W_{k,x_1} x_1^2 (1 - x_1^2) \; ; \; \frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial \omega_{\bar{k}}}{\partial \bar{k}} \frac{\partial}{\partial \bar{r}} \tag{1.9}$$

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and if $\Delta\omega/\omega \ll 1$ the waves mainly interact without significantly changing their angular distribution. This is quite the opposite to the induced scattering on ions. This implies that if process (1.8) predominates, waves excited in some definite direction do not change that direction (to a first approximation) during the non-linear energy transformation.

Second, we should mention that the process (1.8) approximately conserves the energy of the waves. That means that (1.8) affects only the spread of the energy of the waves (for any given direction). Because the conservation laws forbid this process for frequencies higher than (1.15), we can say that at the limiting frequency given by (1.15) there exists a barrier, so that the turbulent energy flows towards lower frequency. However, as we shall see below, the lower the frequency the weaker the non-linear interaction. This means that a quasi-stationary spectrum is first created for the highest frequencies satisfying (1.15), and this formation then propagates like a "wave" towards lower frequencies. Two different physical situations can occur, in which for any given direction ion sound waves are either damped or excited. If they are damped there must be a constant flow of energy from some higher frequency Ω_*. From the mathematical point of view we then have as a boundary condition a given value W_{Ω_*}. The stationarity of the spectra results from a balance between the non-linear flow towards lower frequencies and the damping. If the waves are excited, they grow until they have enough energy that the non-linear spread becomes important. In this case the balance is between the creation of waves from the instability with the downward flow away from the region of their creation by non-linear interaction. From this picture it is obvious that the spectra created in both cases must be similar if the frequency dependence of the damping and the growth rate is the same. The first case may, for example, correspond to the turbulence created by non-linear generation by Langmuir waves, and the second to current driven instability. One may find that there are two stages of development of the instability. The first is the stage in which the non-linear interaction (1.8) spreads the frequency without changing the direction. As the "wave" in ω-space propagates towards lower ω it slows down and the total energy of turbulent motion is raised. One can then find the (lower) frequency which this "wave" reaches in the characteristic time for scattering on ions. From then on the next stage of the development of the turbulence begins in which the angular spread occurs. This change in direction seems to be a possible mechanism of absorption of the energy of turbulent motion on electrons. One can also expect in this some significant heating of ions, because to change the direction of a wave by an angle of the order of unity the ions must take an appreciable part of the momentum of the wave. It is not at all obvious that the second stage is developed in each case. Indeed, in the case of non-linear driven ion sound turbulence the energy can be absorbed before the second stage arises, and in the case of current driven turbulence the multiple changing of the direction of the waves each time by an angle (1.16), resulting from interaction (1.8), may also simply transfer the wave to a direction in which damping occurs before the scattering on ions becomes important.

2. CORRELATION EFFECTS IN ION SOUND TURBULENCE

Owing to the possibility of the decay (1.8) we must modify the method used in MT. We shall not reproduce here the whole calculation which is similar to MT but merely show the essentially new points. Suppose all non-linear current components $S_{k,\,k_1,\,k_2}$ and

$\Sigma_{k,\ k_1,\ k_2,\ k_3}$ are known:

$$j_k = \sigma_k E_k + \int S_{k,k_1,k_2} E_{k_1} E_{k_2} \delta(k-k_1-k_2)dk_1dk_2$$
$$+ \int \Sigma_{k,\ k_1,k_2,k_3} E_{k_1} E_{k_2} E_{k_3} (k-k_1-k_2-k_3)dk_1dk_2dk_3 \tag{2.1}$$

and modified in such a way as to take into account the resonance broadening for $\omega = \bar{k}.\bar{v}$ as was done in RT. The non-linear Maxwell's equation for the stochastic part of the field then becomes

$$\nabla \varepsilon_k^{\ell} E_k = \frac{4\pi i}{\omega} \int S_{k,\ k_1,k_2} (E_{k_1} E_{k_2} - < E_{k_1} E_{k_2} >) \ \delta\ (k-k_1-k_2)dk_1dk_2$$
$$+ \frac{4\pi i}{\omega}\int \Sigma_{k,k_1,k_2,k_3} (E_{k_1} E_{k_2} E_{k_3} - E_{k_1} < E_{k_2} E_{k_3} >$$
$$- < E_{k_1} E_{k_2} E_{k_3} >) \ \delta\ (k-k_1-k_2-k_3)dk_1dk_2dk_3 \tag{2.2}$$

