CLM - P248

United Kingdom Atomic Energy Authority

RESEARCH GROUP

Preprint

ABSORPTION OF PLASMA WAVES IN COLLISIONAL MIRROR PLASMAS

J. G. CORDEY

Culham Laboratory
Abingdon Berkshire

1970

CLM - P 248

ABSORPTION OF PLASMA WAVES IN COLLISIONAL MIRROR PLASMAS

by

J.G. Cordey

(To be submitted for publication in Physics of Fluids)

A B S T R A C T

The equations for the equilibrium density and temperature profiles and the self consistent electric potential are derived for mirror contained collisional distributions. The equations are solved in two limiting cases: (a) the normal mirror reactor equilibrium where the electrons are contained by the electrostatic potential ($e\varphi \gg \frac{1}{2} mV_e^2$); (b) electron heated plasmas where $e\varphi \ll \frac{1}{2}mV_e^2$. The critical magnetic field scale length in the mirror throat necessary for plasma wave absorption is found to be in case (a)

$$L_T > a_i \left(\frac{T_e}{T_i}\frac{M_i}{M_e}\right)^{\frac{1}{2}} \exp(2e\varphi/3mV_e^2) ,$$

and in case (b)

$$L_T > a_i \left(\frac{T_e}{T_i}\frac{M_i}{M_e}\right)^{\frac{1}{2}}/(R_o - 1)^{\frac{1}{2}} .$$

Neither of these conditions should pose a serious problem for the design of mirror reactors.

U.K.A.E.A. Research Group,
Culham Laboratory,
Abingdon, Berks.

August, 1970.

CONTENTS

1. INTRODUCTION

The problem of plasma wave reflection and absorption at the ends of mirror plasmas is of great interest since all mirror equilibria are unstable to convective waves[1]. The first discussion of plasma wave reflection at the ends of mirrors was given by Aamodt and Book[2]. These authors assumed that the plasma was locally Maxwellian and found that the reflection coefficient was very small indeed. Several other authors have since refined this basic theory but always taking the distribution as locally Maxwellian and in all cases the reflection coefficient was found to be quite small.

Recent work by Taylor[3] has questioned the assumption made in the previous papers that the particle distributions are Maxwellian. Taylor's[3] conclusion was that when the electron distribution function $f(\varepsilon)$ has a sharp cut off in energy at the peak of the electrostatic potential φ_m, plasma waves are reflected from the ends of the mirror. Berk[4] has since shown that if the sharp cut off in $f(\varepsilon)$ is relaxed to a linear fall off with energy (due to collisions) then the ends may absorb plasma waves. The point which is made by both these papers is that wave absorption will be critically dependent upon the shape of the distribution.

In this paper we examine the absorption properties of collisional distributions in a mirror machine. In Sec.2 the velocity distributions and the self consistent electrostatic potential for a collisional mirror equilibrium are derived. In Sec.3 the absorption properties of mirror equilibria will be discussed for two limiting cases, first the usual reactor collisional equilibrium where the electrons are confined by the electrostatic forces $(e\varphi_m \gg \frac{1}{2}mV_e^2)$ and second electron heated plasmas where the electrons may be contained magnetically ($\frac{1}{2}mV_e^2 \gg e\varphi_m$). For each case the magnetic field scale lengths required for absorption are given.

2. EQUILIBRIUM SOLUTION

The equilibrium distribution function in a mirror machine is obtained by a solution of the relevant Fokker Planck equations and Poisson's equation.

Most of the analytical and numerical work on the Fokker Planck equation has been concerned with solutions in a mirror in which the central field is uniform and at the mirrors there is a step function in the magnetic field. This approximation which is known as the square well approximation is thought to be a good approximation since most of the collisional processes take place in the central high density region, the shape of the magnetic field at the mirrors having only a small effect on the particle distributions in the centre. A calculation by Marx[5] in which the variation in the magnetic field and electrostatic potential between the mirrors is taken into account appears to agree quantitatively with the results of the square well approximation. Bearing this in mind in this paper we shall assume that the distribution at the centre of the machine is determined by the solution of the Fokker Planck equations in the square well approximation and then determine the distribution in the mirror regions using the constants of energy and magnetic moment.

It has been shown by Watson[6] that a good analytic approximation to the electron distribution is to take the distribution as a Maxwellian for energies less than $e\varphi_m$ and as a collisional loss-cone distribution with effective mirror ratio $R_e = R_o/(1 - e\varphi_m/\varepsilon)$

for energies greater than $e\varphi_m$. The ions are assumed to be a collisional loss-cone distribution with mirror ratio R_o.

