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THE STRUCTURE OF HYDROMAGNETIC SHOCK WAVES

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THE STRUCTURE OF HYDROMAGNETIC SHOCK WAVES

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1. Introduction

We discuss the structure of hydromagnetic shock waves of various types. The plasma is treated in the two fluid approximation and collisional transport coefficients are used. This treatment should be valid in a collisional plasma subject to the usual caveat about the use of transport coefficients in situations where the density, temperature etc., may change significantly over a mean free path. The approach may also have some validity for collisionless shocks if the dissipation mechanisms involve instabilities of sufficiently fine scale and high frequency that they can be described by effective transport coefficients.

2. Basic Equations

The shock is taken to be at rest in the y-z plane. The upstream flow is in the x-direction and the magnetic field upstream and downstream is in the x-y plane (see Fig.1). B_o is the magnitude of the upstream magnetic field and θ is the angle between $\underline{B}_o$ and the normal to the shock. The equations are then:

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$$n\,u_x = \text{constant} \tag{1}$$

$$m_i n\, u_x\, \underline{u}' = \underline{j} \wedge \underline{B} - \underline{x}\,\left(p_e + p_i - \frac{4}{3}\mu_i\, u_x'\right)' \tag{2}$$

$$\underline{E} + \underline{u} \wedge \underline{B} = \eta\, \underline{j} + \frac{c}{ne}\left(\underline{j} \wedge \underline{B} - \underline{x}p_e'\right) \tag{3}$$

$$\frac{n^{\gamma}\, u_x}{\gamma - 1}\left(p_{e,i}\, n^{-\gamma}\right)' = \begin{cases} \eta\, j^2 & \text{electrons} \\ \frac{4}{3}\mu_i\, (u_x')^2 & \text{ions} \end{cases} \tag{4}$$

$$\text{curl}\ \underline{B} = 4\pi\, \underline{j} \tag{5}$$

$$\text{curl}\ \underline{E} = 0 \tag{6}$$

$$p_{e,i} = n\, \kappa\, T_{e,i} \tag{7}$$

$$\gamma = 5/3$$

where primes denote differentiation with respect to x and κ is Boltzmann's constant.

Here η is the plasma resistivity and μ_i the ion viscosity. These are taken to have the classical temperature dependence, namely,

$$\eta = \eta_1 \left(\frac{T_e}{T_1}\right)^{-3/2} \tag{8}$$

$$\mu_1 = \mu_1 \left(\frac{T_i}{T_1}\right)^{5/2} \tag{9}$$

where η_1 and μ_1 are the upstream values. The upstream electron and ion temperatures are assumed equal, $T_e = T_i = T_1$. Chapman and Cowling (1939) showed the transport coefficients η and μ_{xx} to be only weakly dependant on the magnetic field. Here they are assumed to be independant of B. Note that in these equations we have included the Hall term but we have neglected the following:

(i) thermal conduction
(ii) electron inertia
(iii) electron viscosity
(iv) thermoelectric effects
(v) shear viscosity

Electron inertia is negligible as long as the scale length for magnetic field variation is large compared with the collision free skin depth, c/ω_{pe}. Shear viscosity is strongly affected by the magnetic field and should be negligible except in some special cases such as switch shocks where there is a large change in the transverse velocity. Electron viscosity is negligible compared with ion viscosity provided that $T_e/T_i < (m_i/m_e)^{\frac{1}{4}}$ (see Chapman and Cowling (1939)). Thermal conduction and thermoelectric effects are probably important in some regions of the shock structure (Woods (1969a)); we neglect them for the present in the interests of simplicity.

3. Structure Equations

The basic equations can be reduced by eliminating n, u_y, u_z and $\underline{E}$, and simplified by normalisation. The units chosen were

length $(c/\omega_{pe})_1$,	the collision free skin depth
density n_1,	the upstream plasma density
temperature T_1,	the upstream ion (electron) temperature
magnetic field B_o,	the total upstream magnetic field
velocity u_1,	the upstream plasma velocity

Define

$$M_{A_1}^2 = \frac{4\pi\, m_i\, n_1\, u_1^2}{B_o^2}$$

$$M_1^2 = \frac{3}{10}\,\frac{m_i\, u_1^2}{\kappa\, T_1}$$

$$V = V_1\left(\frac{T_i}{T_1}\right)^{5/2} , \quad V_1 = \frac{4}{3}\left(\frac{\mu\,\omega_{pe}}{m_i\, n\, u\, c}\right)_1 \tag{10}$$

$$R = R_1\left(\frac{T_e}{T_1}\right)^{-3/2} , \quad R_1 = \frac{1}{4\pi}\left(\frac{\eta\,\omega_{pe}}{u\, c}\right)_1 \tag{11}$$

$$S = S_1\left(\frac{u}{u_1}\right) , \quad S_1 = \left(\frac{m_i}{m_e}\right)^{\frac{1}{2}} \frac{\cos\theta}{M_{A_1}} \tag{12}$$

$$u^* = \frac{\cos^2\theta}{M^2_{A_1}}$$

$$\xi = \frac{x}{(c/\omega_{pe})}$$

M_{A_1} and M_1 are respectively the Alfven and sonic mach numbers of the upstream flow. V, R, and S are respectively the viscosity, resistivity, and Hall term coefficients. u^* is the value of (u_x/u_1) when u_x has the local normal Alfven speed b_x

$$b_x^2 = \frac{B_o^2 \cos^2\theta}{4\pi\, m_i\, n}$$

Henceforth a symbol will always signify the normalised quantity (i.e. T_e means T_e/T_1 etc.)

The equations take the form

$$V\frac{du}{d\xi} = u-1 + \frac{1}{2M^2_{A_1}}\left[\, B_y^2 + B_z^2 - \sin^2\theta \,\right] + \frac{3}{10\, M_1^2}\left[\frac{T_e + T_i}{u} - 2\right] \tag{13}$$

$$R\frac{dB_y}{d\xi} + S\frac{dB_z}{d\xi} = B_y\,(u-u^*) - \sin\theta\,(1-u^*) \tag{14}$$

$$-S\frac{dB_y}{d\xi} + R\frac{dB_z}{d\xi} = B_z\,(u-u^*) \tag{15}$$

$$\frac{3}{2}\frac{dT_e}{d\xi} = -\frac{T_e}{u}\frac{du}{d\xi} + \frac{10}{3} R \frac{M_1^2}{M_{A_1}^2}\left[\left(\frac{dB_y}{d\xi}\right)^2 + \left(\frac{dB_z}{d\xi}\right)^2\right] \qquad (16)$$

$$\frac{3}{2}\frac{dT_i}{d\xi} = -\frac{T_i}{u}\frac{du}{d\xi} + \frac{10}{3} V M_1^2 \left(\frac{du}{d\xi}\right)^2 \qquad (17)$$

where the suffix x has been dropped from the velocity u_x.

4. Linearisation about the Singular Points

In the flow through the shock there are two singular points $x = -\infty$ and $x = +\infty$ representing the upstream and downstream flows respectively. The conditions at these singular points are related through the Rankine-Hugoniot equations (Anderson (1962)) with one important exception. For a given upstream flow the sum $(T_e+T_i)_2$ of the downstream temperatures is given but not their ratio $(T_e/T_i)_2$.

To find the nature of the solution near the singular points we linearise about them and assume the small perturbations ($\tilde{u}$, $\tilde{B}_y$, $\tilde{B}_z$, $\tilde{T}_e$ and T_i) vary like $\exp(\lambda\xi)$ in the neighbourhood of such a point. The perturbations have to satisfy

$$\tilde{T}_e = -\frac{2}{3}\frac{T_e}{u}\tilde{u} \qquad (18)$$

$$\tilde{T}_i = -\frac{2}{3}\frac{T_i}{u}\tilde{u} \qquad (19)$$

$$\begin{bmatrix} A - V\lambda & \frac{D}{B_y} & 0 \\ B_y & E - R\lambda & -S\lambda \\ 0 & S\lambda & E-R\lambda \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \tilde{B}_y \\ \tilde{B}_z \end{bmatrix} = 0 \qquad (20)$$

where

$$A = 1 - 1/M^2 \qquad (21)$$

$$E = u - u^* \qquad (22)$$

$$D = B_y^2/M_{A_1}^2 \qquad (23)$$

and M is the local sonic mach number

$$M^2 = M_1^2 \frac{2u^2}{(T_e + T_i)} \tag{24}$$

λ must satisfy the cubic

$$a_1 \lambda^3 + a_2 \lambda^2 + a_3 \lambda + a_4 = 0 \tag{25}$$

where

$$\begin{aligned} a_1 &= V(R^2 + S^2) \\ a_2 &= - A(R^2 + S^2) - 2ERV \\ a_3 &= R(2AE-D) + E^2 V \\ a_4 &= - E(AE-D) \end{aligned} \tag{26}$$

To determine whether or not solution curves exist it is not necessary to solve the cubic (25) but merely to know how many of the roots have positive (negative) real part. The number of roots having positive real part is given by the number of sign changes in the sequence

$$a_1,\ a_2,\ \frac{a_2 a_3 - a_1 a_4}{a_2},\ a_4$$

In this case $a_1 \geq 0$

$$\frac{a_2 a_3 - a_1 a_4}{a_2} = (2AE-D)R - \frac{E}{a_2} (a_1 D + 2E^2 RV^2) \tag{27}$$

and

$$(AE-D) = \frac{1}{u^3} (u^2 - c_f^2)\ (u^2 - c_s^2) \tag{28}$$

where c_f and c_s are the (normalised) fast and slow magneto-acoustic wave speeds.

There are four different singular points (Anderson (1962)) each having different flow conditions as shown in Table 1.

Table 1

Singular point	Flow conditions
I	$M > 1,\ u > c_f > b_x > c_s$
II	$c_f > u > b_x > c_s$
III	$c_f > b_x > u > c_s$
IV	$M < 1, c_f > b_x > c_s > u$

We must now determine the number n of eigen values at each singular point which have positive real parts since Woods (1969b) has shown that a <u>unique</u> shock solution can exist only if

$$n_1 - n_2 = 1$$

where n_1 and n_2 are the values of n at the upstream and downstream points respectively.