It is necessary to work near the resonance $\varepsilon_k^{\ell} \simeq 0$ (all the intermediate waves are approximately resonant when process (1.8) is included) where the corrections of the order of W have the same smallness as ε_k^{ℓ}. Let us introduce formally the non-linear dielectric constant

$$\varepsilon_k^N = \frac{4\pi i}{\omega} \int \Sigma'_{k,k_1} I_{k_1} dk_1 \tag{2.3}$$

where Σ'_{k,k_1} is a function to be found, and I_{k_1} is the correlation function for the electric fields:

$$< E_{k_1} E_{k_2} > = - I_{k_1} \delta(k_1 + k_2) \tag{2.4}$$

Let us add $\varepsilon_k^N E_k$ to both sides of (2.2)

$$(\varepsilon_k^{\ell} + \varepsilon_k^N) E_k = \frac{4\pi i}{\omega} \int S_{k,k_1,k_2} (E_{k_1} E_{k_2} - < E_{k_1} E_{k_2} >)$$
$$\times\ \delta(k-k_1-k_2)dk_1dk_2 + \frac{4\pi i}{\omega}\int \Big\{ \Sigma_{k,\ k_1,k_2,k_3} (E_{k_1} E_{k_2} E_{k_3}$$
$$- E_{k_1} < E_{k_2} E_{k_3} > - < E_{k_1} E_{k_2} E_{k_3} >) + \Sigma'_{k,k_1} E_{k_3} < E_{k_1} E_{k_2} > \Big\}$$
$$\times\ \delta(k-k_1-k_2-k_3)dk_1dk_2dk_3 \tag{2.5}$$

One can then follow a procedure similar to that in MT but using $(\varepsilon^{\ell}+\varepsilon^N)$ instead of ε^{ℓ}. The last stage is to determine the Σ'_{k,k_1} in ε^N. Multiplying by E_k, and taking the ensemble average we obtain an equation similar to that in MT. As ε represents the diagonal terms in the field E_k we require that, at least to first order, the diagonal terms must vanish on the right hand side of the equation for the correlation function I_k

(the arguments are similar to those used in RT). Thus the expression for Σ'_{k,k_1} becomes

$$\Sigma'_{k,k_1} = \frac{1}{2} (\Sigma_{k,k_1,k,-k_1} + \Sigma_{k,k_1,-k_1,k}) - \frac{8\pi i \; S_{k,k_1,k-k_1} \; S_{k-k_1,k,-k_1}}{(\omega-\omega_1) \qquad (\varepsilon^{\ell}_{k-k_1} + \varepsilon^{N}_{k-k_1})} \tag{2.6}$$

From equations (2.6) and (2.3) we see that ε^N is determined by an integral equation. Although the general formulation appears complicated one can find an approximate solution appropriate to the problem in question (see below). We should mention that the argument that the integrals containing $1/(\varepsilon^{\ell}+\varepsilon^{N})$ have no irregularities applies equally to this case as in MT. Hence the solution has the form:

$$I_k = \frac{32\pi^2}{\omega^2} \int \frac{|S_{k,k_1,k_2}|^2 \; I_{k_1} I_{k_2} \delta(k-k_1-k_2) dk_1 dk_2}{|\varepsilon^{\ell}_k + \varepsilon^N_k|^2} \tag{2.7}$$

The difference between the problem considered here and that in MT lies in the determination of ε^N. Owing to the resonance behaviour of $(\varepsilon^{\ell} + \varepsilon^N)^{-1}$ for the decay process we can to a first approximation neglect the first two terms in (2.6) and put

$$\Sigma'_{k,k} \approx - \frac{8\pi i \; S_{k,k_1,k-k_1} \; S_{k-k_1,k,-k_1}}{(\omega-\omega_1) \; (\varepsilon^{\ell}_{k-k_1} + \varepsilon^{N}_{k-k_1})} \tag{2.8}$$

Away from the resonance, however, both terms of (2.6) are significant and together represent the scattering of ions. Hence (2.7) gives a general description of ion sound turbulence including the KADOMTSEV interaction. For the resonant case the equation contains only those terms for which a definite expression can easily be found for $kV_{Ti} \ll \omega \ll kV_{Te}$:

$$S_{k,k_1,k_2} = - \frac{e^3}{m_i^2} \frac{(\bar{k}\bar{k}_1) \; (\bar{k}\bar{k}_2) n}{k_1 k_2 k \; \omega\omega_1\omega_2} \tag{2.9}$$

By expanding the dominator of (2.7) near the linear resonance

$$\mathrm{Re}\, \varepsilon^{\ell}_k = (\omega-\omega_{\bar{k}}) \frac{\partial \varepsilon^{\ell}}{\partial \omega}\Big|_{\omega=\omega(\bar{k})}$$

we get an approximate expression for the correlation curve near resonance

$$I_{k\omega} \sim \frac{1}{(\omega-\omega^N_{\bar{k}})^2 + (\gamma^N_{\bar{k}})^2} \tag{2.10}$$

$$\text{where} \quad \gamma^N_{\bar{k}} = \frac{\mathrm{Im}\, (\varepsilon^N_k + \varepsilon^{\ell}_k)}{\frac{\partial \varepsilon^{\ell}_k}{\partial \omega}\Big|_{\omega=\omega_{\bar{k}}}} \; ; \text{ and } \quad \omega^N_{\bar{k}} = \omega_{\bar{k}} + \frac{\mathrm{Re}\varepsilon^N_k}{\frac{\partial \varepsilon^{\ell}_k}{\partial \omega}\Big|_{\omega=\omega_{\bar{k}}}} \tag{2.11}$$