For high mirror ratios ($R_o > 1 \cdot 5$) the electron distribution may be written in the form:

$$f_e(\varepsilon , u) = K \exp(-\varepsilon/\varepsilon_o) \; S(u) \qquad \ldots (1)$$

where

$$S(u) = 1 + \log_e(1 - u^2)/\log_e R_e \quad \text{for} \quad \varepsilon > \chi_m$$
$$= 1^- \quad \text{for} \quad \varepsilon < \chi_m$$

is an approximate solution of Legendre's[7] equation, $\varepsilon = \frac{1}{2} mv^2 + \chi \equiv \frac{1}{2} m(v_{\shortparallel}^2 + v_{\perp}^2) + e\varphi$, ε_o is the mean electron energy and K is a normalising constant.

The expression for $S(u)$ the solution of Legendre's given in Eq.(1) is accurate to within 10% over the whole range of u for mirror ratio's greater than $1 \cdot 5$. For mirror ratio's less than $1 \cdot 5$ use is made of the series solution of Legendre's equation and to obtain accuracy of 10% it is sufficient to consider the first two terms only

$$S(u) = 1 - \frac{R_o}{R_o - 1} u^2 \quad \text{for} \quad R_o < 1 \cdot 5 \qquad \ldots (1a)$$

In the remainder of this Section and Sec.3 to simplify the analysis we discuss fully only the high mirror ratio case (distribution (1)), however where it is appropriate the results for the low mirror ratio distribution (1a) will be given.

In the discussion in the next Section on the absorption of plasma waves the equilibrium number density and mean parallel energy are required. Using Eq.(1) the expressions for number density and mean velocity are

$$n_e = \frac{K}{\sqrt{2m^3}} \int_{\chi}^{\infty} \exp(-\varepsilon/\varepsilon_o)\, d\varepsilon \int_{\mu_L}^{(\varepsilon-\chi)/B} \frac{S\left\{\left(\frac{\varepsilon - \mu B_o}{\varepsilon}\right)^{\frac{1}{2}}\right\}}{(\varepsilon - \mu B - \chi)^{\frac{1}{2}}}\, d\mu$$

$$n_e\, \bar{v}_{\parallel}^2 = K\sqrt{\frac{2}{m^3}} \int_{\chi}^{\infty} \exp(-\varepsilon/\varepsilon_o)\, d\varepsilon \int_{\mu_L}^{(\varepsilon-\chi)/B} (\varepsilon - \mu B - \chi)^{\frac{1}{2}}\, S\left\{\left(\frac{\varepsilon - \mu B_o}{\varepsilon}\right)^{\frac{1}{2}}\right\} d\mu$$

respectively, where $\mu = \frac{1}{2} m v_{\perp}^2/B$, B_o is the field in the centre of the mirror. The lower limit of the μ integration μ_L is zero if $\varepsilon < \chi_m$ and $\mu_L = (\varepsilon - \chi_m)/B_m$ if $\varepsilon > \chi_m$, the suffix m means maximum value. Splitting the ε integration into two regions $o < \varepsilon < \chi_m$ and $\chi_m < \varepsilon < \infty$ and then completing the μ integration gives

$$n_e = K\sqrt{\frac{2}{m^3}} \left[\int_{\chi}^{\chi_m} \exp(-\varepsilon/\varepsilon_o)(\varepsilon - \chi)^{\frac{1}{2}} d\varepsilon + \int_{\chi_m}^{\infty} \frac{\exp(-\varepsilon/\varepsilon_o)}{\log_e R_e} (\varepsilon - \chi)^{\frac{1}{2}} \left\{ \log_e\left(\frac{1+\alpha}{1-\alpha}\right) - 2\alpha \right\} d\varepsilon \right] . \quad \ldots (2)$$

$$n_e\, \bar{v}_{\parallel}^2 = \frac{2}{3}\sqrt{\frac{2}{m^3}}\, K \left[\int_{\chi}^{\chi_m} \exp(-\varepsilon/\varepsilon_o)(\varepsilon - \chi)^{3/2}\, d\varepsilon + \int_{\chi_m}^{\infty} \frac{\exp(-\varepsilon/\varepsilon_o)(\varepsilon - \chi)^{3/2}}{\log_e R_e} \left\{ \log_e\left(\frac{1+\alpha}{1-\alpha}\right) - 2\alpha - \frac{2}{3}\alpha^3 \right\} d\varepsilon \right] , \quad \ldots (3)$$

where $\alpha = \left\{1 - \frac{\varepsilon - \chi_m}{(\varepsilon - \chi)R}\right\}^{\frac{1}{2}}$ and R is the local mirror ratio $R = B_m/B$.