Wood's argument is that if the equation for the eigen values is of degree N then there are n_1 and $N-n_2$ amplitude factors to be determined at the upstream and downstream singular points. There will be N variables to be matched at some arbitrary plane between $x = \pm\infty$. One degree of freedom must be left, essentially the position of the shock in the x-direction. Thus for a unique solution we require,

$$n_1 + N-n_2 - N = 1$$

$$n_1 - n_2 = 1.$$

In our more general case there is a slight modification to the argument - we have an additional free parameter at the back singular point, namely, the ratio of the electron and ion temperatures. But since we must match this ratio as well as the other parameters, the final condition on n is unchanged.

5. <u>The Possible Shock Transitions</u>

The nature of the eigen values at the singular points depends on which dissipative mechanisms are present and also on whether or not $B_y = 0$ at one of the singular points. There are five cases to consider.

(i) <u>Resistivity and viscosity</u> ($R \neq 0$, $V \neq 0$)

No matter what value of S is used, the allowed transitions are:

$$\begin{array}{ll} \mathrm{I} \rightarrow \mathrm{II} & \text{(fast)} \\ \mathrm{II} \rightarrow \mathrm{III} & \text{(intermediate)} \\ \mathrm{III} \rightarrow \mathrm{IV} & \text{(slow)} \end{array}$$

Since $n_1 - n_2 = 2$ for I → III, II → IV shocks a single infinity of integral curves exists and no unique structure exists. If $S = 0$, one of these structures maintains $B_z = 0$, so if we imposed the condition $B_z = 0$ (as done by Germain (1960), Kulikovskii and Lyubimov (1961) and Anderson (1962)) unique structure exist for

$$\begin{array}{ll} I \rightarrow II & \text{(Slow)} \\ I \rightarrow III & \text{(Intermediate)} \\ II \rightarrow IV & \text{(Intermediate)} \\ III \rightarrow IV & \text{(Fast)} \end{array}$$

and the $II \rightarrow III$ shock becomes impossible. Notice that in this case the cubic (25) reduces to the quadratic

$$RV\,\lambda^2 - (RA + VE)\lambda + (AE-D) = 0 \qquad (29)$$

with n given by the number of sign changes in the sequence

$$RV,\ -(RA + VE),\ (AE-D)$$

(ii) <u>Resistivity but no viscosity</u> ($R \neq 0$, $V = 0$)

For any value of S, λ is given by the quadratic

$$A\,(R^2 + S^2)\,\lambda^2 - R\,(2AE-D)\lambda + E(AE-D) = 0 \qquad (30)$$

and n by the number of sign changes in the sequence

$$A(R^2 + S^2),\ -R(2AE-D),\ E(AE-D)$$

The allowed transitions are

$$\begin{array}{l} I \rightarrow II\ (M > 1) \\ II(M > 1) \rightarrow III\ (M > 1) \\ III(M < 1) \rightarrow III\ (M < 1) \\ III(M < 1) \rightarrow IV \end{array}$$

and two transonic shocks

$$\begin{array}{l} I \rightarrow III\ (M < 1) \\ II(M > 1) \rightarrow IV \end{array}$$

With $V = 0$ equations (13) - (17) yield

$$\frac{du}{dx}\left(\frac{1}{M^2} - 1\right) = \frac{1}{2M_{A_1}^2}\frac{d}{dx}\left(B_y^2 + B_z^2\right) + \frac{2}{3}\frac{R}{u\,M_{A_1}^2}\left[\left(\frac{dB_y}{dx}\right)^2 + \left(\frac{dB_z}{dx}\right)^2\right] \qquad (31)$$

		$R \neq 0$ $V \neq 0$ any S	$R \neq 0$ $V \neq 0$ $S = 0$ $B_z \equiv 0$	$R \neq 0$ $V = 0$ any S	$R \neq 0$ $V = 0$ $S = 0$ $B_z \equiv 0$	$R = 0$ $V \neq 0$ $S \neq 0$	$R = 0$ $V \neq 0$ $S = 0$ $(B_z \equiv 0)$
Order of System		3	2	2	1	3	1
n at singular points	I $(M > 1)$	3	2	2	1	3	1
	II $(M > 1)$	2	1	1	0	2	0
	II $(M < 1)$	2	1	2	1	2	0
	III $(M > 1)$	1	1	0	0	1	1
	III $(M < 1)$	1	1	1	1	1	1
	IV $(M < 1)$	0	0	0	0	0	0

Table 2. Number n of eigen values having positive real part for different restraints on the system.

Upstream \ Downstream	II ($M > 1$)	II ($M < 1$)	III ($M > 1$)	III ($M < 1$)	IV ($M < 1$)
I ($M > 1$)	($R \neq 0$) or ($V \neq 0$)	($V \neq 0$)	($B_z \equiv 0, R \neq 0$)	($B_z \equiv 0, R \neq 0, V \neq 0$) or ($V = 0$)	($B_z \equiv 0, R \neq 0$, $V = 0$)
II ($M > 1$)	$\ne$	$\ne$	($R \neq 0$) or ($S \neq 0$)	($R \neq 0$, $V \neq 0$) or ($S \neq 0$, $V \neq 0$)	($B_z \equiv 0, R \neq 0$, $V \neq 0$) or ($V = 0$)
II ($M < 1$)	*	$\ne$	*	($R \neq 0$) or ($S \neq 0$)	($B_z \equiv 0, R \neq 0$)
III ($M > 1$)	*	*	$\ne$	$\ne$	($V \neq 0$)
III ($M < 1$)	*	*	*	$\ne$	($R \neq 0$) or ($V \neq 0$)

Table 3. Restrictions necessary for shock transitions.

hence I $\rightarrow$ II, II $\rightarrow$ III, III $\rightarrow$ IV shocks cannot cross $M = 1$ without an infinite gradient in u. This point was first noted by Geiger et al (1962). With II $\rightarrow$ IV and I $\rightarrow$ III shocks, crossing $M = 1$ removes their degree of freedom by imposing the condition that when $M = 1$ the right hand side of (31) is also zero and a unique structure results.

The limitations on M lead to critical Mach numbers. For example transitions of the type I $\rightarrow$ II ($M > 1$) only exist over a limited range of M_{A_1}. Above a critical value for the upstream Alfvén Mach number the Rankine-Hugoniot relations require $M < 1$ at singular point II. This question has been discussed in detail by Woods (1969a) for normal shocks.

If $S = 0$ and we insist $B_z \equiv 0$, from (29)

$$\lambda = \frac{AB-D}{RA}$$

and the allowed transitions are

I $\rightarrow$ II ($M > 1$)   I $\rightarrow$ III ($M > 1$)

III ($M < 1$) $\rightarrow$ IV   II ($M < 1$) $\rightarrow$ IV

plus the transonic intermediate shock

I $\rightarrow$ IV .

(The II ($M < 1$) $\rightarrow$ III ($M > 1$) is eliminated since the entropy must increase through the shock). For the I $\rightarrow$ IV shock in (i) $n_1 - n_2 = 3$ giving two degrees of freedom to the integral curve. However in this case, since the flow speed crosses the sound speed in the absence of viscosity and also $B_z \equiv 0$, two constraints have been imposed on the system and a unique structure exists.

(iii) <u>Viscosity but no resistivity</u> ($R = 0$, $V \neq 0$)

If $S \neq 0$ the allowed transitions are

I $\rightarrow$ II

II $\rightarrow$ III

III $\rightarrow$ IV

If however $S = 0$, then from (15) and (20) $B_z \equiv 0$ and from (29)

$$\lambda = \frac{AB-D}{VB} ,$$

and the allowed transitions appear to be

$$I \rightarrow II, \; III \rightarrow IV \text{ and } I \rightarrow IV.$$

But from (14) with $R = S = 0$

$$B_y(u - u^*) = \sin\theta\,(1 - u^*) \qquad (32),$$

so no intermediate shock is possible, and the $I \rightarrow IV$ shock has to be eliminated.

(iv) No Dissipation

λ is given by

$$\lambda^2 = -\frac{E(AE-D)}{AS^2} \qquad (33).$$

Therefore either one root is positive and the other negative or both are imaginary. No shocks are possible but provided $S \neq 0$ there may be a wave which stands in the flow. The wave number $k = i\lambda$ so, on eliminating A,E,D and S the wavelength $(2\pi/k)$ is given in terms of the normalised speeds.

$$\text{WAVELENGTH} = 2\pi \left[\frac{({m_i}/{m_e})\ u^3 b_x^2 (u^2-a^2)}{(u^2-c_f^2)(u^2-b_x^2)(u^2-c_s^2)}\right]^{\frac{1}{2}} \qquad (34),$$

which is equivalent to the dispersion relation given by Stringer (1963) for low frequency waves.

(v) Switch Shocks ($B_y = 0$)

If $B_y = 0$ at one of the singular points, we may have the limiting cases of switch-on and switch-off shocks.

The switch-on shock is a fast shock propagating parallel to the magnetic field in $\beta < 2/\gamma$ plasma. Behind the shock a transverse magnetic field (B_y) is switched on and $u = b_x$ i.e., $E = 0$.

The switch off shock is the strongest possible slow shock ahead of which $u = b_x$ ($E = 0$) and behind $B_y = 0$.

When $E = 0$, λ is given by

$$\lambda = 0 \text{ or } c_1\lambda^2 + c_2\lambda + c_3 = 0$$

where

$$c_1 = V(R^2 + S^2) \geq 0$$
$$c_2 = -A(R^2 + S^2)$$
$$c_3 = -RD \leq 0$$

There is always only one sign change in the sequence (c_1, c_2, c_3) so there is one zero, one positive, and one negative eigen value. Since we need exactly one negative eigen value behind a fast shock and one positive eigen value in front of the slow shock, a unique solution curve exists for both switch-off and switch-on shocks.

If we put viscosity (V) equal to zero, then when $E = 0$

$$\lambda = 0 \text{ or } \lambda = -\frac{RD}{A(R^2+S^2)}$$

and the switch shocks are only possible provided we have $M > 1$ ($A > 0$) throughout the switch-on, and $M < 1$ ($A < 0$) throughout the switch-off shock, so again transonic shocks are not allowed.