In these last expressions we use $\omega \approx \omega_{\bar{k}}$, which is permissible near resonance. We can also neglect the difference between ω and $\omega_{\bar{k}}$ in the factor connecting $W_{\bar{k}\omega}$ with $I_{\bar{k}\omega}$, and so get the result (1.5) with $\Delta\omega_{\bar{k}} = \gamma_{\bar{k}}^{N}$ and $\omega(\bar{k}) = \omega_{\bar{k}}^{N}$.

The width of correlation curve depends on the turbulent energy, as can be seen from (2.6). Now in the equation already derived not only the correlation function is broadened, but also the δ-function governing the conservation laws (1.14) for the decay. The most difficult thing to derive exactly in the present theory seems to be the non-linear equation for ε^{N}. But, as we have mentioned, we do not need the exact expression but, according to (2.10), only an approximate one with $\omega=\omega(\bar{k})$. This arises from a solution of an approximate balance equation. The exact balance equation can be found from (2.7) by multiplying by

$\mathrm{Im}\ (\varepsilon_{k}^{N} + \varepsilon_{k}^{\ell})\omega = \gamma_{\bar{k}\omega}^{N} \frac{\partial \varepsilon_{k}^{\ell}}{\partial\omega}$ and integrating both sides over ω:

$$\int \gamma_{\bar{k}\omega}^{N} W_{\bar{k}\omega} d\omega = \int \frac{\frac{\partial \varepsilon_{\bar{k}}^{\ell}}{\partial\omega} 4\pi\gamma_{\bar{k}\omega}^{N} | S_{k,\ k_1,k_2}|^2 I_{k_1} I_{k_2} \delta(k-k_1-k_2) d\omega dk_1 dk_2}{\omega|\varepsilon_{k}^{\ell} + \varepsilon_{k}^{N}|^2} \tag{2.12}$$

The left hand side of (2.12) contains both the linear damping or growth rate and the non-linear effect which is proportional to the energy W_{k} in the turbulent motions. The whole non-linear effect is due to induced scattering and induced decay. The right hand side of (2.12) includes the spontaneous non-linear decay process.

3. APPROXIMATE BALANCE EQUATION AND SPECTRA OF TURBULENCE

Let us now consider the resonant decay process (1.8). The balance equation described by (2.12) differs from the usual one by having a broad dependence on $\omega(k)$ for all three interacting waves, so that instead of $\delta(\omega-\omega(\bar{k}))$ we have $\dfrac{\gamma_{\bar{k}}^{N}}{\pi(\omega-\omega(\bar{k}))^2 + \gamma_{\bar{k}}^{N}}$. Suppose that the broadening is sufficiently great that we can neglect the curvature of the dispersion curve $\omega(\bar{k})$ and set $\omega(\bar{k}) = kv_s$. If, nevertheless, $\gamma_{\bar{k}}^{N} = \Delta\omega_{k} \ll \omega(\bar{k})$, which implies that the turbulence is weak, it is possible approximately to use $\delta(\omega-\underline{k}v_s)$ instead of the proper resonance profiles. Thus we can obtain the approximate balance equation which can be written in terms of the probability of the decay process:

$$\gamma_{\bar{k}} N_{\bar{k}} = \int \frac{d\bar{k}_1 d\bar{k}_2}{(2\pi)^6} \Big\{ W(\bar{k},\bar{k}_1,\bar{k}_2)\ (N_{\bar{k}_1} N_{\bar{k}_2} - N_{\bar{k}} N_{\bar{k}_1} - N_{\bar{k}} N_{\bar{k}_2})$$

$$+\ 2W(\bar{k}_1,\bar{k},\bar{k}_2)\ (N_{\bar{k}} N_{\bar{k}_1} + N_{\bar{k}_1} N_{\bar{k}_2} - N_{\bar{k}} N_{\bar{k}_2}) \Big\} \tag{3.1}$$

$$W(\bar{k},\bar{k}_1,\bar{k}_2) = \frac{\pi}{4n\,T_e} (2\pi)^3\, \delta(\bar{k}-\bar{k}_1-\bar{k}_2)\ \delta(kv_s - k_1 v_s - k_2 v_s)$$

$$\times \frac{(\bar{k}\bar{k}_1)^2}{k^2 k_1^2} \frac{(\bar{k}\bar{k}_2)^2}{k^2 k_2^2}\ \omega\omega_1\omega_2 \tag{3.2}$$

This expression for the W differs from that which can be found using the general formula for the probability of the decay process derived in TSYTOVICH (1967) by a factor 1/2. This arises from using only half of the correlation curve, since formally $\int_0^\infty \delta(x)dx = \frac{1}{2}$. In the more general case of non-stationary turbulence the term $\frac{\partial N_{\bar{k}}}{\partial t} + v_s \frac{\bar{k}}{k} \frac{\partial N_{\bar{k}}}{\partial \bar{r}}$ must be added to the left-hand side of equation (3.1).

