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by

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ABSTRACT

The essential requirements for a stable diffuse pinch configuration are derived by considering the hydromagnetic energy principle. The two basic variables are the radial variation of the pitch of the field lines and the pressure gradient. If the radial variation of the pitch exhibits a minimum then the configuration will be unstable. In order to confine a plasma of appreciable beta (10's of %) with a vacuum region outside the plasma, it is shown that the axial field must be reversed and the axial flux must not reverse. Alternatively, stability is possible if large axial currents flow outside the main plasma column. It is proved that the value of beta with respect to the field produced by the current must be less than unity for stability; the maximum value for stability is found numerically to be about one half.

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C O N T E N T S

1. INTRODUCTION

The hydromagnetic stability of a linear pinch with the plasma current confined to an infinitely thin layer, has been studied in detail (TAYLOR, 1957; TAYLOR, 1958; ROSENBLUTH, 1958). However all the structure of the fields is confined to this thin layer and here we shall consider the stability of the diffuse linear pinch where this structure is all important.

To consider the features of a field configuration which are desirable for stability we will use the hydromagnetic energy principle (BERNSTEIN 1958). From this variational principle can be formulated necessary and sufficient conditions for the stability of a diffuse linear pinch (NEWCOMB, 1960; SUYDAM, 1958). In particular a necessary criterion for the stability of localised displacement has been obtained (SUYDAM, 1958). However this criterion is not a sufficient one and in this paper we consider the non-localised displacements or 'kink' modes, together with the localised displacements in an attempt to determine the essential requirements for stability. This is usually attempted using the theorems of Newcomb (1960) relating to the solutions of the Euler-Lagrange equations associated with the energy principle. Here we obtain our stability conditions by simpler methods and only use the theorems in considering numerical examples.

A number of model field configurations have been studied in detail, in particular the force-free Bessel function model (VOSLAMBER, CALLEBAUT, 1962) and the force free paramagnetic model (KADOMTSEV, 1962; WHITEMAN, 1962) which compares well with experiment (BURTON, 1962). A modification of the latter model shows that the configuration can support a small plasma pressure (WHITEMAN 1965) and furthermore if the criterion for local stability is not violated near the magnetic axis then a whole range of stable field configurations is possible, containing moderately high plasma pressures (β on axis $\sim$ 20-30%) (ROBINSON amd KING, 1969) .

Experimentally relatively stable diffuse pinches are known to exist, where the residual instabilities are not thought to be hydromagnetic (ROBINSON and KING, 1969; OHKAWA, 1963; BOBELDIJK et al., 1969). However one cannot exclude the fact that in some of these cases toroidal curvature may be playing a beneficial role even though the current is above the Kruskal-Shafranov limit.

In Section 2 we shall establish a bound on the potential energy available to drive an instability which leads us to distinguish between current and pressure driven instabilities. By considering all the possible radial pitch variations we establish which forms can be stable.

The energy integral limits the value of beta with respect to the field produced by the current, and in Section 3 this is derived together with an integral condition for stability, which can be close to sufficient. Section 4 describes a simple method for estimating the position of the conducting wall for stability and this is supported by numerical calculations. Finally higher mode numbers are considered under conditions when the first mode ($m = 1$) is not permitted due to periodicity.

2. ENERGY INTEGRALS

2.1

We consider the usual idealised model of a cylindrically symmetrical plasma with infinite conductivity surrounded by a perfectly conducting wall - the columnar pinch of Newcomb (1960).

For such a system the magnetohydrostatic equilibrium condition is

$$\frac{dp}{dr} + B_z \frac{dB_z}{dr} + \frac{B_\theta}{r} \frac{d}{dr} r B_\theta = 0 \tag{1}$$

where p is the scalar plasma pressure, B_z the axial field and B_θ the azimuthal field in our cylindrical coordinate system. The field lines form a system of helices with pitch length

$$2\pi\ r\ B_z/B_\theta\ .$$

We consider the plasma to be subject to a small displacement from this equilibrium state and according to the energy principle, if the energy integral is positive for all displacements satisfying the boundary conditions the system is stable. For a cylindrical system the displacement can be analysed in the form

$$\xi_{r,\theta,z}\ \exp(im\theta + ikz)$$

where m and k are the azimuthal and axial mode numbers respectively. The energy integral can now be minimised with respect to $\xi_{\theta,z}$ which shows that minimisation is effected by incompressible perturbations. The expression for the energy integral can then be written

$$W_A(\xi) = \pi/2 \int_0^b r dr \left[\left\{ (kr\,B_z + mB_\theta) \frac{d\xi}{dr} + (kr\,B_z - mB_\theta) \frac{\xi}{r} \right\}^2 /(m^2 + k^2 r^2) + \frac{\xi^2}{r^2} \left[(kr\,B_z + mB_\theta)^2 - 2\,B_\theta \frac{d}{dr} (rB_\theta) \right] \right] \tag{2}$$

where $\xi = \xi_r$ and b is the radius of the conducting wall. The sufficient condition for stability, that $|rB_\theta|$ be a decreasing function of r, everywhere, can never be satisfied in a columnar pinch. This minimisation is not correct if there exist finite regions of constant pitch, since in this case the minimising perturbations are compressible. For a region of constant pitch, $c < r < d$, where the perturbation helix exactly matches the pitch of the magnetic field lines, the energy integral becomes

$$W(\xi) = -\pi/2 \int_c^d r\,dr \left[2B_\theta \frac{d}{dr}\frac{B_\theta}{r} + 4B_\theta^4 / r^2 (B_\theta^2 + B_z^2 + \gamma p) \right] \xi^2 \qquad (3)$$

where γ is the ratio of specific heats. Note that the integrand vanishes if the region of constant pitch is a pressureless one.

If we use the boundary condition that $\xi(b) = 0$, equation (2) can be integrated by parts to give an alternative expression for $W(\xi)$:

$$W_B(\xi) = \frac{\pi}{2} \int_0^b \left[f \left(\frac{d\xi}{dr}\right)^2 + g\,\xi^2 \right] dr$$

$$f = r \frac{(krB_z + mB_\theta)^2}{m^2 + k^2 r^2}$$

$$g = \frac{2k^2 r^2}{m^2 + k^2 r^2} \frac{dp}{dr} + \frac{1}{r} (krB_z + mB_\theta)^2 \frac{k^2 r^2 + m^2 - 1}{m^2 + k^2 r^2} +$$

$$+ \frac{2k^2 r}{(m^2 + k^2 r^2)} (k^2 r^2 B_z^2 - m^2 B_\theta^2) \qquad (4)$$

For some situations, for example, when the effect of an insulating wall inside a conducting shell is included, it is convenient to combine (2) and (4) so that

$$W(\xi) = \underset{0 < r < a}{W_B(\xi)} + \frac{\pi}{2} \frac{(k^2 a^2 B_z^2(a) - m^2 B_\theta^2(a))}{m^2 + k^2 a^2} \xi^2(a) + \underset{a < r < b}{W_A(\xi)} \qquad (5)$$

where the integrals, W_A, W_B, are to be taken over the indicated ranges. The Euler-Lagrange equation

$$\frac{d}{dr}\left(f \frac{d\xi}{dr} \right) - g\,\xi = 0 \qquad (6)$$

must be satisfied by ξ to make the energy integral, $W_B(\xi)$, an extremum. This equation is singular wherever

$$krB_z + mB_\theta = 0 \qquad (7)$$

and the behaviour of the solutions near the singularity gives rise to Suydam's necessary condition for stability (SUYDAM, 1958)

$$\frac{rB_z^2}{8}\left(\frac{P'}{P}\right)^2 + \frac{dp}{dr} > 0 \qquad (8)$$

where P is the pitch, defined as $P = rB_z/B_\theta$. This criterion limits only the local pressure gradient and thereby does not give the limit to the value of β that a given field configuration can confine.

2.2 A Lower Bound for the Potential Energy

Let us consider the m = 1 mode for which the boundary conditions on ξ are $\xi(b) = 0$, $\left.\frac{d\xi}{dr}\right|_{r\to 0} = 0$. In this case g (equation (4)) can be written in terms of the pitch.

$$g = \frac{2k^2\,r^2}{1 + k^2\,r^2}\frac{dp}{dr} + \frac{k^2 rB_\theta^2}{(1 + k^2r^2)^2}\left[(kP + 1)(kP(3 + k^2r^2) + k^2r^2 - 1)\right] \qquad (9)$$

For a confined plasma the first factor is negative when $\frac{dp}{dr}$ is negative, which is usually the case over much of the plasma. It can only make a positive contribution to the potential energy for a displacement which is localised near regions of positive pressure gradient. Stability therefore depends crucially on the second factor, especially where $\frac{dp}{dr}$ is small.