If we put the resistivity (R) equal to zero then when $E = 0$

$$\lambda = 0 \text{ or } \lambda = A/V,$$

and transitions are only possible if they are transonic. The shocks are however not switch-shocks, but purely gas dynamic. Putting $u = b_x$ where the flow is parallel to the field does not uniquely determine a switch shock, for there is also a gas shock satisfying the same criterion. The fact that a switch-on shock cannot exist if the resistivity is zero can be seen by casting equations (14) and (15) in the form

$$S \frac{d}{dx} (B_y^2 + B_z^2) = 0 \qquad (35)$$

when $R = 0$ and $\theta = 0$.

It is clear that all the results of this section are unchanged if we neglect the Hall term ($S = 0$).

The results of (i) (ii) (iii) are summarised in Tables 2 and 3. The values of n are listed in Table 2. Clearly point I can never be a downstream point, while IV can never be an upstream one. Table 3 shows what restrictions are necessary for each shock transition to be possible, and also which transitions can never occur. Those marked * correspond to decreasing entropy, and those marked $\ddagger$ can never satisfy $n_1 - n_2 = 1$.

6. Computing Shock Wave Structures

To compute a shock structure we must integrate the full non-linear shock structure equations from one singular point to another. In order to start the computation, we leave a singular point using perturbations of the parameters (B_y, B_z, u, T_e and T_i) that satisfy the linearised equations (18) - (20). The properties of the eigen values at the singular points determine the method of computation for we cannot enter a singular point unless the roots are such that all perturbations decay as the solution approaches it. For example, in the case of a fast (I $\rightarrow$ II) shock (Table 2) the roots upstream all have positive real parts, while downstream two are positive and one is negative. In this case, it is necessary to compute from the downstream point to the upstream one. A further complication is that the Rankine-Hugoniot relations give us the sum of the electron and ion temperatures downstream, but not their ratio. Thus we must guess a value for the ratio, compute towards the upstream point, and if the electron and ion temperatures do not approach the same upstream value, we must correct the guess and calculate the structure again. In the case of the slow (III $\rightarrow$ IV) shock, all the downstream eigen values have negative real part, and it is possible to compute from the upstream point (where there is one positive eigen value) to the downstream point in a straightforward manner.

For the intermediate (II $\rightarrow$ III) shock of 5(i), the situation is complicated by the appearance at each singular point of two eigen values giving growing perturbations. Thus a solution curve cannot enter either point and a degree of freedom is allowed on the perturbation used to leave either point - the proportion of the two eigen-perturbations. We again have the problem that we do not know the separate electron and ion temperatures at the downstream point, but in order to simplify the problem we impose the constraint $T_e = T_i$ in this case. By integrating from the front and back points and varying the perturbation, the solution curve can be found by matching in the middle.

7. Results

With the exception of the intermediate (II → III) shock, the parameters were chosen to match those of Paul et al (1965) for normal shocks and Robson and Sheffield (1968) for oblique shocks. For the intermediate shock they were chosen arbitrarily to make all the scale lengths similar in order to ease the matching problem.

(a) Fast Shocks

Computed structures for the normal case with the Alfvén Mach number less than and greater than the critical value (M_A^*) are shown in Figures 2 and 3. Note that in both cases the B_z component of the magnetic field remains zero throughout the structure. This is to be expected since the Hall term (S) is zero for perpendicular propagation.

In the case $M_A < M_A^*$ the electrons receive the bulk of the irreversible heating, the ion heating being mainly due to adiabatic compression with the ion viscosity playing a negligible role. The velocity and magnetic field vary smoothly through the shock on the same scale length, determined essentially by the resistivity(R). By contrast when $M_A > M_A^*$, the ions receive substantial irreversible heating so that $T_i > T_e$ at the back of the shock. The velocity undergoes a sharp change as the flow becomes subsonic and the scale length of this 'sub-shock' is determined by the viscosity V. Since $R = R_1 \, T_e^{-3/2}$ and $V = V_1 \, T_i^{5/2}$ we have in this case $R \sim V$ in the region of the subshock, and a clear distinction between the effects of resistivity and viscosity is not possible in Figure 3. However, by making $V \ll R$, such a distinction becomes very clear.

In the experiments of Paul et al (1965, 67, 68, 69) the electron temperature is measured directly, whereas the ion temperature must be inferred by subtraction from the Rankine-Hugoniot value for the sum of the two temperatures. Analysed in this way the experiments show that the irreversible heating goes entirely into the electrons for $M_A < M_A^*$, while for $M_A > M_A^*$ an increasing fraction goes into the ions. In this general sense the computed structures and experiments agree. However, the computed scale length for the magnetic field variation is approximately an order of magnitude less than the experimental value. This is natural since we have taken the classical resistivity in the computation, whereas the resistivity in the experiments is enhanced by the electron-drift excitation of the ion waves (Sagdeev (1967)).

In the case of oblique shocks the computed structures (Figures 4 and 5) show rotation of the magnetic field vector through the shock as represented by oscillating B_y and B_z components. This is a whistler wave standing in the shock front which agrees with the experimental observations of Robson and Sheffield (1968). The computed wavelength agrees well with the linear theory of equation (34). We note that for high Mach numbers and low β plasma the whistler wavelength is approximately

$$\lambda_w = 2\pi \sqrt{\frac{m_i}{m_e}} \; \frac{\cos\theta}{M_{A_1}} = 2\pi S \qquad (36).$$

As with the normal shock, a sub-shock occurs when $M_{A_1} = 4.0 > M_A^*$, but the effect is more marked in this case. The velocity changes rapidly over a scale length determined by the ion viscosity, whereas the magnetic field is sensibly constant. Corresponding to this sub-shock there is strong irreversible ion heating. Note, however, that the pre-heating of the electrons by the currents in the whistler is such that even when $M_{A_1} > M_A^*$ the electron temperature still exceeds the ion temperature at the back of the shock. In the case $M_{A_1} = 2.0 < M_A^*$ only a vestigial whistler oscillation occurs, essentially because of the strong resistive damping. If the value of R is increased sufficiently the oscillation can be damped out so that only a monotonic transition in B_y occurs.

In summary we can say that there are three lengths in the problem corresponding to resistivity (R) viscosity (V) and Hall effect (S). The viscosity controls the scale length over which the flow velocity u can vary, while the resistivity and Hall effect control the magnetic field. If the resistive length is substantially longer than the "Hall length" there will be no whistler oscillations. If the opposite is true, there will be a large number of oscillations roughly proportional to the ratio of the resistive length to the wavelength. Note that since resistivity and viscosity depend sensitively on temperature, the effective damping lengths will show considerable variation over their upstream values. Thus it is not possible in some cases to predict the form of the structure simply from a knowledge of the upstream values.

A polar diagram showing the critical Alfvén Mach number M_A^* as a function of θ for different upstream plasma conditions is shown in Figure 6. Note that the critical Mach number is reduced as the ratio β ($\beta = 16\pi n_1 kT_1/B_o^2$) of the upstream plasma is increased.

(b) <u>Switch-on Shocks</u>

An example of a switch-on shock structure is given in Figure 7. Most of the heating goes into the electrons. There is again a whistler oscillation at the front of the shock and the wavelength agrees well with equation (34) which for parallel propagation takes the form

$$\lambda_w = 2\pi \sqrt{\frac{m_i}{m_e}} \frac{M_{A_1}}{M_{A_1}^2-1} = 2\pi S \frac{M_{A_1}^2}{M_{A_1}^2-1} \qquad (37).$$

Because, in this case, the downstream flow is subsonic, a viscous subshock appears at the back.

(c) <u>Intermediate Shocks</u>

(i) <u>II → III shock</u>

In finding the structure of a II → III shock (with electron and ion temperatures assumed equal) we examined the integral curves leaving the downstream point for different (linear) perturbations. The projections of some of these curves on the (u, B_y) and (B_z, B_y) planes are shown in Figures 8 and 9. The curves cannot enter the "saddle point" at II so they diverge to end with $u \to 0$ or on another singular point, namely point I. Two solution curves were found which went within a small perturbation of the conditions at point II, hence the "matching in the middle" could be performed at the front point. There is clearly an infinite set of solution curves joining III to I showing explicitly the non-uniqueness of the I → III shock structure. (We also see the unique I → II transition).

To understand the extent of the non-uniqueness of the II → III transition, consider the case S = 0. From (15)

$$\frac{dB_z}{d\xi} = B_z \,(u - u^*) \qquad (38),$$

hence $|B_z|$ grows to some maximum value where the flow speed crosses the normal Alfvén speed and then decays to zero. Since B_z only occurs squared in the other equations, it is only determined in absolute value. Therefore, there are two solutions having opposite polarisation. This is to be expected since a large amplitude intermediate wave may have either polarisation. Thus, there is a unique structure for each polarisation.

The two solutions are shown in Figures 10 and 11. They have opposite polarisation at the front where the non-linear terms dominate the dispersive effects. At the back, however, (where the dispersive effects dominate) they are both polarised in the opposite sense to that of the whistler. This is due to a higher frequency intermediate wave which follows the shock. Since its velocity is slightly greater than the sound speed, it is identified from the linear theory of Stringer (1963) as an ion cyclotron/acoustic wave. The wave is strongly damped because the local value of V (~280) is larger than the wavelength (~235) given by (34). With reduced dissipation we have shown that the wave becomes a much more dominant feature with the wavelength agreeing with (34) when $V, R \ll S$ at the back of the shock.

We also see that due to the effects of the Hall term, the right hand polarised case shows an expansion preceding the compression.

(ii) I → III Shock

By neglecting the Hall term and imposing the condition $B_z \equiv 0$ we found the I → III structure shown in Figure 12. In this case the transverse magnetic field reverses by decreasing through zero to its final value. The shock is first expansive and then compressive as predicted by Anderson (1962).

Note however that this is only one of the infinite set of I → III shock structures which exist when $S = 0$, namely, the one which has $\tilde{B}_z = 0$ when leaving point III.

(d) Switch-off Shocks

The structure of a switch-off shock is shown in Figure 13. The flow is supersonic upstream and subsonic downstream, so a viscous subshock occurs, this time at the front of the shock. Consequently, the ions are initially heated more than the electrons, but subsequent ohmic heating makes the electrons hotter at the back of the shock. The oscillations in B_y and B_z are in the opposite sense to the whistler wave, and are caused by a heavily damped electro-acoustic wave (Stringer, 1963). The wavelength agrees well with (34) which gives the dimensionless value of 208.