A rough estimate of the characteristic time for the non-linear energy transformation described by (3.1) can be found from

$$\frac{1}{\tau_d} \approx \int W(\bar{k},\bar{k}_1,\bar{k}_2)\frac{d\bar{k}_1 d\bar{k}_2}{(2\pi)^6} N_{\bar{k}_1} \approx \omega \frac{W}{nT_e} \tag{3.3}$$

which is much larger than the time for the scattering of waves on ions (1.9)

$$\frac{1}{\tau_{sc}} \approx \omega \frac{T_i}{8T_e} \frac{W}{nT_e} \tag{3.4}$$

This confirms the above hypothesis that process (1.8) is the most effective one.

Let us now consider a case of axially symmetric turbulence in which the spectrum $W_{\bar{k}}$ depends only on $|\bar{k}|$ and the angle θ between $\bar{k}$ and this axis. Let us introduce $W_{\Omega,\xi}$ where $\xi = \cos\theta$, $\Omega = kv_s$, normalized by the expression

$$\int_0^\infty d\Omega \int_{-1}^{1} d\xi \, W_{\Omega,\xi} \, d\Omega = W \tag{3.5}$$

Then equation (3.1) can be written in the form

$$\begin{aligned}\gamma_{\Omega,\xi} W_{\Omega,\xi} = \frac{\pi}{4n_o T_e} \Bigg\{ \int_0^\Omega d\Omega_1 &\left[\frac{\Omega^3}{\Omega_1(\Omega-\Omega_1)} W_{\Omega,\xi} W_{\Omega-\Omega_1,\xi} \right. \\ &\left. - W_{\Omega,\xi} \frac{(\Omega-\Omega_1)^2}{\Omega_1} W_{\Omega_1,\xi} - \frac{\Omega_1^2}{\Omega-\Omega_1} W_{\Omega,\xi} W_{\Omega-\Omega_1,\xi} \right] \\ + 2\int_\Omega^{\Omega_*} d\Omega_1 &\left[\frac{(\Omega-\Omega_1)^2}{\Omega_1} W_{\Omega,\xi} W_{\Omega_1,\xi} + \frac{\Omega^3}{\Omega_1(\Omega_1-\Omega)} W_{\Omega_1,\xi} W_{\Omega_1-\Omega,\xi} \right. \\ &\left. - \frac{\Omega_1^2}{\Omega_1-\Omega} W_{\Omega_1-\Omega,\xi} W_{\Omega,\xi} \right]\end{aligned} \tag{3.6}$$

where Ω_* is the highest frequency for which the decay process is allowed

$$\frac{\Omega_*}{\omega_{pi}} \approx \sqrt{\frac{\gamma^N_{\Omega_*}}{\Omega_*}} \tag{3.7}$$

For many cases of interest $\gamma_{\Omega,\xi}$ is a linear function of Ω

$$\gamma_{\Omega,\xi} = \Omega \cdot \gamma_\xi \tag{3.8}$$

For Landau damping in the case of an isotropic distribution of electrons

$$\gamma = \sqrt{\frac{\pi}{2}\frac{m_e}{m_i}} \tag{3.9}$$

In the case of Maxwellian distribution displaced by a mean electron drift velocity u we have

$$\gamma = -\frac{u}{v_s}\sqrt{\frac{\pi}{2}\frac{m_e}{m_i}} \tag{3.10}$$

In the more realistic case in which the angular electron distribution during a current driven instability is determined by quasi-linear relaxation in the turbulent fields created by the non-linear effects considered here the expression for γ_ξ is more complicated, but the linear dependence on Ω remains.

From a dimensional analysis in the case of a dependence of the form of (3.8) it can be seen that the solution of (3.6) must be of the form

$$W_{\Omega,} = \frac{1}{\Omega}\frac{4nT}{\pi}\left|\gamma_\xi\right| \psi_\pm\left(\frac{\Omega}{\Omega_*}\right) \tag{3.11}$$

where $\psi(\lambda)$ $(\lambda < 1)$ is a slowly varying logarithmic time-dependent function of λ that satisfies the equation

$$\psi_\pm(\lambda) = \lambda^2\varphi_\pm(\lambda)$$

$$\varphi_\pm(\lambda) = \lambda^2\frac{\int_0^1 \varphi_\pm(\lambda x)\,\varphi_\pm\left[\lambda(1-x)\right]dx + 2\int_\lambda^1 \frac{1}{x^2}\,\varphi_\pm\left(\frac{\lambda}{x}\right)\cdot\varphi_\pm\left[\frac{\lambda}{x}(1-x)\right]dx}{\pm 1+2\lambda^2\int_0^1 (1-x)^2\,\varphi_\pm(\lambda x)\,dx + 2\lambda^2\int_\lambda^1 \frac{1}{x^4}\left\{\varphi_\pm\left[\frac{\lambda}{x}(1-x)\right] - (1-x)^2\,\varphi_\pm\left(\frac{\lambda}{x}\right)\right\}dx} \tag{3.12}$$