The ions are assumed to be contained magnetically and hence the ion velocity distribution will be identical to Eq.(1) for the electron velocity distribution with R_e replaced by R_o. The ion number density can then be written in the form

$$n_i = \sqrt{\frac{2}{m^3}}\, K \frac{\left[\log_e \left\{\frac{1 + (1 - 1/R)^{1/2}}{1 - (1 - 1/R)^{1/2}}\right\} - 2(1 - 1/R)^{1/2}\right]}{\log_e R_o} \int_0^\infty \exp(-\varepsilon/\varepsilon_o)\,\varepsilon^{1/2} d\varepsilon \quad .$$

Finally completing the ε integration then gives

$$n_i = N_o \left[\log_e \left\{\frac{1 + (1 - 1/R)^{1/2}}{1 - (1 - 1/R)^{1/2}}\right\} - 2(1 - 1/R)^{1/2}\right] / \gamma \ , \qquad \ldots (4)$$

where

$$\gamma = \log_e \left\{\frac{1 + (1 - 1/R_o)^{1/2}}{1 - (1 - 1/R_o)^{1/2}}\right\} - 2(1 - 1/R_o)^{1/2} \quad .$$

For low mirror ratios ($R_o < 1{\cdot}5$) using distribution (1a) gives

$$n_i = N_o \frac{R}{R_o} \left(\frac{1 - 1/R}{1 - 1/R_o}\right)^{3/2} \quad . \qquad \ldots (4a)$$

From Eq.4, one may easily deduce simple useful analytic expressions for the variation of n_i with R at the centre and ends of the mirror. At the centre of the mirror $1/R$ is small and expanding in $1/R$ gives $n_i \propto \log 4R - 2$. At the ends of the mirror $R \to 1$ hence expanding in $(1 - 1/R)$ we find that $n_i \propto (1 - 1/R)^{3/2}$. Thus the density varies slowly with R in the central portion of the mirror with a steeper variation at the ends.

To obtain the self consistent potential χ as a function of R and hence as a function of z the quasi-neutral approximation is used

$$n_i = n_e \quad . \qquad \ldots (5)$$

This approximation will be good as long as the scale length for the variation of χ is large compared with the electron debye length. Using Eqs(2), (4) and (5) one may in principle obtain an expression for χ in terms of R. It is difficult to see though how one can make further analytic progress with Eq.(2) for general χ and ε_o. However, Eq.(2) may be considerably simplified in the limits of either pure electrostatic confinement of electrons ($\chi_m \gg \varepsilon_o$) or magnetic confinement of electrons ($\chi_m \ll \varepsilon_o$). The first limit (electrostatic confinement of electrons) is appropriate for reactor conditions where the ion energy, electrostatic potential, and the electron energy are in the ratio 1 : 0·5 : 0·1 respectively. The magnetic confinement approximation is applicable to some experimental conditions in which the electrons are microwave heated.

A. Electrostatic Confinement of Electrons ($\chi_m \gg \varepsilon_o$)

For $\chi_m \gg \varepsilon_o$ the expressions for n and $n_e\bar{v}_{\parallel}^2$ (Eqs.(2) and (3)) may be considerably simplified. At the centre of the mirror where $\chi = 0$,

$$n_e = \sqrt{\frac{2}{m^2}}\, K \int_{\chi}^{\chi_m} e^{-\varepsilon/\varepsilon_o}(\varepsilon - \chi)^{\frac{1}{2}} d\varepsilon + O(\exp(-\chi_m/\varepsilon_o)) \ ,$$

and

$$n_e\bar{v}_{\parallel}^2 = \frac{2}{3}\sqrt{\frac{2}{m^5}}\, K \int_{\chi}^{\chi_m} e^{-\varepsilon/\varepsilon_o}(\varepsilon - \chi)^{3/2} d\varepsilon + O(\exp(-\chi_m/\varepsilon_o)) \ .$$