(e) Slow Shocks

Computed structures for a subsonic and a supersonic slow shock are shown in Figures 14 and 15. Both show changes over a scale length defined by the resistivity, with the second one having a subshock as the flow becomes subsonic. The electrons receive most of the irreversible heating. B_z varies slightly through both shocks in the same sense as the electro-acoustic wave (Stringer, 1963). By decreasing the resistivity, a wave can be seen following the shock. The predicted wavelengths (Eqn.34) in the two cases are 126 and 137, so they are damped out by resistivity in the case considered here.

Note that the magnetic field decreases across this shock.

Conclusions

We have shown that the only shocks to have a unique structure are the I → II fast shock, the right and left hand polarised II → III intermediate shocks, the III → IV slow shock, and the switch shocks. We have computed the structure in each case. All the other transitions are only possible with special limitations, and so must be considered structurally unstable.

With one exception (when an anomalous resistivity coefficient would have been more realistic) the magnetic field varies over a scale long compared with the collision-free skin depth, thus justifying the neglect of electron inertia. In some cases the neglect of other physical effects such as thermal conduction and shear viscosity is probably not justifiable, but their inclusion would clearly complicate an already extremely complex multi-parameter problem.

Good agreement with experimental data on fast shocks can be obtained, although to do this it is sometimes necessary to use fitted (enhanced) values for the effective resistivity. As yet there are no experimental results for switch-on, intermediate, switch-off and slow shocks.

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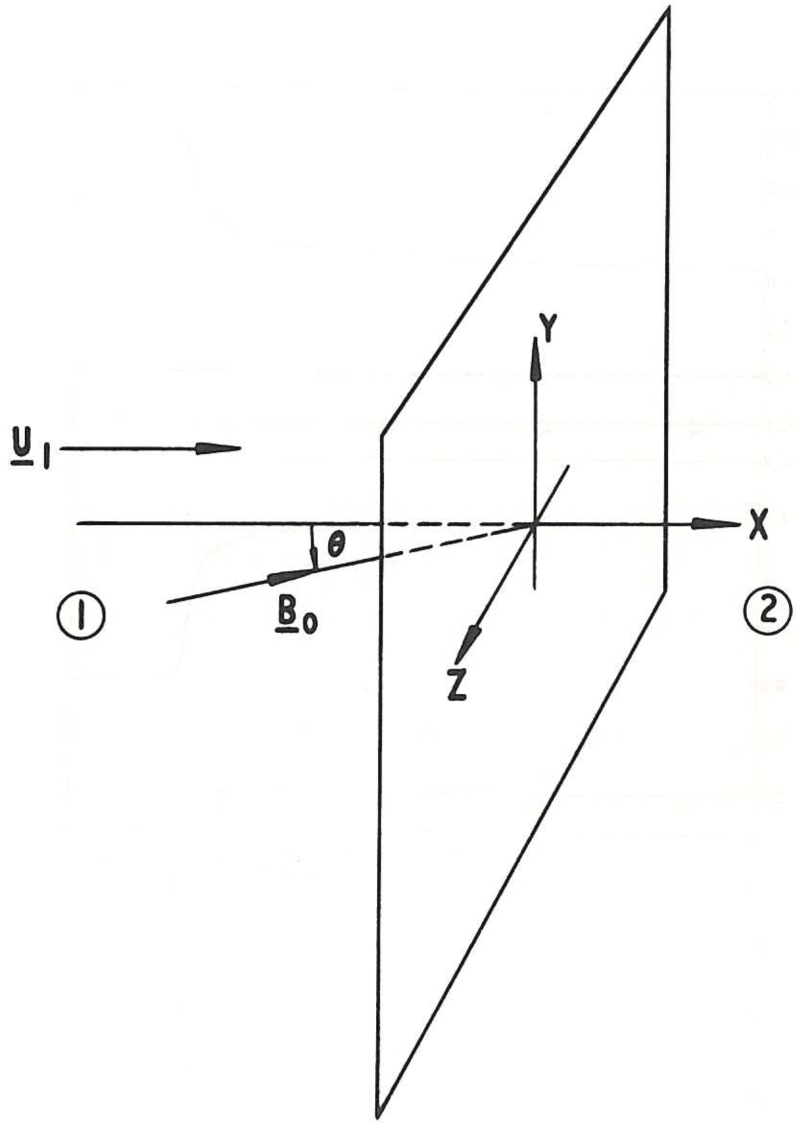