The $\pm$ sign is to be taken for $\gamma > 0$ and $\gamma < 0$ respectively. Equation (3.12) is valid only in the region in which there has been enough time for a stationary spectrum to be established, i.e. the smallest λ in (3.12) corresponds to the time of observation, or the flow time of the plasma through the shock front. Although the exact solution of (3.12) can be found only by numerical computation, it is not necessary to know the exact behavious of ψ for small λ to derive the anomalous plasma conductivity, as we shall see later. The lowest frequency in (3.12) for $\gamma < 0$ can be found by the following physical arguments.

If u, the mean drift velocity of electrons, is large enough compared with v_s, the most important effect is the scattering of electrons by the turbulence, rather than their heating. That means that during the quasi-linear transformation, the electrons mainly change their angular distribution and therefore the ratio γ/Ω does not change. However the time for non-linear energy transfer is proportional to Ω^{-1}, so that one can find some low enough Ω for which this rate of energy transfer equals that of the quasi-linear heating. In this region the quasi-linear transfer mixes Ω and the angular dependence, and equation (3.12) is no longer valid. On this time scale the quasi-linear heating is more important than the non-linear transfer. The heating rate is of order

$$\gamma_{heat} \sim \frac{W}{nT} \; \Omega_* \sqrt{\frac{m_e}{m_i}} \tag{3.13}$$

Since we have for the non-linear transfer $\gamma \sim \Omega W/nT$ we may roughly estimate

$$\Omega_{min} \gtrsim \Omega_* \sqrt{\frac{m_e}{m_i}} \tag{3.14}$$

We can say that the energy is transferred without a significant change of direction until it finds an absorption region as a result of a significant quasi-linear change of γ. As we show later, the anomalous resistivity does not depend on Ω_{min} or the lack of stationarity of turbulence at low frequencies*.

It is now possible to estimate the turbulent energy that occurs in the first stage in the range $\Omega_{min} < \Omega < \Omega_*$:

$$\frac{W}{nT} \simeq \frac{8}{\pi} \int_0^1 \gamma_\xi \, d\xi \int_{\Omega_{min}}^{\Omega_*} \frac{d\Omega}{\Omega} \; \psi(\Omega) \simeq \frac{8}{\pi} \bar{\gamma} \; \ln^2 \sqrt{\frac{m_e}{m_i}} \tag{3.15}$$

where $\bar{\gamma}$ is the mean value of γ_ξ: If we put $\bar{\gamma} \sim \sqrt{\frac{m_e}{m_i}} \frac{u}{v_s}$ we find a value that is on a factor $T_i/8T_e$ smaller than (1.20).

From (3.11) it is possible to find the correlation time $\tau = 1/\gamma^N$

$$\gamma_\pm^N = \Omega \gamma_\xi \left[\int_0^1 dx \; (\mp 1 + 4\psi_\pm(\lambda x)) + 4 \int_0^1 \frac{\psi_\pm(\lambda x) \, dx}{x^2} \right.$$

$$\left. + \; 8 \int_1^{\frac{1}{\lambda}} \frac{\psi_\pm(\lambda x)}{x} \, dx \right] \tag{3.16}$$

The largest term is the second in the square brackets. The minimum value of Ω is given by $\Omega_{min}/\Omega_* \lambda = \Omega_{min}/\Omega$ and therefore (3.16) becomes approximately

$$\gamma^N \simeq \frac{u}{V_s} \frac{\Omega^2}{\Omega_{min}} \sqrt{\frac{\pi}{2} \frac{m_e}{m_i}} \tag{3.17}$$

The last estimate is written for a current-driven instability. By putting $\Omega = \Omega_*$ in (3.17) it is easy to see that $\gamma/\Omega \sim 1$ if $\Omega_{min} \sim \sqrt{m_e/m_i}(u/v_s)$.

Thus we find that the smaller is Ω_{min} the greater the correlational broadening for the highest frequency $\Omega \sim \Omega_*$, and that the real Ω_{min} must be higher than (3.17) if $u > v_s$. This argument is independent of the quasi-linear estimates given above. On the other hand, one can see also that the correlational broadening introduces a new type of heating.