By completing the square in KP in equation (9) we can obtain a lower bound for g. This gives an estimate for the maximum potential energy available to drive an instability.

$$g(r) > 2\frac{dp}{dr}\frac{k^2r^2}{1+k^2r^2} - \frac{4B_\theta^2}{r}\frac{k^2r^2}{(1+k^2r^2)^2\,(3+k^2r^2)} \qquad (10)$$

which is minimised w.r.t.k for $k^2 = \frac{0.69}{r^2}$ and thus

$$g(r) > +\,2\frac{dp}{dr} - \frac{B_\theta^2}{3.8r} \qquad (11)$$

$$\text{and} \quad g(r) > g_b = \left(2\frac{dp}{dr} - \frac{B_\theta^2}{3.8r}\right)_{min}$$

where the minimum value of the expression on the right hand side is to be taken. For higher m numbers (m ≥ 2) this lower bound becomes

$$g_b = \left(2\frac{dp}{dr} - \frac{B_\theta^2}{3m^2r}\right)_{min} \qquad (12)$$

$$W \geq \frac{\pi}{2} g_b \int_o^b \xi^2\, dr$$

From equations (11), (12) we see that an instability can be driven by the pressure gradient, either in the form of a localised Suydam mode or otherwise, and by the current (B_θ). For higher modes the pressure gradient predominates but for m = 1, the current driven modes are most important, provided $\beta_\theta = \frac{2r}{B_\theta^2}\frac{dp}{dr} < 1$ (see also (WARE, 1964))

It follows directly from equation (10) that

$$\frac{r}{B_\theta^2}\frac{dp}{dr} > \frac{2}{(1+k^2r^2)\,(3+k^2r^2)} \qquad (13)$$

everywhere, is a sufficient condition for stability. It might seem that by combining a central region where equation (13) is satisfied and an outer region where the sufficient condition from equation (2) is satisfied, we could produce stable configurations with the value

of beta limited only by equilibrium (LAING, 1958).
However, this is not the case as the energy integral now involves the surface term, equation (5), which is only non-negative if $B_\theta(a) = 0$. Such a configuration could not exist in equilibrium in a torus.

2.3 Radial Pitch Variations and Stability

We now consider the detailed behaviour of the term in equation (9), which gives rise to the current driven modes; for m = 1

$$g_j = \frac{k^2 r B_\theta^2}{(1 + k^2r^2)^2} (kP + 1)\ (kP(3 + k^2r^2) - 1 + k^2r^2) \tag{14}$$

For $kP < -1$ or $kP > \dfrac{1 - k^2r^2}{3 + k^2r^2}$ g_j is positive and if this is true everywhere the particular mode, k, will be stable. The region over which g_j is negative diminishes as k^2r^2 is increased as the two roots for $g_j = 0$ approach each other.

Let us now consider all the possible radial pitch variations, some of which are illustrated in Fig.1. As $\left.\dfrac{dP}{dr}\right|_{r\to 0} = 0$ for cases where the pitch is finite on axis, then either the pitch decreases or increases (Figs.1(a), 1(b), 1(d)) monotonically or it can possess a minimum (Fig.1(c)) or maximum*. In the latter case it must also possess a minimum on axis, which for our purposes is covered by the monotonically increasing case Fig.1(b). A second class of pitch variations comprises the case when the pitch can go to infinity over some core region near the axis (i.e. a case where there is no axial current in the core), but Figs.1(a), 1(c), 1(d) cover the three possibilities in this case also.

Fig.1(a) shows a pitch variation which is similar to that of the force free paramagnetic model (KADOMTSEV, 1962).

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* A point of inflection is considered as covered by Fig.1(a).

The behaviour of g_j as a function of k and r for this case is seen in Fig.2, which shows a region of negative g_j concentrated near small k values, i.e. long wavelength instabilities. If we consider a perturbation with wavelength λ_1 such that there is a singular point $\frac{2\pi}{\lambda_1} P(r_s) = -1$ at r_s, then there is an inner region $0 < r < r_s$ where $g_j > 0$, an outer region where $g_j < 0$ initially and becomes positive again at r_2 when $|kr| \gtrsim 1$. In infinite conductivity theory the regions either side of the singularity can be considered separately, consequently the inner region is stable and the outer may be unstable depending on the position of the conducting wall. For large k, i.e. small λ, the region of negative g_j between r_s and r_2 becomes localised close to the singularity at r_s, permitting only a localised instability of the Suydam type. For small k, $g_j/f \sim k^2$ and thus ξ must be almost uniform to satisfy equation (6) and consequently instability is only possible for a conducting wall positioned at large radii. Therefore the most unstable wavelength for a 'kink' instability will be $\lambda \sim 2\pi P(o)$ i.e. $kP(o) = -1$ (most unstable in the sense that the radius of the conducting wall for stability is least compared with other modes, see Fig.7). In Section 4 we will demonstrate how to estimate the critical wall position for stability to such modes.

If there is a central core region to the configuration where B_θ is negligible and the pitch becomes infinite, then g_j is positive for such a region, as indicated by the line passing through r_1 on Fig.2. Because of this the configuration is less likely to have a negative value for $W(\xi)$ for a given conducting wall position, however such a situation may only be a transient one because of field diffusion.

Fig.1(b) shows an example in which the pitch increases away from the axis, as is thought to be the case in Tokamak (ROBINSON et al. 1970). In this case we can show that, in a cylinder, the configuration is unstable to an m = 1 instability. Choose a trial displacement ξ as shown in Fig.1(b) which vanishes as we approach the singular point r_s, defined by $\frac{2\pi}{\lambda_2} P(r_s) = -1$, ε is assumed to be very small then

$$W_B = \frac{\pi}{2} \left\{ \int_0^{r_s} g \, dr + \theta \int_{r_{s-\varepsilon}}^{r_s} (r - r_s)^2/\varepsilon^2 \, dr \right\} \xi^2 \underset{\varepsilon \to 0}{\rightarrow} \frac{\pi}{2} \xi^2 \int_0^{r_s} g \, dr \tag{15}$$

where f has been expanded near the singularity as $\theta(r - r_s)^2$. If λ_2 is chosen so that $kP \not> \frac{1 - k^2 r^2}{3 + k^2 r^2}$, which is always possible then $g_j < 0$ in the interval $0 < r < r_s$ and therefore $W_B < 0$ and the configuration is unstable. A positive pressure gradient in this region could stabilise such a mode. Note that the instability would be partially localised near the axis.

In Fig.1(c) we have a pitch variation showing a minimum which will occur when there is a vacuum region outside a plasma, such as in the ordinary stabilised pinch, where the pitch is proportional to r^2. In this case we choose a trial function ξ which is non-zero between the two singularities r_{s1}, r_{s2} for a wavenumber $2\pi/\lambda_3$ as shown. In the region $r_{s1} < r < r_{s2}$, $g_j < 0$ provided λ_3 is chosen so that $kP \not> \frac{1 - k^2 r^2}{3 + k^2 r^2}$, which again is always possible. As in equation (15) $W_B < 0$ and the configuration is unstable to a fairly localised instability.

The fourth figure, 1(d), shows a configuration with axial field reversal, which can have a vacuum layer outside the plasma without introducing a minimum in the pitch and therefore instability as in

Fig.1(c). In this case the regions of $g_j < 0$ are indicated in Fig.3, where k^2r^2 has been assumed small for simplification of the diagram. A larger value of k^2r^2 reduces the area where $g_j < 0$. The diagram for negative k is essentially as in the lower half of Fig.2 except that the boundaries associated with

$$kP = \frac{1 - k^2r^2}{3 + k^2r^2} \qquad \text{and} \qquad kP = -1$$

approach each other at the field reversal point, r_1.

When k is small and positive g_j is negative out to the point $kP = -1$. In this case a trial function of the form shown in Fig.1(b) is possible and therefore the positive contribution to $W(\xi)$ from f for small negative values of k is no longer possible for small positive values of k. If we choose k so that $kP(o) < \frac{1}{3}$ and if $kP(r_s) = -1$ is such that r_s is less than the radius of the conducting wall, then $g_j < o$ everywhere and, as in equation (15), the configuration will be unstable to a large scale instability. This gives us the simple necessary condition for stability

$$P(r) > -3\,P\,(o) \tag{16}$$

where $P(r)$ is the pitch at the outer conducting boundary. This implies that at the outer wall B_θ cannot be too small relative to B_z or B_z too strongly reversed. For example, taking the force free Bessel function model with $B_\theta = J_1(r)$, $B_z = J_o(r)$, equation (16) then gives instability when $r > 3.35$, the actual stability criterion is $r < 3.176$ (VOSLAMBER, CALLEBAUT, 1962).
Condition (16) is far from a sufficient condition for such configurations as we shall demonstrate in Section 3. Evidently the most unstable wave number for reverse field configurations is about $\frac{1}{3}P(o)$. Note that this is opposite in sign to that associated with field

configurations possessing no reversal. Inclusion of a core region where B_θ vanishes again makes g_j positive for small radii, and equation (16) is not valid.

It is possible to formulate this type of argument in an exactly equivalent way starting from the expression for W_A rather than W_B. However this approach is less illuminating, apart from indicating that an outer region where the current is reversed is helpful for stability, both with respect to localised (ROBINSON and KING, 1968) and non-localised modes. A sufficient condition for stability to the mode which we consider to be one of the most dangerous for reversed field configurations, namely $k = -\frac{1}{P(r_w)}$ is from the two terms involving ξ^2 in equation (2) just

$$j_z < \frac{1}{2}\frac{B_\theta}{r}\left(-\frac{P(r)}{P(r_w)} + 1\right)^2$$

everywhere. For small r, if j_z is non zero this requires

$$\left|\frac{P(o)}{P(r_w)}\right| > 1$$

and for large r that $j_z \leqslant o$. These conditions are certainly not necessary ones but are found to give a good guide since all the configurations of this type which have been shown to be stable approximately satisfy these conditions, for example $\left|\frac{P(o)}{P(r_w)}\right|$ can fall to a value of 0.7.