Fig.1 The frame of reference CLM - P 246

$\theta = 90.0^0 \quad M_{A1} = 2.5000 \quad M_{a1} = 13.16 \quad M_{a2} = 1.13$

$R = 6.90 \quad V = 0.0005$

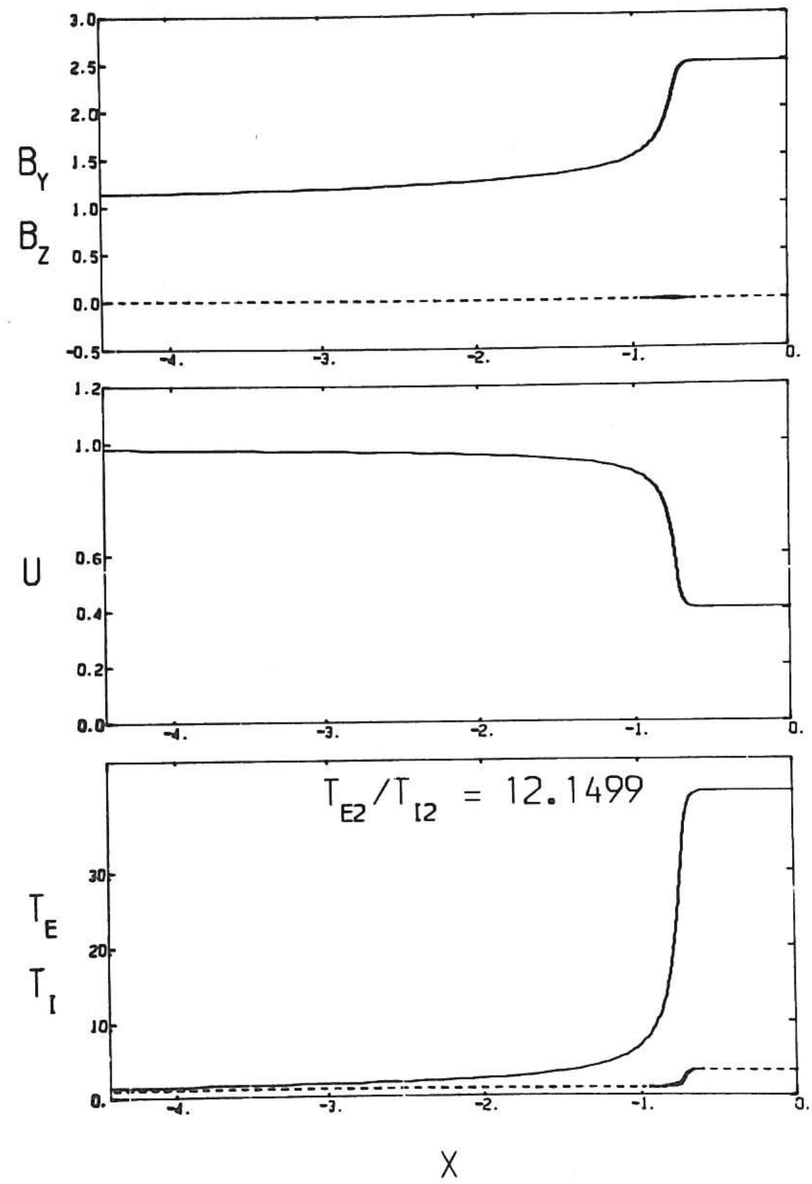

Fig.2 Supersonic normal shock CLM-P 246

$\theta = 90.0^0 \quad M_{A1} = 3.7000 \quad M_{a1} = 12.76 \quad M_{a2} = 0.73$

$R = 6.90 \quad V = 0.0052$

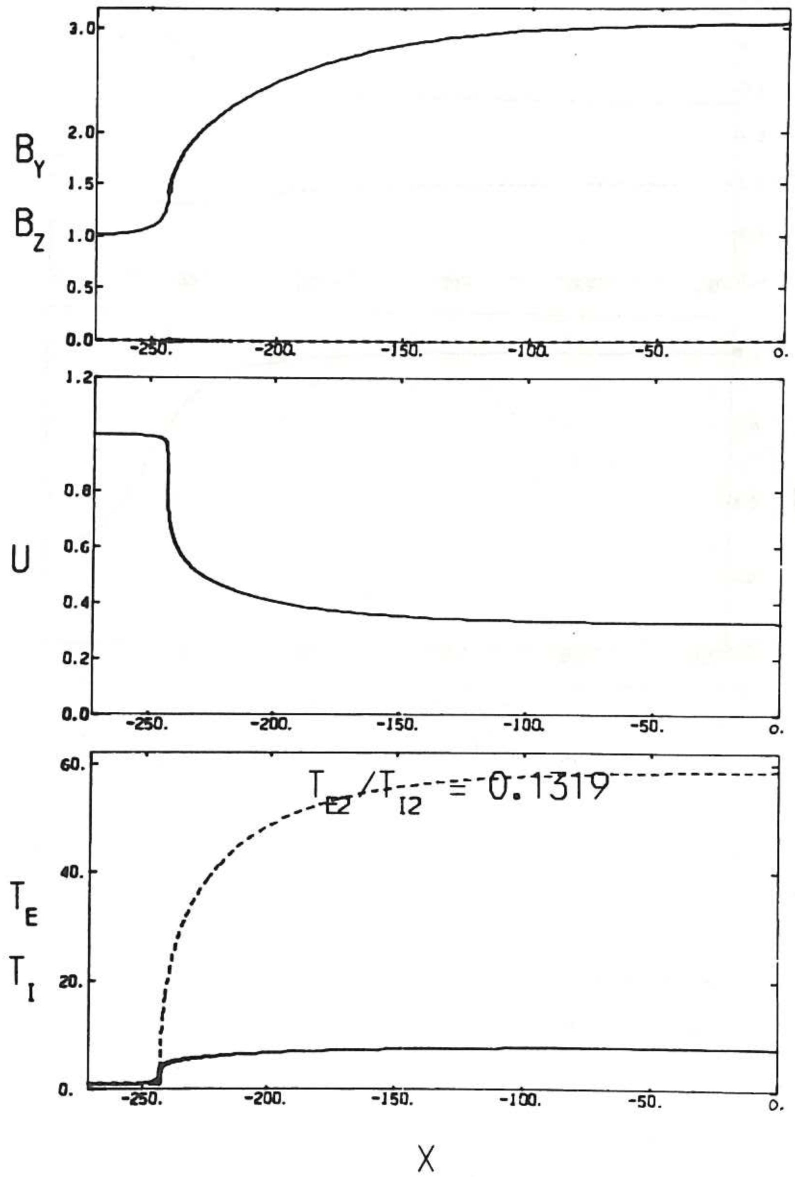

Fig.3 Transonic normal shock $\left(M_A > M_A^*\right)$ CLM - P 246

$\theta = 57.3^0 \quad M_{A1} = 2.0000 \quad M_{a1} = 5.56 \quad M_{a2} = 1.32$

$R = 50.00 \quad V = 0.0050$

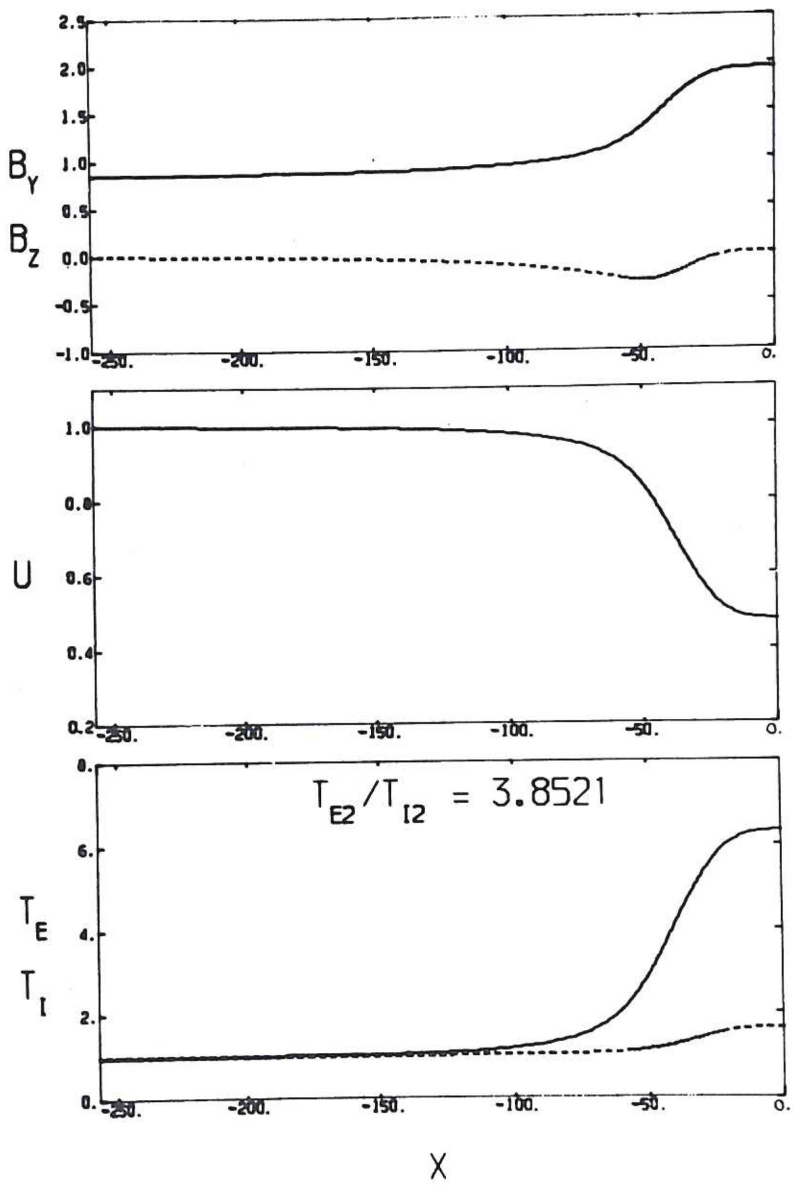

Fig.4 Fast oblique supersonic shock CLM - P 246

$\theta = 57.3^0$ $M_{A1} = 4.0000$ $M_{a1} = 11.11$ $M_{a2} = 0.63$