Indeed, the approximate expression for the correlational broadening can be now used in the exact equation to find more precisely the spectrum and time development of the ion sound instability. Of course, this interaction includes the scattering on ions and

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* The quasi-linear transfer for $\Omega < \Omega_{min}$ is qualitatively different from that considered by RUDAKOV and KORABLEV (1966) as a result of the in flux of turbulent energy from higher Ω.

their heating during this scattering. The time scale of this process is much longer than that considered above. We restrict ourselves here with only one comment which shows what kind of new effects appear in the general equation (2.17) of ion-sound turbulence apart from the well known effects of ion-scattering. For example, if one considers current driven ion sound turbulence, excitation occurs for waves in the Cerenkov cone along the applied electric field. The final stationary turbulent state must include conversion of these waves to the damping region to balance the power generated. Since, for current-driven ion-sound turbulence, the growth rate exceeds the damping ($u > v_s$) for all Ω the usual reduction of frequency cannot give a stationary turbulent state. However, the sign of γ changes with increasing angle to the direction if u. Therefore the non-linear angular transformation can transfer the turbulent pulsations to the damping region. The correlation width allows the interaction of waves with

$$\Delta\theta \sim \frac{\Delta\Omega}{\Omega} \tag{3.18}$$

To change the direction of the energy by an angle $\Delta\theta \sim 1$ it is necessary to have $1/\Delta\theta \simeq \Omega/\Delta\Omega$ steps. This can give a rough estimation of the time needed to transfer the energy to the dissipation region

$$\frac{1}{\tau_c} \simeq \frac{\Omega^2}{\Delta\Omega} \quad \frac{u}{v_s} \quad \frac{8}{\pi} \left(\frac{\ell n^2 \Omega}{\Omega_{min}}\right) \sqrt{\frac{m_e}{m_i}} \simeq \Omega_{min} \; \ell n^2 \left(\frac{\Omega_*}{\Omega_{min}}\right) \tag{3.19}$$

If $\Omega_{min} \sim \sqrt{\frac{m_e}{m_i}} \; \Omega_*$, we have $\frac{1}{\tau} \sim \frac{1}{5} \; \Omega_*$. The characteristic time for non-linear transfer by ion scattering for $\Omega = \Omega_*$ is of order

$$\frac{1}{\tau_i} \leq \frac{\Omega_* T_i}{5T_e} \sqrt{\frac{m_e}{m_i}} \; \ell n^2 \; \frac{\Omega_*}{\Omega_{min}} \tag{3.20}$$

At least $\tau_i \geq \tau_c \; 10 \; T_e/T_i$. Moreover τ_i can also exceed the absorption time

$$\frac{1}{\tau} \sim \sqrt{\frac{m_e}{m_i}} \; \Omega_* \tag{3.21}$$

This means that the turbulent energy can be transferred to the absorption region, and there absorbed by the electrons before the ion scattering becomes important. In this case the ion heating must be small.

4. ANOMALOUS CONDUCTIVITY

Although a truly stationary turbulent spectrum is reached only after the waves reach the absorption region, the anomalous conductivity reaches a steady value when the spectra is formed for the highest Ω of the order of Ω_*. We conclude this from the expression for the quasi-linear diffusion coefficient, which is proportional to

$$\int \Omega \cdot W_{\Omega,\xi}\, d\Omega \qquad (4.1)$$

This shows that one can consider the anomalous conductivity as defined even when the spectrum is formed only for frequencies of the order of ~ $(1/5 - 1/10)\Omega_*$, and that (4.1) is not sensitive to the low frequency changes in the spectrum. We shall therefore consider here only the conductivity during the first stage of the development of the ion sound turbulence. The general quasi-linear term can be written as follows:

$$\frac{\partial}{\partial v_i} D_{ij} \frac{\partial \phi}{\partial v_j} \qquad (4.2)$$

$$D_{ij} = \frac{\pi}{n} \frac{v_{Te}^2}{m_e} \int k_i\, k_j \quad W_{\bar{k}}\, \delta(\omega - \bar{k}\bar{v})\, d\bar{k} \qquad (4.3)$$

where ϕ is the regular part of the electron distribution function. In the first approximation the collision of an electron with a sound wave is elastic, i.e. electrons are scattered and lose their momentum but not their energy. The distribution function, due to the axial symmetry along the direction of the electric field, depends on v and x = cos $(\bar{E},\bar{v})$

$$\phi = \phi(v,x) \qquad (4.4)$$

If the mean drift velocity u lies in the interval

$$v_s \ll u \ll v_{Te} \qquad (4.5)$$

the biggest term in (4.2) arises from the derivative $\partial/\partial x$ and (4.2) can be written in the form

$$\frac{v_{Te}^2\, \pi}{nm_e v^2} \frac{\partial}{\partial x} \int k^2 \xi^2 \quad W_{\bar{k}}\, \delta(\bar{k}\bar{v})\, d\bar{k}\, \frac{\partial \phi}{\partial x} \qquad (4.6)$$

where ξ = cos $(\bar{k},\bar{E})$, and W depends only on ξ and $k = \Omega/v_s$. After integration over the angle φ between the components of $\bar{k}$ perpendicular to $\bar{E}$ and $\bar{v}$, we can find the equation which describes the quasi-stationary distribution in which the electric force is balanced by the turbulent collisions:

$$eEx \frac{\partial \phi_o}{\partial v} = \frac{2\pi v_{Te}^2}{nv_s v^3} \frac{\partial}{\partial x} \int_0^{\infty} d\Omega \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} d \frac{\Omega W_{\Omega,\xi}\, \xi^2 d\xi d\Omega}{\sqrt{1-x^2-\xi^2}} \quad \frac{\partial \phi_1}{\partial x} \qquad (4.7)$$