3. INTEGRAL CONDITIONS FOR STABILITY

The considerations of the previous section essentially relate to current driven instabilities which do not depend on the value of β. However by considering the pressure gradient term in equation (9) we can obtain estimates for the maximum values of β that configurations (a) and (d) of Fig.1 will confine.

Let us first consider $W_A(\xi)$ and a wavenumber such that $kP(r_w) = -1$ where r_w is the boundary of the plasma. The values of k may be either positive or negative depending on whether the axial field reverses or not. For this wavenumber we can choose a trial function as in Fig.1(b), which satisfies our boundary conditions then

$$W_A(\xi) = \frac{\pi}{2}\,\xi^2 \int_0^{r_w} \frac{dr}{r}\left[-\,2B_\theta \frac{d}{dr}\, rB_\theta + (krB_z + mB_\theta)^2 + \frac{(krB_z - mB_\theta)^2}{m^2 + k^2r^2}\right] \tag{17}$$

3.1 Stability Limit to β_θ for m = 1

For m = 1 from equation (17) we obtain

$$W_A(\xi) = \frac{\pi}{2}\,\xi^2 \int_0^{r_w} \frac{dr}{r}\left[-\,2B_\theta \frac{d}{dr}\, rB_\theta + 2k^2r^2B_z^2 + 2B_\theta^2 - \frac{k^2r^2(krB_z - B_\theta)^2}{1 + k^2r^2}\right]$$

the last term is always negative, hence a necessary criterion is that the sum of the first three terms should be positive.

Integrating the first and third terms we find that

$$W_A(\xi) \leq \pi\,\xi^2 \left[-\,\frac{B_\theta^2(r_w)}{2} + k^2 \int_0^{r_w} rB_z^2\right]$$

For $W_A(\xi)$ to be positive, it follows from integration by parts and using pressure balance that

$$r_w^2\, B_\theta^2(r_w) > 4 \int_0^{r} r\, p dr \tag{18}$$

If we write the Bennett relation for a diffuse pinch as

$$\beta_\theta\, I^2 = 2NkT = 4\pi \int_0^{r_w} r\, p\, dr$$

where I is the total axial current, T the average temperature and N the line density, then equation (18) gives the necessary criterion for stability

$$\boxed{\beta_\theta < 1} \tag{19}$$

Note that we have made no assumptions about the configuration apart from the existence of a wavenumber such that $kP(r_w) = -1$, and $m = 1$. Criterion (19) will always be satisfied for a resistively heated diffuse pinch (BUTT and PEASE, 1963).
but for a screw pinch with compressional heating the criterion can be violated, leading to instability.

3.2 Stability Limit to β_θ for m = 0

$W_A(\xi)$ can also be used to establish a limiting β for m = 0 perturbations in the case when the axial field is reversed. Suppose the field reverses at r_o then choosing a trial function $\xi = \lambda r$, $0 < r < r_o - \varepsilon$ and $\xi = (r_o - \varepsilon) \frac{\lambda}{\varepsilon} (r_o - r)$, $r_o - \varepsilon < r < r_o$ and taking the limit $\varepsilon \to 0$ we obtain for W_A

$$W_A(\xi) = \frac{\pi}{2} \lambda^2 \int_0^{r_o} r\, dr \left[4B_z^2 - 2B_\theta \frac{d}{dr} rB_\theta + k^2 r^2 B_z^2 \right]$$

which for $k \to 0$ requires

$$r_o^2 B_\theta^2 (r_o) > 8 \int_0^{r_o} r\, p\, dr - 4 r_o^2 p(r_o)$$

and if $p(r_o) \approx 0$ then using the Bennett relation requires

$$\beta_\theta < \tfrac{1}{2} \qquad (20)$$

A completely stable configuration with $B_z = 0$ at the boundary and containing a β_θ close to this limit is known to exist (RUSBRIDGE, 1964).

3.3 Stability Limits for Reverse Field Configurations

Turning now to $W_B(\xi)$ for a trial function as in Fig.1(b) we have the necessary criterion for stability

$$\int_0^{r_s} dr \left[\frac{2k^2 r^2}{1 + k^2 r^2} \frac{dp}{dr} + \frac{k^2 r B_\theta^2}{(1 + k^2 r^2)^2} \cdot (kP + 1)(kP(3 + k^2 r^2) + k^2 r^2 - 1) \right] > 0 \qquad (21)$$

where k, r_s are defined by $kP(r_s) = -1$. This necessary criterion can now be applied to reverse field configurations to obtain the maximum possible value of β for a given compression ratio and degree of field reversal. In addition it can also be applied to field configurations showing no reversal, for it allows us to obtain the critical pressure gradient for a region where $g_j > 0$ or $kP < -1$.

The application of the equilibrium equation to (21) gives no simple expression, so we will demonstrate the essential features of (21) by choosing a model for the field configurations.

$$0 < r < r_o - \delta$$

$$B_z = B_{zo}$$

$$B_\theta \text{ small}$$

$r_o - \delta < r < r_o + \delta$ $\qquad$ B_z reverses, B_θ rises and there is a pressure gradient.

$$r_o + \delta < r < r_w$$

$$B_z = B_z(r_w)$$

$$B_\theta = B_\theta(r_o)\,\frac{r_o}{r}$$

and $\delta << r_o$.

If we assume $k^2 r_w^2$ to be small then we have

$$\tfrac{3}{4}\, k^4 r_o^4\, B_{zo}^2 - k^2 B_\theta^2 (r_o)\, r_o^2 \left[\log \frac{r_w}{r_o} + \frac{r_w^2 - r_o^2}{r_w^2} - \tfrac{3}{4}\left(\frac{r_w^4 - r_o^4}{r_w^4} \right) \right] - 2k^2\, r_o^2\, p > 0$$

or if $\beta = \dfrac{2P}{B_\theta^2(r_o)}$

$$-\beta + \tfrac{3}{4}\left(\frac{r_o^2}{r_w} \right)\left(\frac{B_{zo}^2}{B_z^2\,(r_w)} - 1 \right) - \left(\log \frac{r_w}{r_o} + \tfrac{1}{4} - \frac{r_o^2}{r_w^2} \right) > 0$$

$\frac{r_w}{r_o}$ is the compression ratio and $\left|\frac{B_{zo}}{B_z(r_w)}\right|$ the degree of field reversal. This illustrative example is shown in Fig.4 where we see that large compression ratios necessitate small degrees of field reversal and the stability boundary is not very sensitive to β. It is easy to verify for this example that if the total axial flux $\int_0^{r_w} r B_z \, dr$ is negative then there will be instability and this is also borne out by numerical calculations (for an other example see Fig.8). Increasing $k^2 r_w^2$ has a stabilising effect but this is equivalent to decreasing the degree of field reversal and this wave-number is no longer the most unstable. It is interesting to note that the optimum heating requirements for such a plasma are incompatible with the conditions for stability.

3.4 Proximity to Sufficiency for Modes Having Integral Conditions for Stability

If the necessary criterion (21) is to be of practical use then it must be close to sufficient for such modes. Thus we have to demonstrate that the trial function of Fig.1(b) is close to the displacement which minimises $W_B(\xi)$.

Let ξ have the following form

$$\xi = \xi_o + \frac{\delta r}{r_w - \varepsilon} , \quad 0 < r < r_w - \varepsilon$$

and

$$\xi = (\xi_o + \delta) \frac{(r_w - r)}{\varepsilon} , \quad r_w - \varepsilon < r < r_w$$

where δ is assumed to be small, then

$$W_B(\xi) = \frac{\pi}{2} \int_0^{r_w} \left[g \, \xi_o^2 + 2\delta \, \frac{r}{r_w} \, g \, \xi_o + \delta^2 \, \frac{(f + g \, r^2)}{r_w^2} \right] dr \qquad (22)$$

(The boundary condition at r = 0 can be satisfied by choosing a small region over which the gradient in ξ vanishes, which does not contribute to the above integral in the limit). For small and large

k^2r^2, $\int_0^{r_w} (f + g\, r^2)\, dr > 0$ and for $k^2r^2 \gtrsim 1$ the region over which $f + g\, r^2$ can be negative is small, consequently the third term on the right of equation (22) is usually positive. Plainly if $\int_0^{r_w} r\, g\, dr \geq 0$ then $W_B(\xi)$ is positive, whereas if $\int_0^{r_w} g\, dr = 0$ it is negative. If the term in δ^2 is positive then $\int_0^{r_w} r\, g\, dr \geq 0$ is a sufficient condition for stability and as g is a function which is strongly weighted to large r ($\propto r\, B_\theta^2$) then the two integrals will have similar values and thus the necessary condition $\int_0^{r_w} g\, dr \geq 0$ forms a reasonably accurate necessary and sufficient condition for such modes (gives the wall position to an accuracy of about 7%). This is borne out by numerical calculations, as illustrated in Fig.5, which shows the minimal ξ as the wavenumber approaches that given by $kP(r_w) = -1$. These ξ's are very close to our simple trial function and thus equation (21) will give necessary and sufficient conditions for stability for this particular wavenumber. We have already shown that for reverse field configurations this wavenumber is approximately the most unstable.