$R = 25.00$ $V = 0.0025$

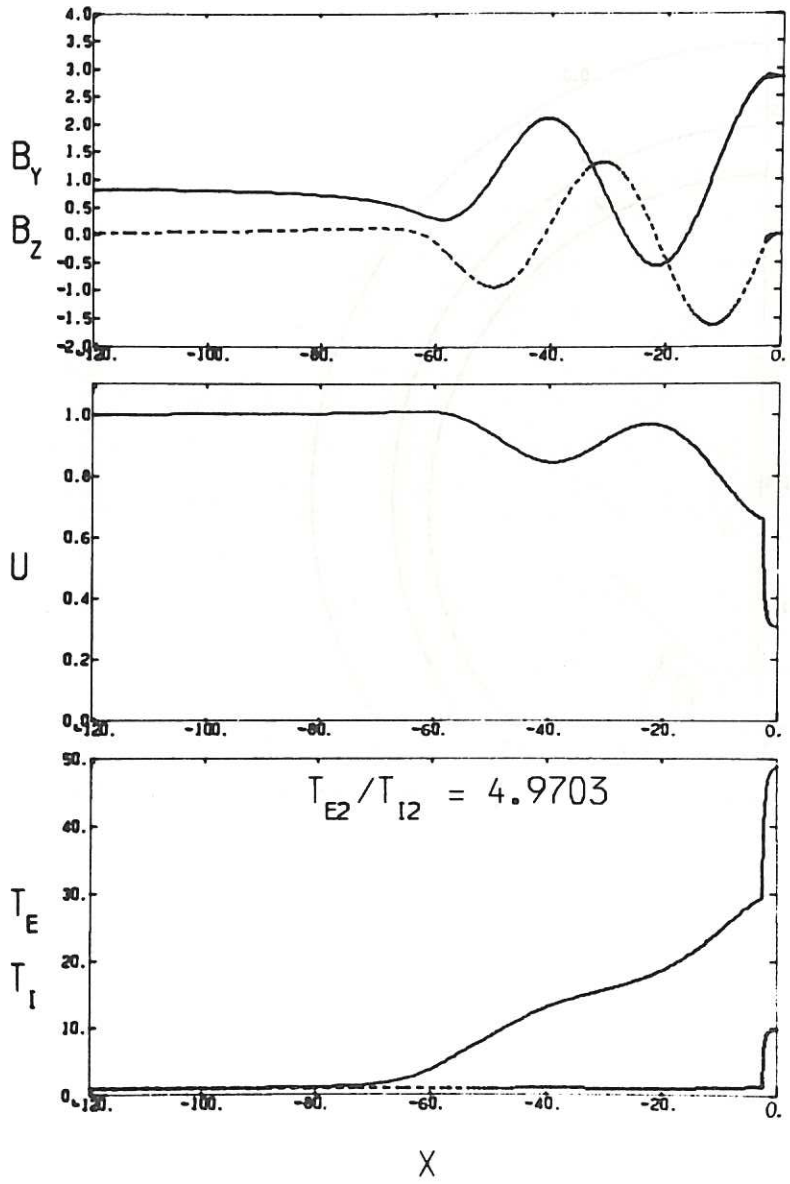

Fig.5 Fast oblique transonic shock CLM - P 246

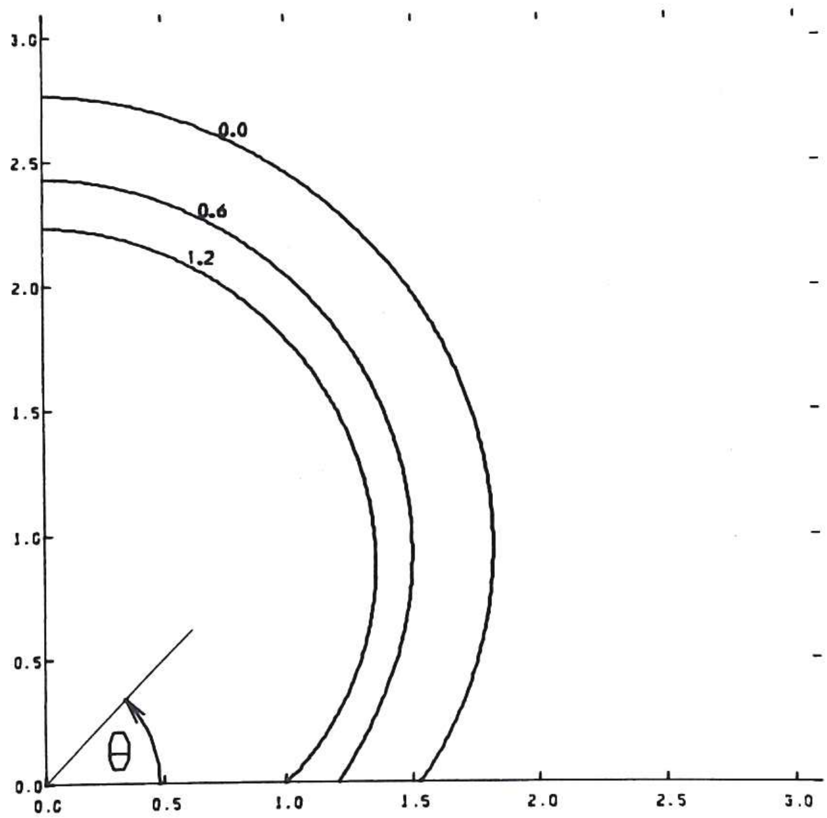

Fig.6 Polar plot of M_A^* for different values of β CLM-P 246

$\theta = 0.0^0 \quad M_{A1} = 1.5000 \quad M_{a1} = 4.17 \quad M_{a2} = 0.89$

$R = 66.70 \quad V = 0.0067$

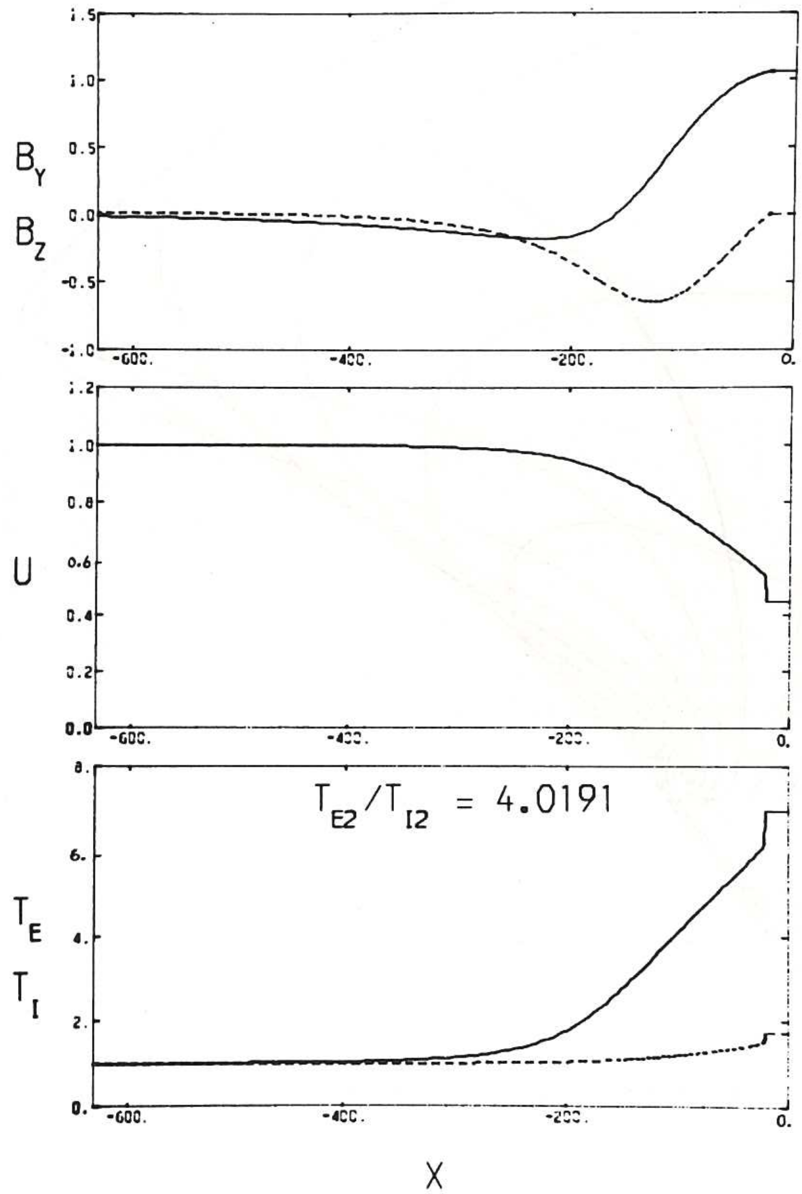

Fig.7 Transonic switch-on shock CLM-P 246

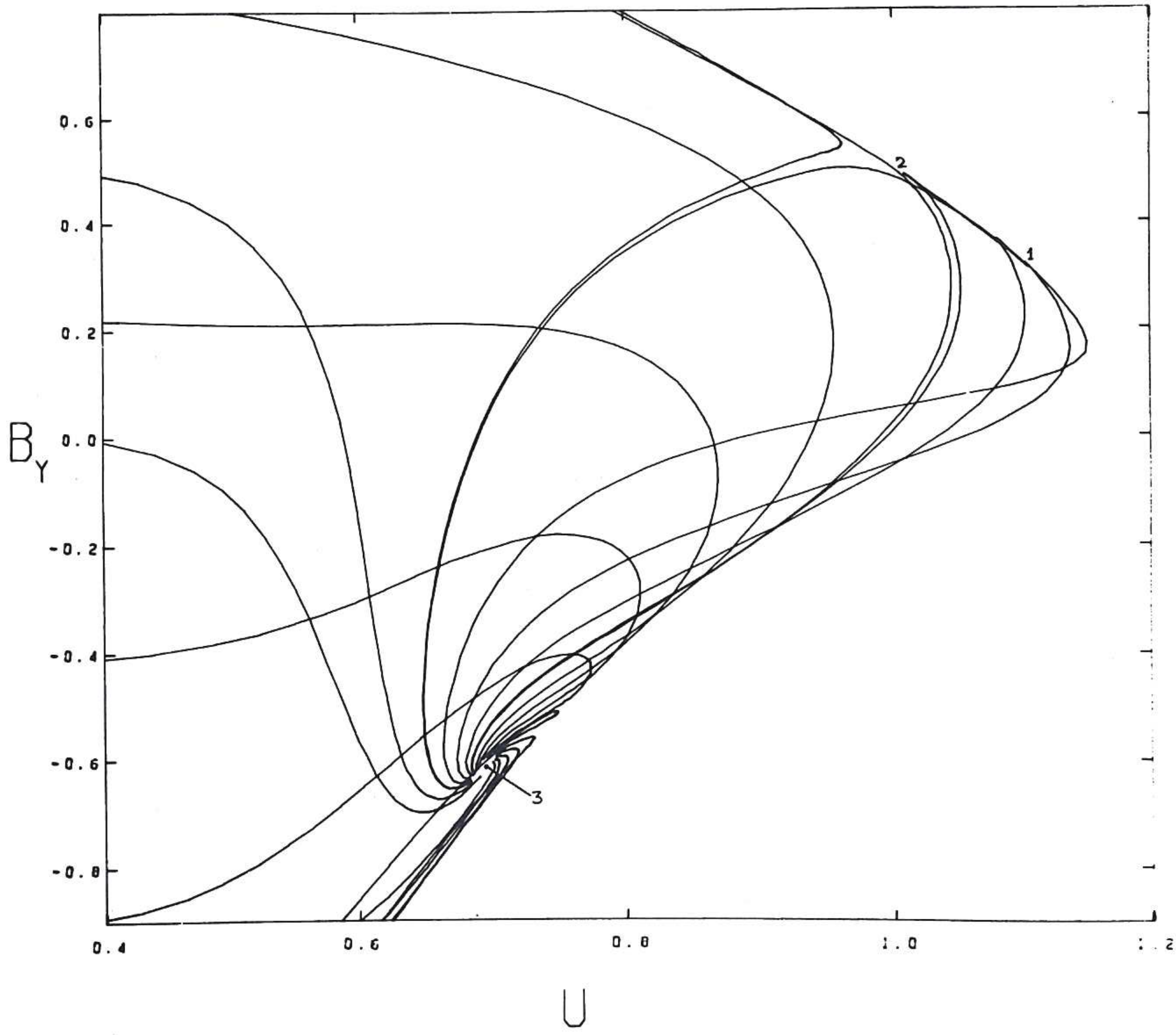

Fig.8 Projections of the integral curves on the (By , u) plane CLM - P 246

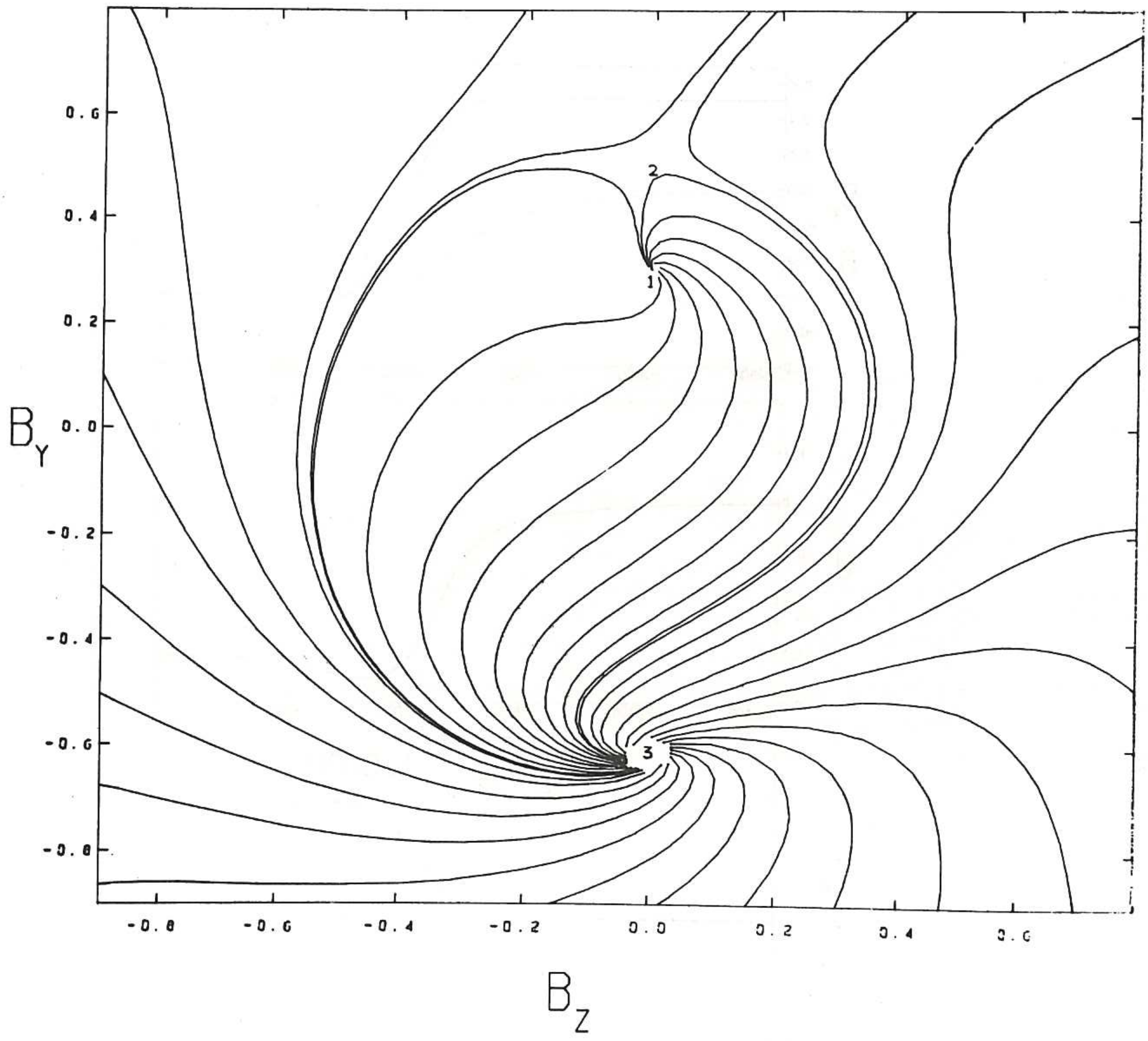

Fig.9 Projections of the integral curves on the (B_y, B_Z) plane CLM-P 246

$\theta = 30.0^0$ $\quad M_{A1} = 0.9500$ $\quad M_{a1} = 2.12$ $\quad M_{a2} = 1.06$