Here the distribution function is divided into two parts

$$\phi = \phi_o(v) + \phi_1(v,x) \qquad (4.8)$$

where ϕ_o is the isotropic, and ϕ_1 the small anisotropic parts

$$\phi_1 \ll \phi_o \tag{4.9}$$

The right hand side of (4.7) contains only ϕ_1 and ϕ_1 is neglected in the left hand side. The normalisation of W is

$$W = \int_{-1}^{1} W_{\Omega,\xi}\, d\xi d\Omega\, 2\pi \tag{4.10}$$

In a similar way we can write the ion-sound growth rate

$$\gamma = \Omega\gamma_\xi ; \quad \gamma_\xi = \frac{\pi\xi}{n} v_{Te}^2 \int \frac{k}{v}\, \delta(\bar{k}\bar{v})\, \frac{\partial\phi_1}{\partial x}\, d\bar{v} \tag{4.11}$$

This expression is valid if the Landau damping due to ϕ_o can be neglected as is in the conditions of (4.5) and if

$$\xi > \frac{v_s}{u} \tag{4.12}$$

To an accuracy of the order of v_s/u the Cerenkov cone corresponds to $\xi > 0$. Integration of (4.11) over φ gives

$$\gamma_\xi = -\frac{2\pi\xi}{n} v_{Te}^2 \int_{-\sqrt{1-\xi^2}}^{\sqrt{1-\xi^2}} \frac{dx}{\sqrt{1-x^2-\xi^2}}\, \frac{\partial}{\partial x} \int_0^\infty \phi_1\, dv \tag{4.13}$$

In the approximation considered in Section 3, ξ does not change during the non-linear interaction, and the turbulent energy exists only in the region of excitation, that is to an accuracy of order v_s/u in the region where $\xi > 0$. We get then

$$\begin{aligned} W_{\Omega,\xi} &= \frac{2}{\pi^2} n_o T_e \frac{1}{\Omega} |\gamma_\xi| \psi_+(\lambda) \\ &= \frac{4}{\pi} m_e v_{Te}^2 \xi \frac{1}{\Omega} \psi_+(\lambda) \left| \int_{-\sqrt{1-\xi^2}}^{\sqrt{1-\xi^2}} \frac{dx_1}{1-x^2-\xi^2} \frac{\partial}{\partial x_1} \int_0^\infty \phi_1(x,v)\, dv \right| \end{aligned} \tag{4.14}$$

From (4.7) one can then see that ϕ_1 is proportional to $v^3\, \partial\phi_o/\partial v$. Let us define a function g by

$$\phi_1(v,x) = -\frac{v^3}{v_{Te}^3} \frac{\partial\phi_o}{\partial v} v_s\, g(x) \tag{4.15}$$

If we suppose ϕ_o to be a Maxwellian distribution, we then have

$$\int_0^\infty \phi_1(v,x)\, dv = \frac{3}{4\pi} n g(x) \frac{v_s}{v_{Te}^3} \tag{4.16}$$

Thus we have from (4.13) and (4.16)

$$\gamma_\xi = -3\sqrt{\frac{m_e}{m_i}}\, \xi \int_0^{\sqrt{1-\xi}} \frac{dx}{\sqrt{1-x^2-\xi^2}} \frac{\partial g}{\partial x} \tag{4.17}$$

g is an odd function of x, and therefore $\partial g/\partial x$ is an even function.

$$W_{\Omega,\xi} = \frac{6}{\pi^2} m_e \sqrt{\frac{m_e}{m_i}} n\xi \frac{1}{\Omega} \psi_+(\lambda) \int_0^{\sqrt{1-\xi^2}} \frac{dx_1}{\sqrt{1-x_1^2-\xi^2}} \frac{\partial g}{\partial x_1} \tag{4.18}$$

Putting (4.18) and (4.15) into (4.7), and integrating (4.7) over x taking into account that the right hand side of the result vanishes as $x \to 1$ we find

$$|\alpha| (1-x^2) = \int_0^{\sqrt{1-x^2}} \frac{\xi^3 d\xi}{\sqrt{1-x^2-\xi^2}} \int_0^{\sqrt{1-\xi^2}} \frac{dx_1 \frac{\partial g(x_1)}{\partial x_1}}{\sqrt{1-x_1^2-\xi^2}} \frac{\partial g(x)}{\partial x} \tag{4.19}$$

where

$$\alpha = \frac{\pi e E}{6 m_e v_s \Omega_* \int_0^1 \psi_+(\lambda) d\lambda} \tag{4.20}$$