For configurations possessing no reversal equation (21) can still be used to determine the critical pressure gradient. For example, consider a configuration where the pitch becomes infinite on the axis, i.e. $B_\theta = Ar^n$, $n > 1$ for small r, then for r_s small, such that $kr_s \ll 1$, equation (21) becomes

$$k^2 \int_0^{r_s} \left(2r^2 \frac{dp}{dr} + 3k^2\, r^3\, B_z^2 \right) dr > 0.$$

This requires $\frac{dp}{dr} \geq 0$ for small r as k can be chosen to be arbitrarily small, $k = -\frac{1}{P(r_s)}$. For larger r this equation can be solved to give the limiting pressure profile such that this integral is always positive as the wavenumber is varied.

4. POSITION OF THE CONDUCTING WALL FOR STABILITY

In the previous sections we obtained various necessary (and sufficient) conditions for stability. Here we will indicate how to estimate the position of the conducting wall required for stability which previously was only obtainable using Newcomb's analysis (NEWCOMB, 1960). By a change of variables the Euler equation (6), can be rewritten in the form

$$\frac{d^2\psi}{dr^2} + \psi\, A(r, m, k) = 0 \qquad (23)$$

where

$$\psi = r^{\frac{1}{2}} \frac{(kr\, B_z + m\, B_\theta)}{(m^2 + k^2\, r^2)^{\frac{1}{2}}}\; \xi \;,$$

which is closely related to the radial field perturbation. We then apply the following theorem: If A has a maximum positive value A_{max} (the solution of $\frac{d^2y}{dr^2} + A_{max}\, y = 0$ has its first zero at $\pi/A_{max}^{\frac{1}{2}}$) the solution of equation (23) has its first zero at $r > \pi/A_{max}^{\frac{1}{2}}$. The accuracy of the estimate depends on how close ψ approaches to $\sin(A_{max}^{\frac{1}{2}}\, r)$. The boundary for stability is therefore obtained by invoking Newcomb's theorem (NEWCOMB, 1960) concerning the first zero in ξ and maximising A with respect to r and k.

A is a complicated function and given by

$$-A = \frac{m^2 + k^2 r^2}{r^2} - \sigma^2 + \frac{d\sigma}{dr}\frac{(mB_z - kr\, B_\theta)}{mB_\theta + kr\, B_z} - \frac{2\sigma\, m\, k}{m^2 + k^2 r^2} - \frac{[m^4 + 10m^2k^2r^2 - 3k^4r^4]}{4r^2\,(m^2 + k^2r^2)^2}$$

$$+ \frac{2k^2r^2B^2}{(mB_\theta + kr\, B_z)^2} \cdot \frac{1}{B^2 r}\frac{dp}{dr} - \frac{d}{dr}\left(\frac{1}{B^2}\frac{dp}{dr}\right) + \left(\frac{1}{B^2}\frac{dp}{dr}\right)^2 - \frac{(m^2 - k^2r^2)}{m^2 + k^2r^2}\,\frac{1}{rB^2}\frac{dp}{dr}$$

$$- \frac{2}{B^2}\frac{dp}{dr}\left[\frac{\sigma\,(mB_z - kr\, B_\theta)}{mB_\theta + kr\, B_z} + \frac{kB_z - mB_\theta/r}{mB_\theta + kr\, B_z}\right] , \quad \sigma = \underline{J}.\underline{B}/B^2 \qquad (24)$$

This method can be readily applied to the force-free Bessel function model where $\frac{dp}{dr} = 0$, $B_\theta = J_1(r)$, $B_z = J_o(r)$ and therefore

$\sigma = J_z/B_z$, $\frac{d\sigma}{dr} = 0$. This latter property makes equation (23) non-singular at $kr\,B_z + mB_\theta = 0$. In this case A has the simple form

$$- A^{BFM} = \frac{m^2 + k^2 r^2}{r^2} - 1 - \frac{2\,m\,k}{m^2 + k^2 r^2} - \frac{(m^4 + 10m^2 k^2 r^2 - 3k^4 r^4)}{4r^2 (m^2 + k^2 r^2)} . \qquad (25)$$

Maximising this coefficient for m = 0 and 1 gives the position of the first zeros to be not less than 3.3 and 2.97 respectively, and the corresponding wavenumbers as 0 and + 0.5; a conducting wall placed at $r \leq 2.97$ will therefore give stability. That a positive wavenumber should be the most unstable for m = 1 agrees with our general considerations (Section 2) of the stability of reverse field configurations. The exact analytical results for the position of the wall in this case (VOSLAMBER, CALLEBAUT, 1962) are infinity and 3.18 for m = 0 and 1 respectively. However we have made an estimate for the position of the first zero in ψ (not ξ) and as noted by Whiteman (GIBSON and WHITEMAN, 1968) this gives us the conducting wall position for stability to the tearing mode. In this case the exact positions are 3.83 and 3.10 for m = 0 and 1 respectively, which are the numbers to be compared with our estimates. Thus in this case the method is accurate.

For all other field configurations $\frac{d\sigma}{dr} \neq 0$ and A possesses a singular term. Our general considerations showed that the most unstable perturbation for a non-reversed field configuration was one given by $kP(o) = -1$, so the singularity in A occurs at r = 0. Expanding all the quantities in A for small r we have

$$\begin{aligned} j_z &= j_z^o \left(1 - \frac{2r^2}{P(o)^2}(1+\gamma) + \dots\right) \\ \sigma &= \sigma^o \left(1 - \frac{r^2}{P(o)^2}(1+2\gamma) + \dots\right) \\ P &= P(o)\left(1 + \frac{r^2}{P(o)^2}\gamma + \dots\right), \quad \gamma = \left.\frac{P}{2}\frac{d^2P}{dr^2}\right|_{r\to 0} \end{aligned} \qquad (26)$$

where we have assumed $\frac{dp}{dr} = 0$. γ is the curvature of the pitch on axis.

For local stability (ROBINSON, 1969) to m = 1 we have $\gamma > 0$ i.e. a pitch variation as in Fig.1(b), and $\gamma < -\frac{4}{9}$ (Fig.1(a)). When the shear is 'strong' $\gamma < -\frac{1}{2}$, and $\frac{d\sigma}{dr}$ is positive, whereas for $\frac{dP}{dr} > 0$, $\frac{d\sigma}{dr}$ is negative which we already know to be unstable (Fig.1(b)). The singular term in A for small r takes the form $-\frac{(1 + 2\gamma)}{1 + kP(r)}$ which for the perturbation $kP(o) = -1$ becomes $-\frac{4(1 + 2\gamma)}{r^2\gamma}$. This is negative for $\gamma < -\frac{1}{2}$ or > 0 and thus we can apply our theorem to equation (23) for this perturbation which is singular on the axis.

Both the force free Bessel function and paramagnetic models have $\gamma = -\frac{1}{2}$ and so σ is approximately constant for small r. For the paramagnetic case $\sigma = B_z/B^2$, $\frac{d\sigma}{dr} = \frac{B_\theta B_z}{B^4} (\frac{2B_\theta}{r} - 1)$ and maximising A for m = 1 and $k = -\frac{1}{2}$ we obtain 2.82 as the limiting wall position for stability. The numerical solution of the Euler equation gives 3.16 so again the method is reasonably successful. The theorem can be used for examples with finite pressure gradients provided they satisfy $\frac{dp}{dr} \geq 0$ for small r, otherwise they are unstable to a localised displacement near r = 0.

5 NUMERICAL EXAMPLES OF STABLE AND UNSTABLE CONFIGURATIONS

These calculations have been performed using the numerical procedure developed by Copley and Whiteman (COPLEY and WHITEMAN, 1962). First we consider the configurations of Fig.1(a) and (d). Fig.6 shows a force free field configuration derived from the paramagnetic equations by the inclusion of a term $\overline{\sigma_{11} E_{11}}$ (which has the form of a Reynolds stress between the conductivity and electric field). This additional term allows reversal of the axial field, B_z, which can be arbitrarily varied. The resulting stability diagrams for various values of this parameter are shown in Fig.7. For no field reversal, $\alpha = 0$, the most unstable wavenumber is that given by $kP(o) = -1$, or $k = 0.5$; this is also true for the reversed field cases as $P(o) = \frac{2}{1 + \alpha}$. The increased shear produced by the reversal is more stabilising for these negative wavenumbers, but as noted earlier instability is now possible for positive k.

Guided by the necessary and sufficient conditions obtained above we have constructed a field configuration in Fig.8 with a central value of $\beta = 31\%$ which is stable for wall positions up to that marked. Note that the point where the total axial flux vanishes is close to the stability limit. The maximum value for β is probably not more than 40%. The existence of such field configurations has already been pointed out (ROBINSON and KING, 1968) using fields generated from a paramagnetic model by including an azimuthal electric field E_{θ}.