$R = 50.00$ $\quad V = 50.0000$

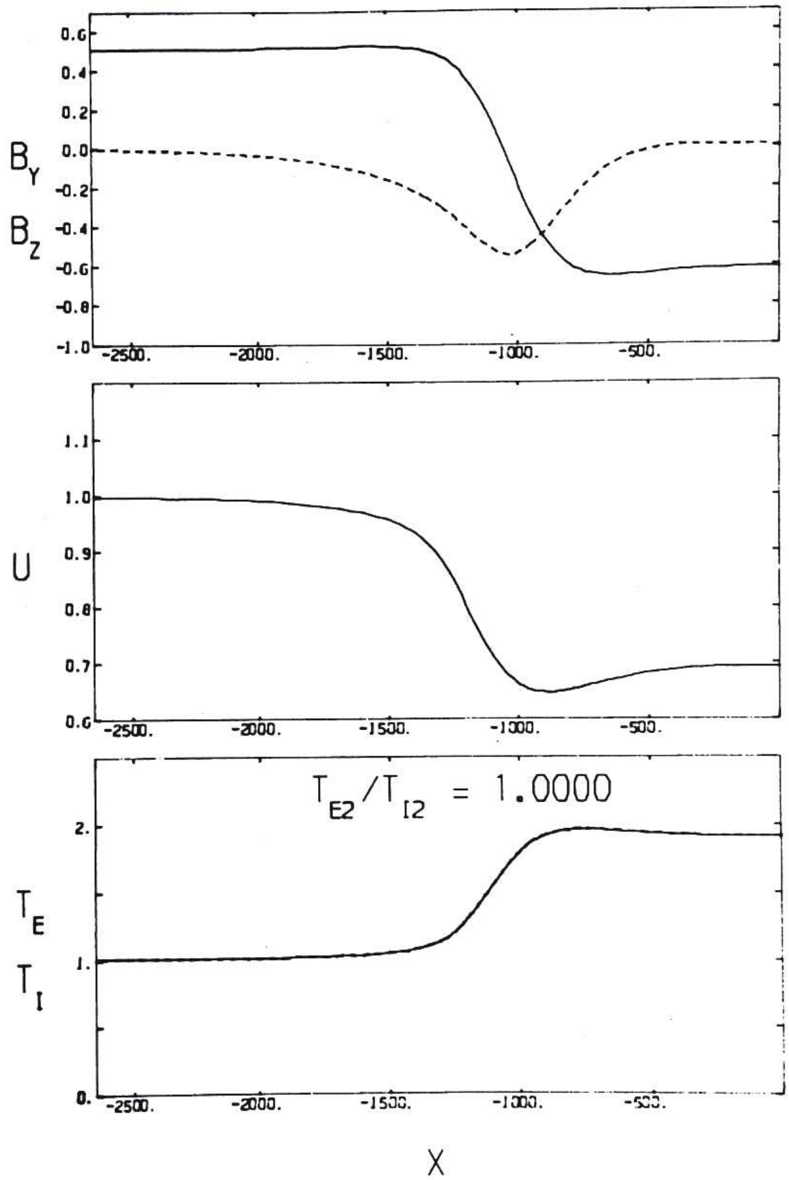

Fig. 10 Left-polarised intermediate II→III shock CLM – P 246

$\theta = 30.0^{0}$ $M_{A1} = 0.9500$ $M_{a1} = 2.12$ $M_{a2} = 1.06$

$R = 50.00$ $V = 50.0000$

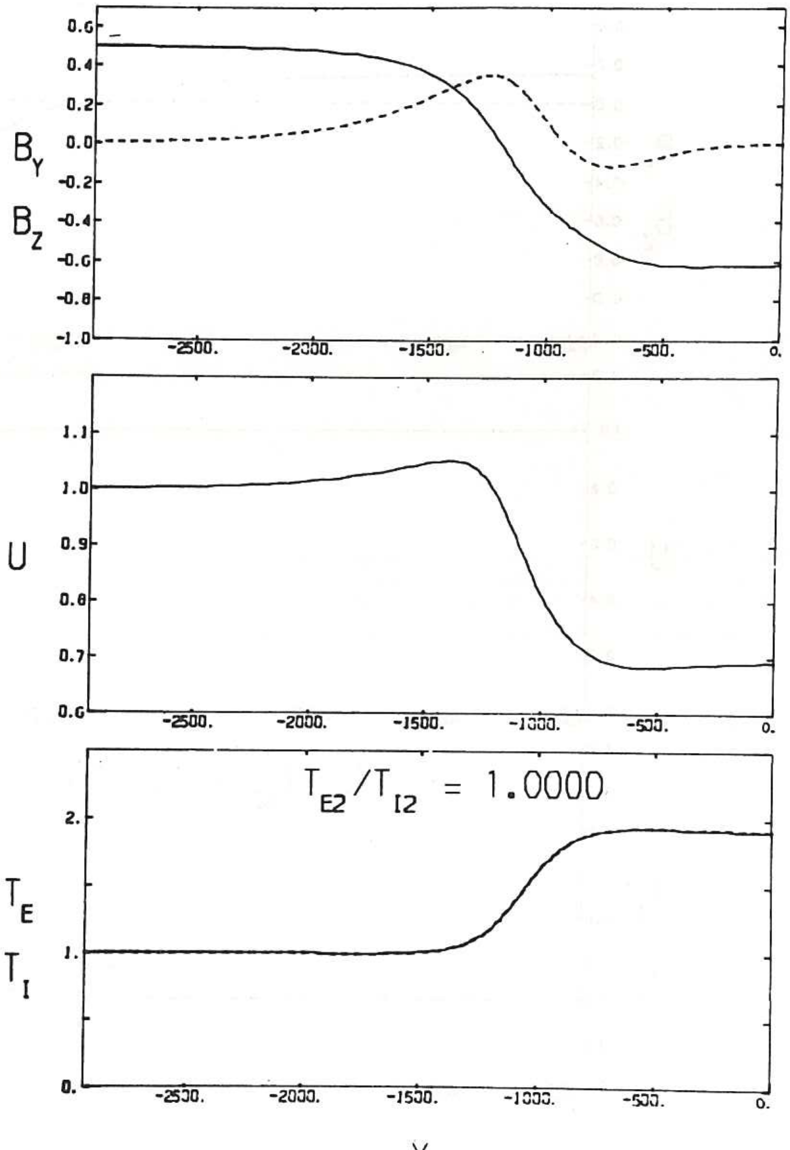

Fig. 11 Right-polarised intermediate II→III shock CLM - P 246

$\theta = 8.6^0 \quad M_{A1} = 1.3000 \quad M_{a1} = 3.61 \quad M_{a2} = 0.99$

$R = 77.00 \quad V = 0.0077$

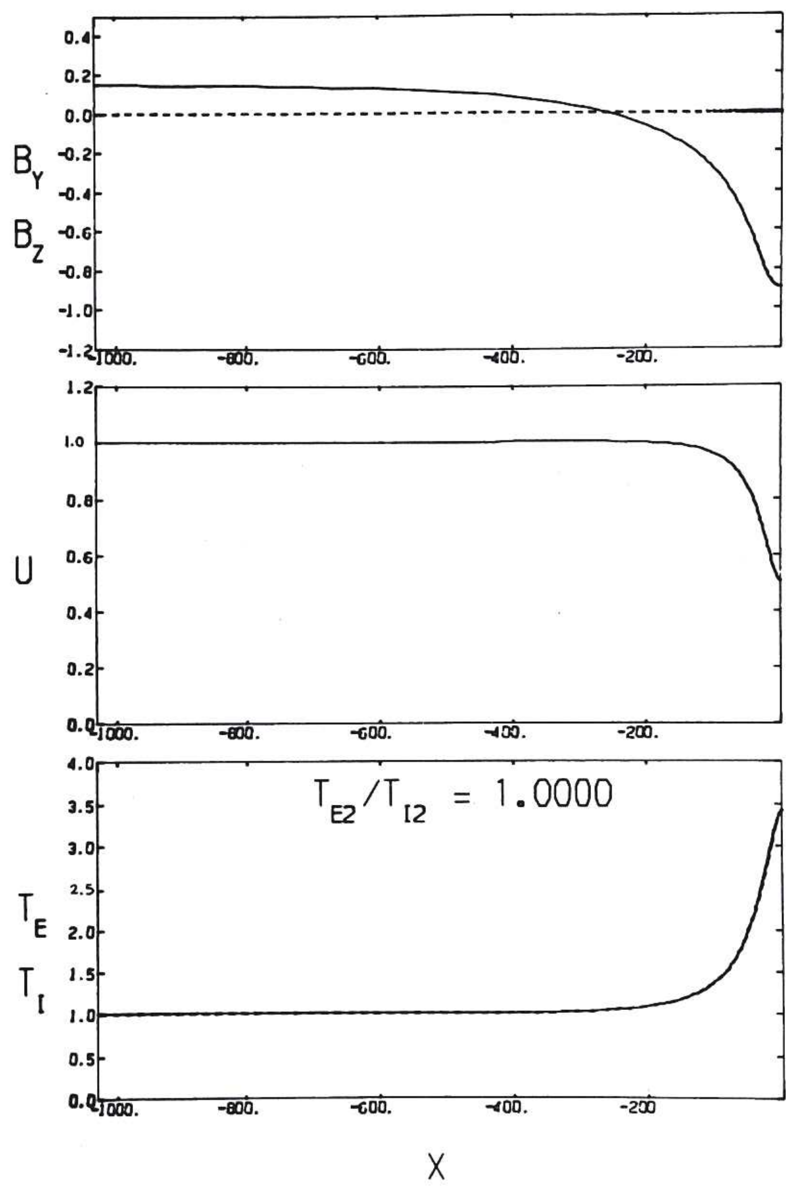

Fig.12 Intermediate I→III shock CLM-P 246