By introducing a new function

$$\frac{\partial g}{\partial x} = \sqrt{|\alpha|} \frac{1}{s} \rho(s); \qquad s = \sqrt{1-x^2} \tag{4.21}$$

and changing the order of integration, equation (4.19) can be written as follows:

$$\frac{s^3}{\rho(s)} = \int_0^1 \frac{\rho(\eta) d}{\sqrt{1-\eta^2}} \left\{ \frac{s^2+\eta^2}{4} \ln \frac{(s+\eta)^2}{(s-\eta)} - s\eta \right\} \tag{4.22}$$

The function $\rho(s)$ can be found numerically. Notice that if s is not small $\rho(s)$ is of the order of 1. If $s < 1$, ρ behaves as $\rho \propto \frac{1}{\sqrt{\ell n\, s}}$, so that it is a slowly varying function of s. The current density j is

$$j = 4\pi e \int_0^1 x dx \int_0^\infty v^3 \phi_1(v,x) dv = \frac{24\pi e v_s}{v_{Te}^3} \int_0^1 x g(x) dx \cdot$$

$$\cdot \int_0^\infty v^5 \phi_o(v) dv = 24\sqrt{\frac{2}{\pi}}\, e n v_s \int_0^1 \frac{s^2 \rho(s) ds}{\sqrt{1-s^2}} \sqrt{|\alpha|} \tag{4.23}$$

Since by definition, the mean drift velocity is

$$u = \frac{j}{en} \tag{4.24}$$

then from (4.23)

$$u = 24\sqrt{\frac{2}{\pi}} \cdot v_s \int_0^1 \frac{s^2 \rho(s) ds}{\sqrt{1-s^2}} \sqrt{|\alpha|} \tag{4.25}$$

Therefore

$$j = \frac{24 e n v_s \alpha}{\sqrt{|\alpha|}} \frac{2}{\pi} \int_0^1 \frac{s^2 \rho(s)\, ds}{\sqrt{1-s^2}} = \frac{8 e^2 n E v_s}{m_e \Omega_* u} \cdot 24 \frac{\left[\int_0^1 \frac{s^2 \rho(s)\, ds}{\sqrt{1-s^2}} \right]^2}{\int_0^1 \psi_+(\lambda)\, d\lambda} \tag{4.26}$$

If we now define the effective collision time τ^* and the effective collision frequency ν^* by

$$j = \sigma E; \quad \sigma = \frac{ne^2}{m_e} \tau^* ; \quad \tau^* = \frac{1}{\gamma^*} \tag{4.27}$$

we find

$$\tau^* = \left[\int_0^1 \frac{s^2 \rho(s)\, ds}{\sqrt{1-s^2}} \right]^2 \frac{v_s}{u} \frac{192}{\Omega_* \int_0^1 \psi_+(\lambda)\, d\lambda} \tag{4.28}$$

This dependence on u is similar to that found in the case of scattering by ions, but the numerical factor is much bigger and in general depends on the turbulent energy. This arises from Ω_* whose maximum value is of the order of ω_{pi}.

5. DISCUSSION OF THE RESULTS

Let us briefly underline the above statements. Firstly, the correlation effects change the non-linear interactions. The new interaction is stronger and therefore gives a more anomalous conductivity. Secondly, the time development of the instability is much faster than for ion scattering, and therefore the spectrum can be formed under conditions in which the turbulence is excited for much shorter time intervals. Thirdly, the present considerations predict that during the first stage of ion-sound turbulence the direction of waves is not much changed. The opposite effect is predicted from the non-linear theory based on induced scattering on ions. In this interaction the direction of the waves can be changed considerably, and if an interaction of this type limited the energy of resonant waves, an appropriate number of waves must be excited in the direction opposite to that of the applied electric field. Therefore, in principle, experimental results can indicate which of the possible types of interaction of ion-sound waves occurs not only by quantitative comparison the measured resistivity with that predicted by theory, but also by checking the qualitative predictions, for example the presence of the waves moving in the direction opposite to the field. Recent results (HAMBERGER and JANCARIK, 1970; PAUL et al, 1970) show that the waves moving in the direction opposite to E are much weaker than in the direction of E. Fourthly, we should mention that according to the theory considered in this paper the energy of the turbulence heats mainly only electrons, and the energy in the ions must approximately disappear as the sound waves are damped. We should also mention that the angular distribution of the waves excited according to (4.18) has a complicated angular dependence not simply $\sim \xi$ as can be found if the electrons have a Maxwell distribution displaced by some drift velocity. Finally, we mention that the correlation time decreases rapidly during the development of instability and therefore the average correlation time is not a good quantity to compare with theory. Some more detailed numerical computations of the general equations found in Section 2 are desirable for comparison with experimental data.

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