Consider now the configuration of Fig.1(c), produced by vacuum fields outside the plasma. This is most conveniently obtained analytically, by supposing that the conductivity decreases radially in the paramagnetic model. Fig.9 shows the stability diagram for a configuration like that of Fig.1(c) obtained using $\sigma_{\parallel} = \sigma_{\parallel}(o)(1 - r^2/21)$. The position of the singularity, $kP(r) = 1$ is denoted by S and is a plot of the inverse pitch. Not only is there instability produced by the minimum in the pitch, as we would expect, but also at the point $kP(o) = -1$. This later instability is a local one due to the fact that the curvature of the pitch on axis $\gamma = \frac{P}{2} \left.\frac{d^2P}{dr^2}\right|_{r \to 0} = -\frac{17}{42}$ fails to satisfy the stability criterion (ROBINSON, 1969) that $\gamma > 0$ or $< -\frac{4}{9}$ for stability. The stability diagram for a pitch variation of the form $P = 0.8 \cos r + 1.2$ is shown in Fig.10. In this case $\gamma < -\frac{4}{9}$ so there is no local instability associated with $k = -0.5$. Again an instability is associated with the minimum in the pitch, and in the usual way with $kP(o) = -1$. A more extreme case where the minimum in the pitch occurs at $r = 2.5$ rather than $r = 4.2$, of Fig.9 is shown in Fig.11 for $\sigma_{11} = \sigma_{11}(o)(1 - r^2/10)$. Here $\gamma = -\frac{3}{10}$ so we have the local instability again, an instability associated with the minimum in the pitch and a third region of instability appearing because the configuration tends to that of Fig.1(b).

In addition to considering these types of field configuration we have also studied those where the pitch tends to infinity on axis, where regions of constant pitch exist (BOBELDIJK et al. 1967). and/or large currents flow outside the plasma. The maximum beta for stability is obtained by optimising the pressure gradient, in the outer regions such that equation (8) is satisfied and near the axis so that the "$\int g\, dr$ criterion" (Section 3) is satisfied. For example, if j_z is assumed to be uniform then the field configuration is determined by these conditions and the maximum value of beta for stability is 35% (RUSBRIDGE, 1964).

Configurations possessing regions of constant pitch outside the plasma and relatively high values of β internally have been studied experimentally (BOBELDIJK et al., 1967) and theoretically (SCHUURMANN et al., 1969 and GOEDBLOED, 1970). Fig.12 shows a field configuration possessing constant pitch and zero pressure gradient everywhere, i.e.

$$B_\theta = \frac{C\, r/P}{1 + r^2/P^2} \quad , \quad B_z = \frac{C}{1 + r^2/P^2}$$

where C is a constant. The stability diagram shown in Fig.13, exhibits instability at all radii close to $k = -1/P = -\frac{1}{2}$.

Actually at $k = -1/P$ the energy integral vanishes, equation (3). Fig.12 also shows a configuration where the central core has been replaced by a pressureless region with $B_\theta = r^2/2$. The stability diagram in this case, Fig.13, shows instability only in the constant pitch region.

In an attempt to find configurations containing an appreciable β that of Fig.14, defined by a pitch variation of the form $P = 2.0 + 1.6/r^3$, and a pressure profile $p(r) = \frac{p(o)}{1 + r^2}$, was studied. Even though the Suydam criterion is satisfied it is unstable for wavenumbers in the range $0 \rightarrow -0.47$ to an instability which arises because the "$\int g\, dr$ criterion" is not satisfied for small radii, as the pressure gradient is too large.

Figs.15 and 16 show the configuration and corresponding stability diagrams obtained by optimising the pressure gradient to satisfy the "$\int g\, dr$ criterion" and Suydam for the same pitch variation. Fig.16 shows the stability diagram for $\beta = 0$ and for the maximum beta value for stability of 5%. This is to be compared with the case of a screw pinch with a sharp boundary which gave $\beta \sim 20\%$ (SCHUURMANN et al., 1969); evidently the diffuse nature of the configuration reduces the critical β for stability.

The numerical results presented here and the general considerations given in the preceding sections indicate that there are only two types of configuration which are stable for appreciable values of β. These are the reversed field configurations with small or zero current near the outer wall and configurations with large currents near the wall. Configurations possessing regions of constant pitch near the outer boundary do not appear to have more favourable stability properties than other configurations with similar large currents near the wall.

6. HIGHER MODE NUMBERS $m \geq 2$

For practical systems without a periodicity restriction the least stable displacement are those with $m = 1$ so there is little need to consider the higher mode numbers. However, the $m = 1$ mode may be absent due to periodicity, for example in a toroid or cylinder of finite length which has a pitch length comparable with the length of the system, and in such cases the hydromagnetic stability depends on the higher modes (SHAFRANOV, 1970) and in particular the Suydam criterion.

Let us now consider a toroid of major radius R so that the wavenumber k has discrete values n/R, $n = 1, 2, \ldots$, and $q = \frac{P}{R}$. The singularity condition becomes $nq + m = 0$ and consequently as q becomes greater than unity at some point the $m = 1$ mode is no longer possible. As we have seen, the configuration of Fig.1(a) is most unstable for $kP(o) = -1$ so that $q(o)$ would have to exceed unity to exclude the $m = 1$ mode. However for the configuration of Fig.1(d) the most unstable mode was that given approximately by $kP(o) \approx 1/3$ so periodicity will strongly affect the results when $q(o) \approx 1/3$ as the most unstable modes are no longer permitted. A similar conclusion is arrived at for configurations possessing core regions where the pitch becomes very large, or the axial current is negligible on axis.

In this notation g (equation (6)) can be written

$$g = \frac{n^2 \, r \, B_\theta}{R^2(m^2 + n^2 \, r^2/R^2)^2} \left\{ (nq + m) \left\{ (nq + m)(m^2 - 1) \frac{m^2 + n^2 \, r^2/R^2}{n^2 \, r^2/R^2} + nq(m^2 + 2 + n^2 \, r^2/R^2) + m^3 - 2m + m \frac{n^2 r^2}{R^2} \right\} + (m^2 + \frac{n^2 r^2}{R^2}) \frac{2r}{B_\theta^2} \frac{dp}{dr} \right\} \quad (27)$$

where we have ignored terms arising from toroidal curvature for the moment. For m = 1 the two boundaries for stability i.e. $g > 0$, are

$$nq < -1 - \left(1 + \frac{n^2 r^2}{R^2}\right)\frac{\beta_\theta}{4} \quad \text{and} \quad nq > \frac{1}{3} + \frac{\beta_\theta}{4}$$

where $\beta_\theta = \frac{-2r}{B_\theta^2}\frac{dp}{dr}$ is assumed to be small.

For $\beta_\theta = 0$, the two roots of the equation $g = 0$ are

$$nq = -m \quad , \quad nq = -m + \frac{4m}{m^2 + 2 + \left(\frac{nr}{R}\right)^2 + (m^2 - 1)\frac{\left(m^2 + \left(\frac{nr}{R}\right)^2\right)}{\left(\frac{nr}{R}\right)^2}} \tag{28}$$

As m increases so the two roots approach each other and the region over which g can be negative decreases. As r/R is small and $nq \approx -m$ then the second root becomes $nq = -m \frac{4}{m(m^2 - 1)}\left(\frac{nr}{R}\right)^2$. If we expand q near the singular point as $q = q(r_s) + \Delta \frac{dq}{dr} + \ldots$ then the distance Δ, over which g is negative is given by

$$\frac{\Delta}{q}\frac{dq}{dr} = \frac{4}{m^2 - 1}\left(\frac{r}{Rq}\right)^2 \tag{29}$$

For $q > 1$ this is very small even for m = 2, provided the shear is not very weak i.e. $\frac{r}{q}\frac{dq}{dr} \approx 1$. Examining the roots of equation (27) for $\beta_\theta \neq 0$ we find that provided $\beta_\theta > \left(\frac{r}{Rq}\right)^2$, the equation for Δ is

$$\frac{\Delta}{q}\frac{dq}{dr} = \frac{2r}{Rq}\frac{\beta_\theta^{\frac{1}{2}}}{(m^2 - 1)^{\frac{1}{2}}} \tag{30}$$

A lower bound on g for this case is

$$g(r) > \frac{B_\theta^2}{r}\beta_\theta\left(\frac{r}{Rq}\right)^2$$

which by comparison with equation (11) and assuming $\beta_\theta \approx 1.$ will give rise to much slower growth rates than for m = 1, by a factor $\left(\frac{r}{Rq}\right)^2$.

Without any additional terms in our energy integral due to toroidal curvature the configuration of Fig.1(c) is unstable. However, our proof that the configuration of Fig.1(b) is unstable is not valid for $m = 2$ as the boundary condition on axis changes, and g is positive for small r. In this case application of a modified form for the integral g condition gives rise to a critical value of the shear for stability if β_θ is small. We choose the perturbation shown in Fig.1(b) except that it goes to zero rapidly near $r = 0$; this it can do and make no contribution to the energy integral as $f \to 0$ as $r \to 0$. Using the g of equation (27) and expanding $q = q_o + ar^2 + \ldots$ we obtain

$$a\, q_o\, R^2 > \frac{2}{m^2 - 1} \quad \text{or} \quad R^2 q_o \left.\frac{d^2 q}{dr^2}\right|_{r\to 0} > \frac{4}{m^2 - 1} \qquad (31)$$

as a necessary criterion for stability, when β_θ is small. This parameter is the quantity γ (ROBINSON, 1969) mentioned earlier, which has to be positive or $< -\frac{4}{9}$ to ensure stability to a localised displacement near $r = 0$. In practice equation (31) would easily be satisfied unless the shear near the axis was very weak (i.e. $\frac{r}{q}\frac{dq}{dr} \sim \left(\frac{r}{Rq}\right)^2$).