$\theta = 45.0^0 \quad M_{A1} = 0.7071 \quad M_{a1} = 1.96 \quad M_{a2} = 0.45$

$R = 141.46 \quad V = 0.0141$

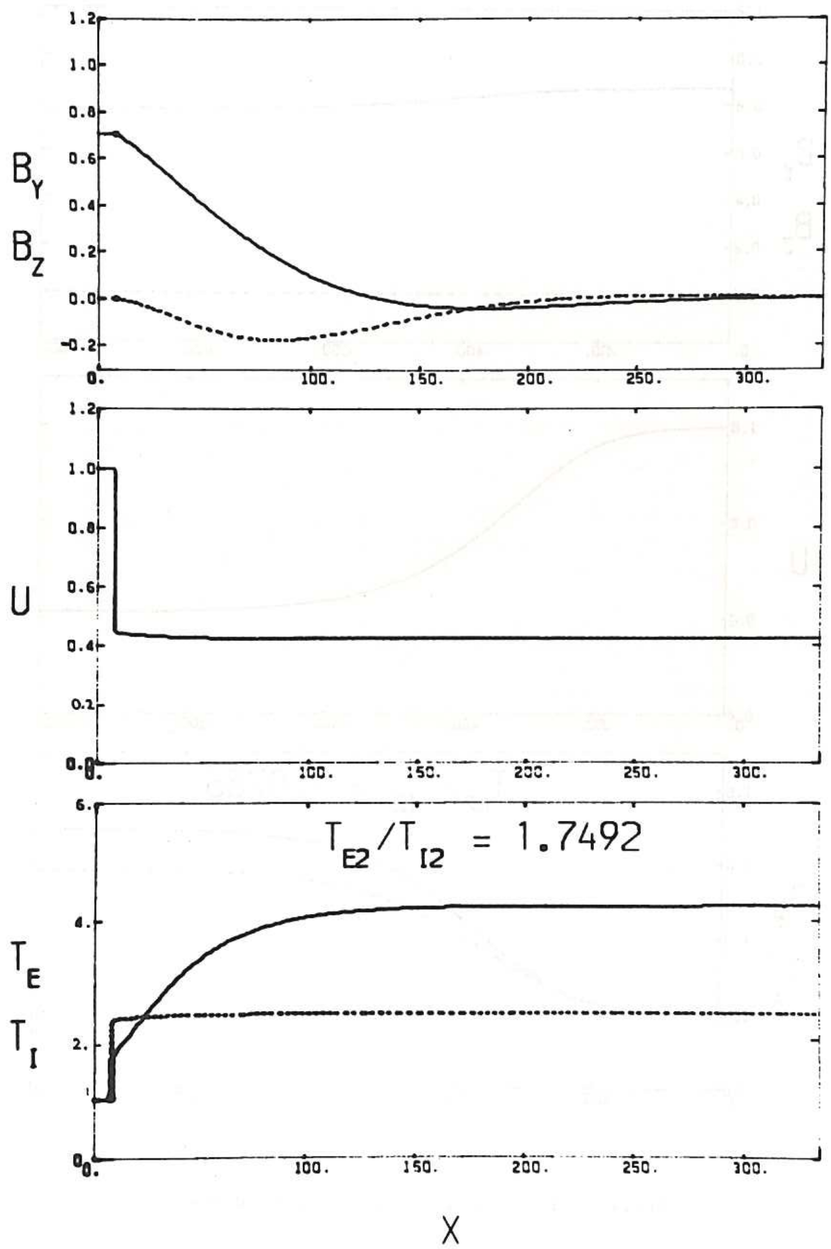

Fig. 13 Switch-off shock CLM-P 246

$\theta = 60.0^0 \quad M_{A1} = 0.2500 \quad M_{a1} = 0.69 \quad M_{a2} = 0.35$

$R = 400.00 \quad V = 0.0400$

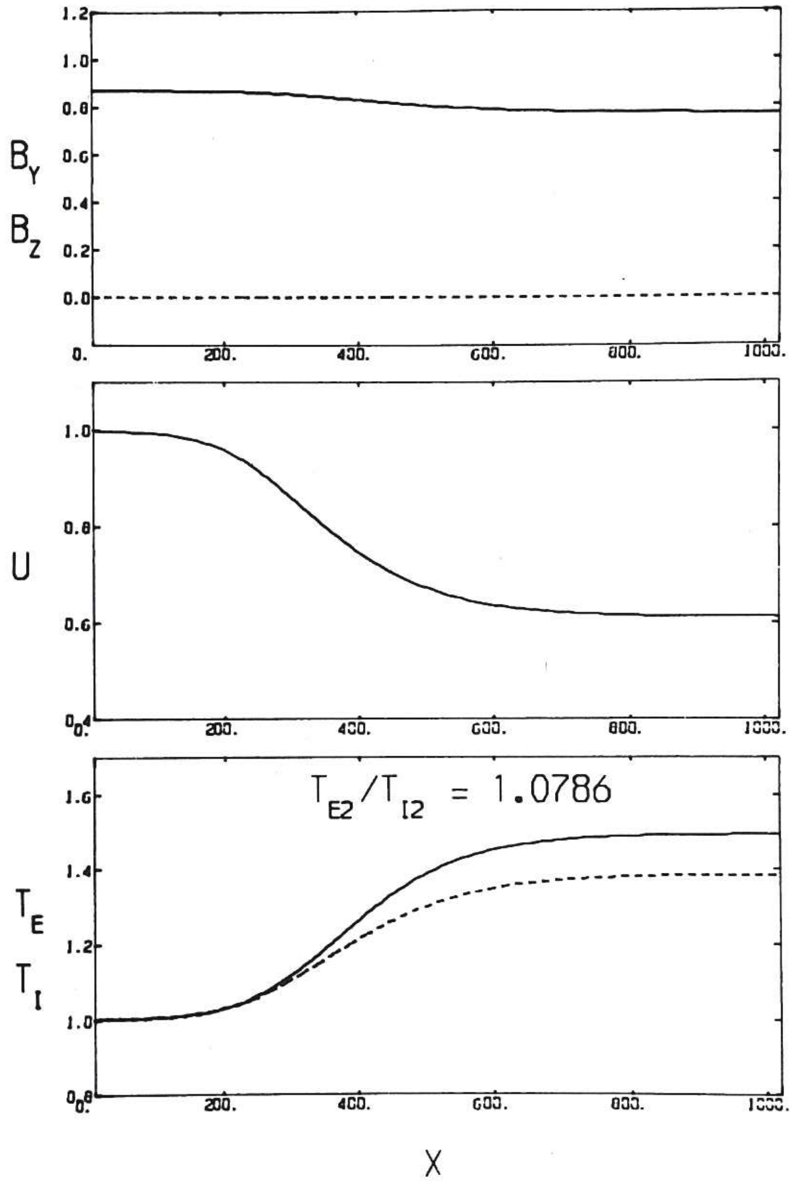

Fig.14 Slow subsonic shock CLM-P 246

$\theta = 60.0^0 \quad M_{A1} = 0.4000 \quad M_{a1} = 1.11 \quad M_{a2} = 0.30$

$R = 250.00 \quad V = 0.0250$

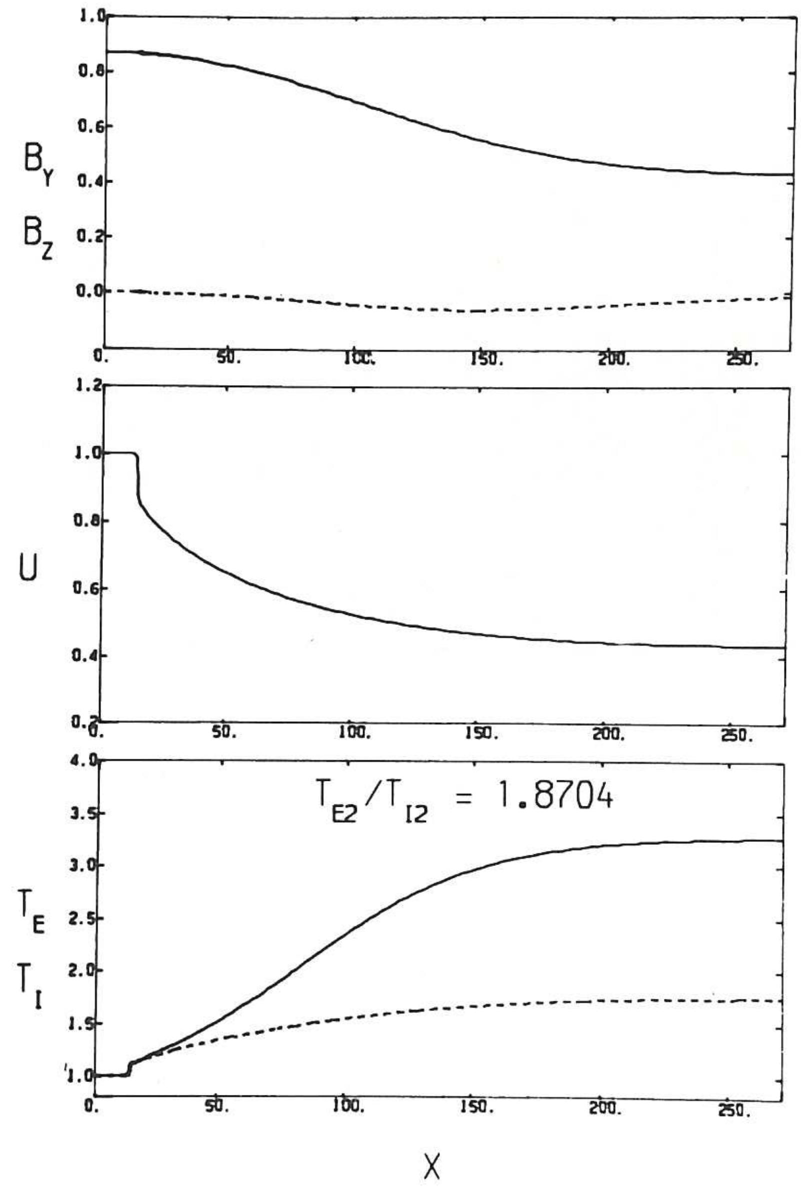

Fig.15 Slow transonic shock CLM-P 246