The inclusion of toroidal curvature into the energy integral introduces a term of the form $\frac{dp}{dr} q^2$, consequently situations which are unstable because the pressure gradient is negative become stable when $q > 1$ (i.e. toroidal curvature greater than field line curvature). Even when β_θ is small there are other toroidal curvature terms associated with the driving force for the possible instability of equation (31), namely $\frac{d}{dr}\frac{J_z}{B_z}$, but we shall not consider the effect of these here (ROBINSON, 1969). Evidently for $q << 1$ there are no toroidal effects associated with the stability criteria that we have derived. However as $q \to 1$ so the "$\int g$ criterion" which depends on a detailed radial balance of positive and negative terms, is likely to be strongly affected by toroidal effects if stability is achieved by having $\int g\, dr \gtrsim 0$. This may be favourable or unfavourable. Certainly the latter is to be expected if the stability largely depends on pressure gradient driven modes.

Consequently for $m \geqslant 2$, only Fig.1(c) will be unstable and then only if β_θ is small enough to allow the current driven instability. Such an instability could be present during the heating phase of a Tokamak when skin currents flow (ROBINSON et al., 1970).

From equation (11) it is seen that the growth rate for an $m = 1$ instability is given approximately by

$$\omega^2 \sim \frac{B_\theta^2}{4\pi\rho\ a^2} \cdot k^2\ a^2 \qquad (32)$$

where a is the radius of the system and ρ the average mass density. For a flute instability in a Tokamak this becomes

$$\omega^2 \sim \frac{B_\theta^2}{4\pi\rho\ a^2} \left(\frac{a}{Rq}\right)^2$$

whereas for a current driven instability the growth rate is given approximately by

$$\omega^2 \sim \frac{B_\theta^2}{4\pi\rho\ a^2} \left(\frac{a}{Rq}\right)^4$$

and ω can be two orders of magnitude slower than that given by equation (32).

7. CONCLUSION

The stability of a diffuse linear pinch system has been studied in order to determine the essential requirements for stability and the maximum values of beta for stability.

By considering a lower bound to the potential energy available to drive an instability we find that the first azimuthal mode ($m = 1$)

is predominantly a current driven mode, whereas for higher mode numbers pressure gradient driven instabilities are more important.

An examination of the potential energy available to drive this $m = 1$ current or kink instability shows that it will always be unstable if the radial variation of the pitch of the field lines possesses a minimum. For stability it is therefore necessary for the pitch to decrease away from the axis. If the pitch changes sign then the system will be unstable if its magnitude at the outer boundary is more than three times the central value.

A general proof was given that the value of β with respect to the field produced by the current i.e. β_θ, must be less than unity for stability.

An analytic method for testing the stability of a given configuration, which is shown to be in good agreement with numerical calculations, is derived. It is formulated in terms of a trial function for the displacement which is uniform, and the resulting integral condition is shown to be close to sufficient for perturbations whose wavelength coincides with the pitch length at some position. If this is on axis (or does not exist) a simple theorem can be used to estimate the position of the conducting wall for stability. Application of these results shows that values of β of 30-40% are possible for two classes of diffuse pinch configurations - those where the axial field reverses outside the plasma or where large currents flow in that region. The former case is the only one in which high-β is possible with a vacuum region outside the plasma.

It is deduced from a numerical study of many different examples of field configurations possessing axial field reversal that for stability the axial flux must not reverse.

Periodicity effects are considered in the limit that the first azimuthal mode is no longer possible and it is shown that any instabilities must then be only local in character. In this case toroidal curvature is the major stabilising influence.

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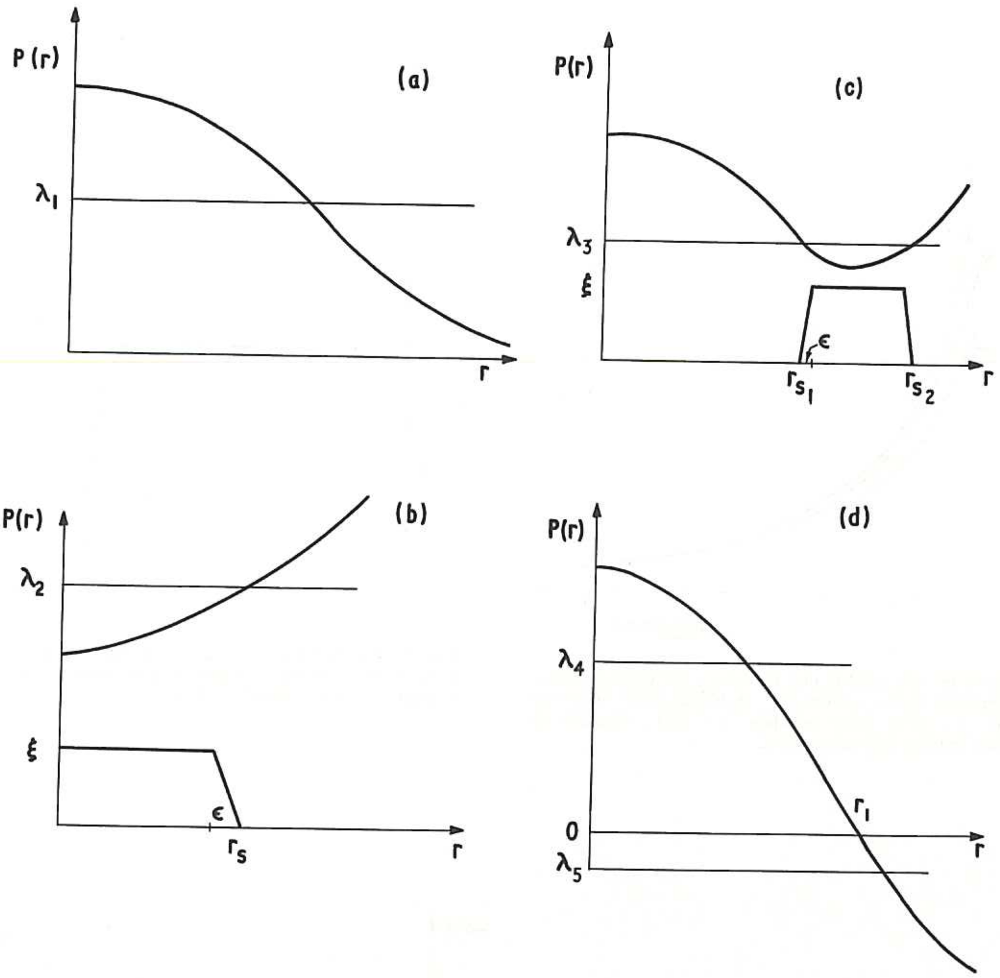

Fig.1 The four basic pitch variations for a non-zero axial current on axis. (a) monotonically decreasing, (b) monotonically increasing, (c) exhibiting a minimum and (d) decreasing and changing sign.

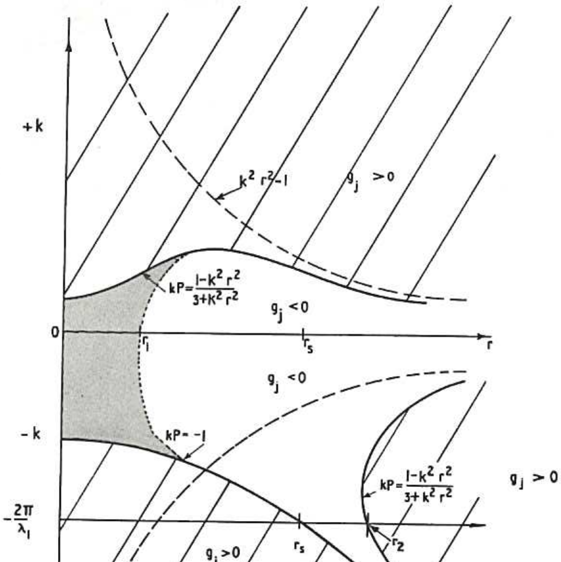

Fig.2 Regions in k-r space where g_j can be negative (unshaded) for the pitch variation of Fig.1(a). r_s is the position of the singularity, and r_2 the point at which $g_j = 0$ that is not a singularity, for a perturbation of wavelength λ_1. r_1 denotes the point at which the axial current vanishes, the dotted line is for $KP = -1$ in this case, and the shaded region to the left has $g_j > 0$.

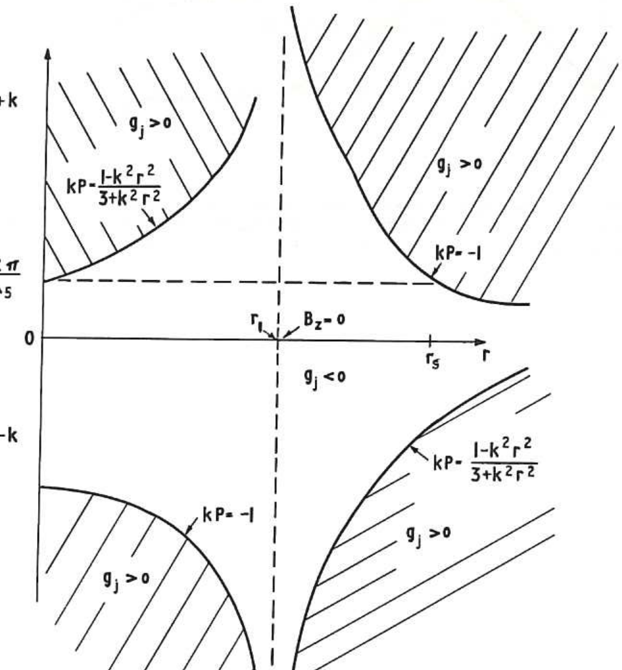

Fig.3 Regions in k-r space where g_j can be negative - (unshaded) for the pitch variation of Fig.1(d). r_s is the position of the singularity to a perturbation of wavelength λ_5 which will be unstable.

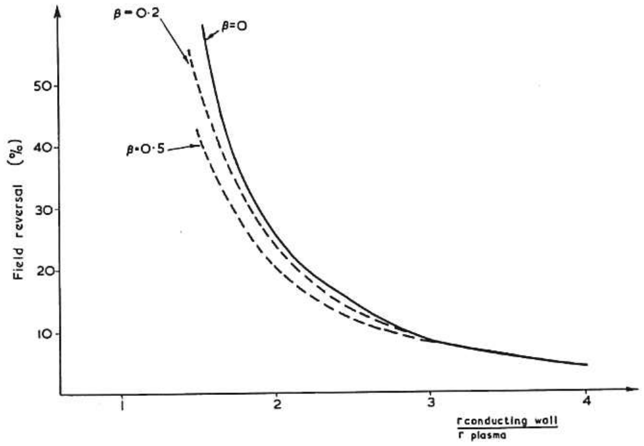

Fig.4 Dependence of the stability boundary on compression ratio and percentage field reversal, $B_z(0)/B_z(r_W)$ for the three values of β. (For this example $\beta = \beta\theta$). Values to the right of the curves are unstable.

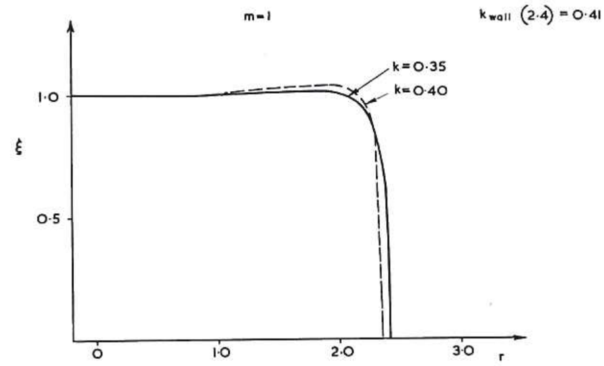

Fig.5 The minimal radial perturbation as a function of radius for two wave numbers approaching that given by $kP(r_W) = -1$. (Kwall = 0·41).

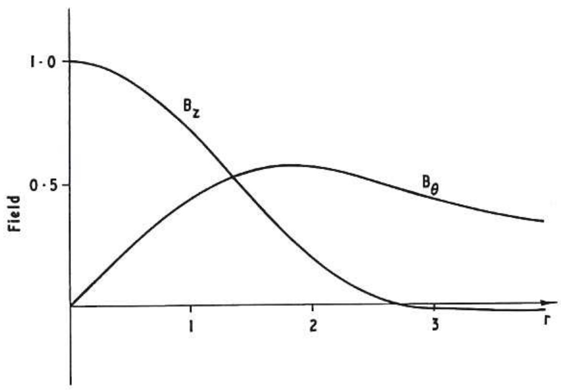

Fig.6 A field configuration, possessing a small axial reversed field, generated from the paramagnetic equations including a term $\alpha = \overline{\sigma_{\parallel} E_{\parallel}/\sigma_{\parallel} E_z}$. In this case $\alpha = 0{\cdot}1$.

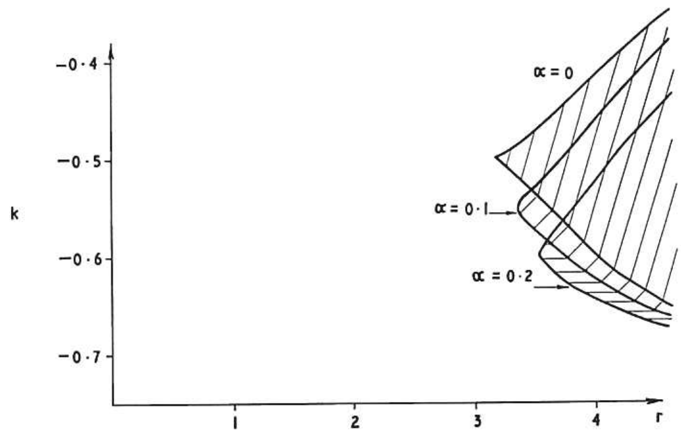

Fig.7 The stability diagrams for various values of α. The shaded areas are the regions of instability and $\alpha = 0$ corresponds to the force free paramagnetic model.

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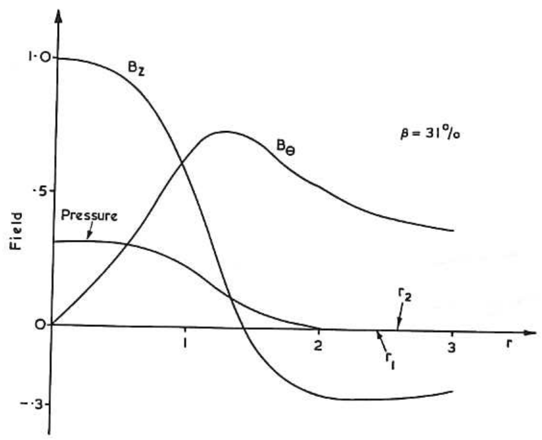

Fig.8 A reversed field configuration with a central β value of 31%. The maximum conducting wall position for complete stability is denoted by r_1. r_2 shows the position at which the total axial flux vanishes.

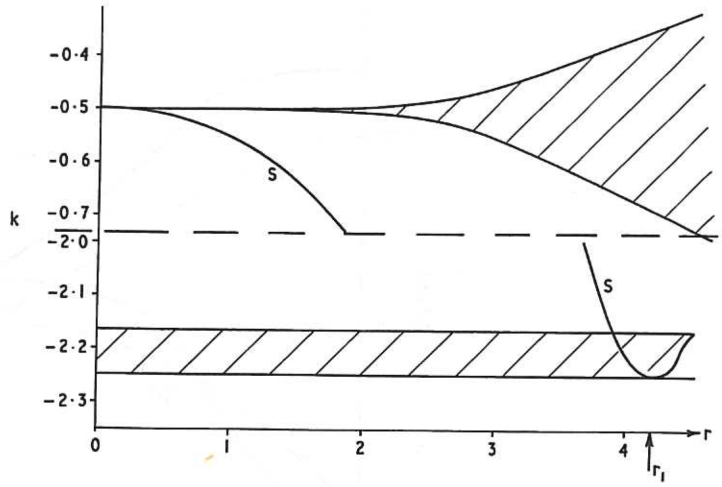

Fig.9 Stability diagram for the paramagnetic model with $\sigma_{\parallel} = \sigma_{\parallel}(0) \times (1 - r^2/21)$, $\sigma_{\perp} = 0$. The hatched regions are unstable for a conducting wall placed at $r = 4{\cdot}5$ and S is the radius at which a singularity occurs for a given wave number. r_1 denotes the position where the shear vanishes.

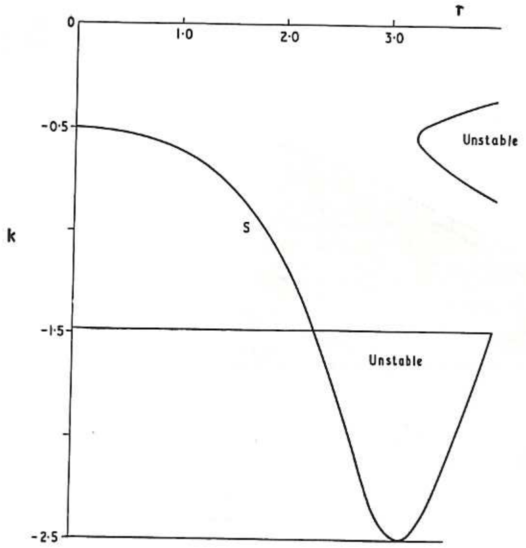

Fig.10 Stability diagram for a pitch variation $P(r) = 0{\cdot}8 \cos r + 1{\cdot}2$ and $\frac{dp}{dr} = 0$. The unstable regions are shown for a conducting wall at $r = 4{\cdot}0$. S denotes the radius at which a singularity occurs.

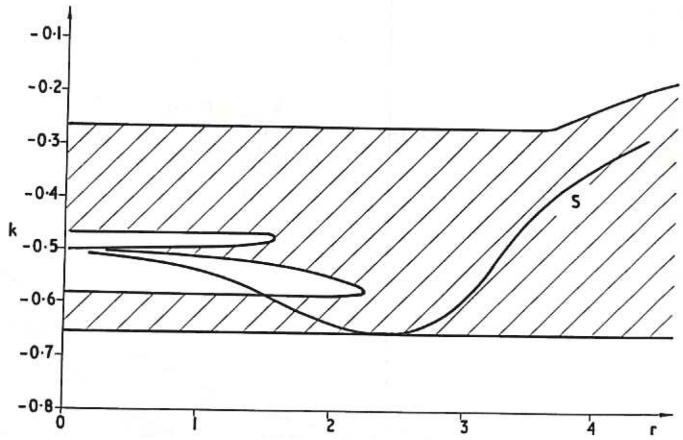

Fig.11 Stability diagram for a paramagnetic model with $\sigma_{\parallel} = \sigma_{\parallel}(0) \times (1 - r^2/10)$, $\sigma_{\perp} = 0$. The hatched regions are unstable for a conducting wall at $r = 4{\cdot}5$, and S is the radius of the singularity.

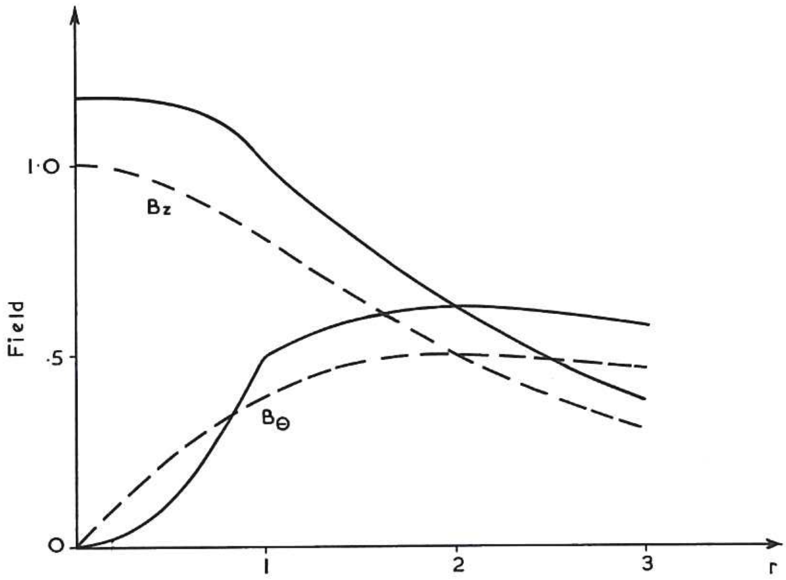

Fig.12 The dotted curves show the constant pitch zero pressure gradient field configuration and the full curves a force free configuration with constant pitch for $r > 1$ and $B_\theta = r^2/2$ for $0 < r < 1$.

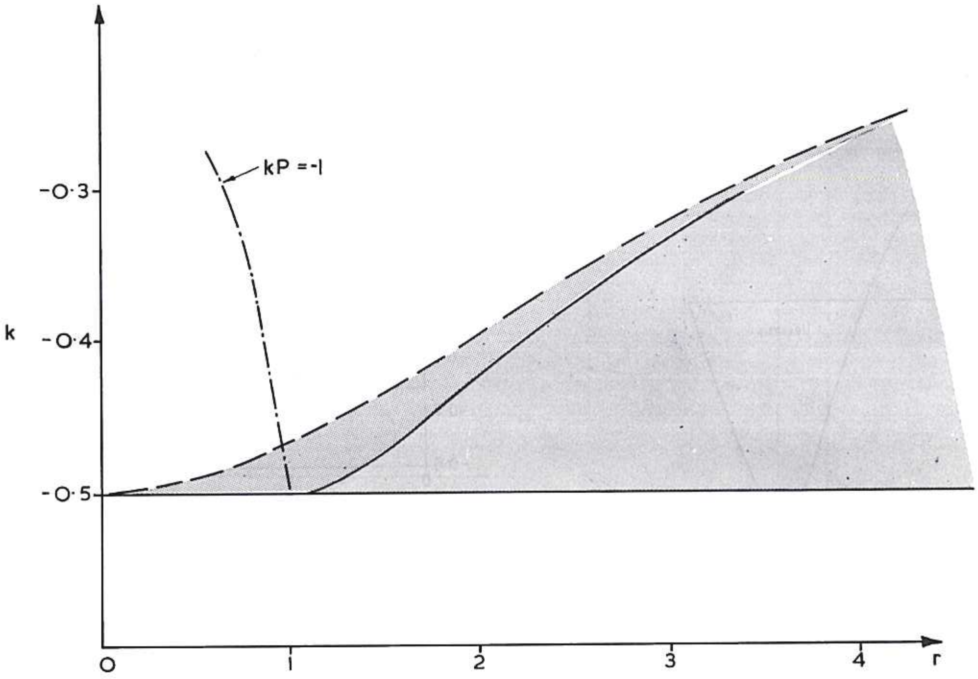

Fig.13 Stability diagram for the two field configurations of Fig.12. The outer limits of the shaded region refer to the constant pitch case and the inner region to the constant pitch configuration with a central core. The line $kP = -1$ denotes the radius at which the singularity occurs for the second field configuration. CLM-P 247

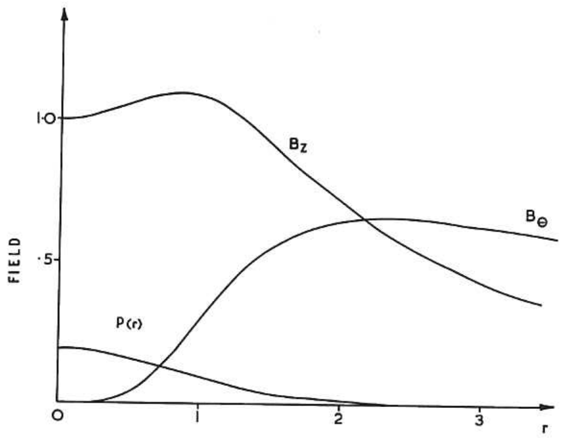

Fig.14 Field configuration and pressure profile for $P = 2{\cdot}0 + 1{\cdot}6/r^3$ which satisfies the Suydam criterion and which has an approximately constant pitch field outside the plasma but is unstable.

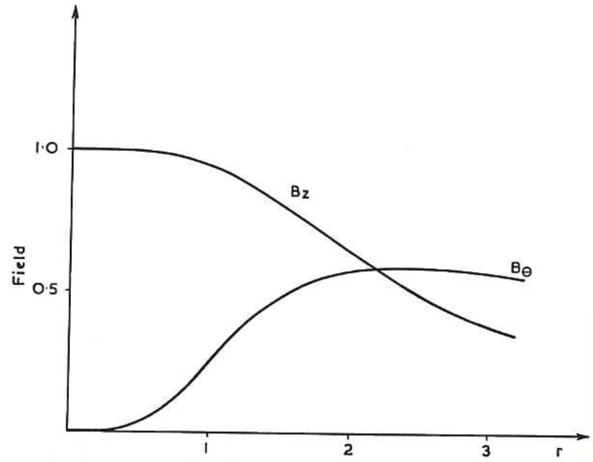

Fig.15 Field configuration with a pitch variation $P = 2{\cdot}0 + 1{\cdot}6/r^3$ and a pressure gradient determined from the integral condition.

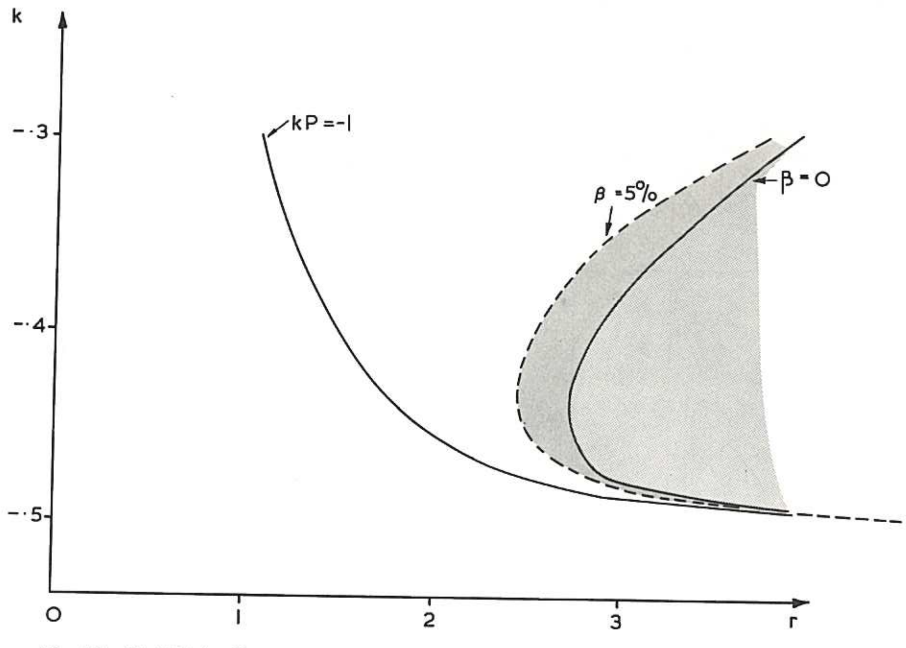

Fig.16 Stability diagram for the configuration of Fig.15 for $\beta = 0$ and 5%. The shaded regions are unstable and $kP = -1$ denotes the radius at which the singularity occurs.