At ends of the mirror where $\chi \to \chi_m$, Eqs.(2) and (3) become

$$n_e = \sqrt{\frac{2}{m^3}}\, K\rho\exp(-\chi_m/\varepsilon_o)\left[\frac{2}{3}(\chi_m - \chi)^{3/2} + O\left\{\chi_m - \chi)^{5/2}\right\}\right] , \qquad \ldots (6)$$

and

$$n_e\bar{v}_{\parallel}^2 = \frac{2}{3}\sqrt{\frac{2}{m^5}}\, K\rho\exp(-\chi_m/\varepsilon_o)\left[\frac{2}{5}(\chi_m - \chi)^{5/2} + O\left\{(\chi_m - \chi)^{7/2}\right\}\right] , \qquad \ldots (7)$$

where $\rho(\chi)$ is a slowly varying function of χ,

$$\rho(\chi) = 1 + \log_e[\log_e\{\varepsilon_o/(\chi_m - \chi)\}].$$

For the usual reactor parameters ($n = 10^{14}$, $B = 50$ kG) $\rho(\chi)$ varies between 1 and 5. Using the equation of quasi-neutrality (5) and Eqs.(4) and (6) we find the following relation between χ and R in the mirror region ($R \to 1$).

$$\chi_m - \chi = (R - 1)\varepsilon_o \exp(2/3\ \chi_m/\varepsilon_o)\gamma^{-2/3}\pi^{1/3} .$$

Hence Eqs.(6) and (7) may be written in the form

$$n_e = \frac{2}{3} N_o(R - 1)^{3/2}\gamma^{-1}\rho , \qquad \ldots (8)$$

$$v_\parallel^2 = \frac{2}{5}(R - 1)\varepsilon_o/m \exp(2\chi_m/3\varepsilon_o)\gamma^{-2/3}\pi^{1/3} .$$

B. Magnetic Confinement of Electrons ($\chi_m \ll \varepsilon_o$)

In this limit $\chi_m = \chi = 0$, completing the integrals in Eqs.(2) and (3) and expanding in $(R - 1)$ to determine the behaviour at the mirrors gives

$$n_e = N_o(R - 1)^{3/2}\gamma^{-1} , \qquad \ldots (9)$$

$$v_\parallel^2 = \frac{(R - 1)}{R_o^{-1}} v_{\parallel o}^2 .$$

For the low mirror ratio case γ in Eq.(9) is replaced by $(R_o - 1)^{3/2}$.

In the next Section we discuss the wave absorption properties of the mirror region for the two types of equilibrium solution.

3. ABSORPTION PROPERTIES OF MIRRORS

The dispersion equation for plasma waves in an inhomogeneous media is

$$k_\perp^2 = \frac{\omega_{pe}^2\, k^2}{\omega^2} + 2\pi^{1/2}\, i\, \frac{\omega_{pe}^2\, \omega}{k_\parallel\, v_{\parallel e}^3} \exp(-\omega^2/k_\parallel v_e^2)$$

$$\ldots (10)$$

where the usual WKB approximation has been made $\varphi = \psi(z)\exp(i\int k_{\parallel}dz - i\omega t)$ and

$$\frac{1}{k_{\parallel}^2}\frac{dk_{\parallel}}{dz} \ll 1 \quad . \qquad \ldots (11)$$

If the inequality (11) is violated reflection may occur and Eq.(10) is no longer valid. In the following it will be assumed that inequality (11) is satisfied and it will be shown that this is the case later.

We now calculate the reduction in wave amplitude caused by Landau damping. Since the imaginary part of Eq.(10) is small one can solve approximately for $k_{\parallel}$

$$\text{Im } k_{\parallel} = -\frac{\omega_{pe}^2}{k_{\perp}^2 v_{\parallel}^3}\,\omega\,\pi^{\frac{1}{2}}\exp(-\omega_{pe}^2/k_{\perp}^2 v_{\parallel}^2) \quad . \qquad \ldots (12)$$

Now the reduction in wave amplitude

$$\delta = \exp(-\int \text{Im } k_{\parallel}dz) \quad . \qquad \ldots (13)$$

We may determine δ for the two types of equilibrium by substituting in Eq.(13) for $I_m k_{\parallel}$ given by Eq.(12) and using Eq.(8) to determine ω_{pe} and $v_{\parallel}$.

A. Electrostatic Confinement of Electrons

The decrement in wave amplitude

$$\delta \simeq \exp\left\{-\frac{3\cdot\omega_{po}\omega\rho}{k_{\perp}^2(\varepsilon_o/m)^{3/2}}\exp(-\chi_m/\varepsilon_o)\int_{-L_T}^{0}\exp\left[\frac{\omega_{po}^2\, m\gamma^{1/3}\rho}{2k_{\perp}^2\varepsilon_o}\,\frac{z}{L_T}\,\frac{5}{3}\right.\right.$$

$$\left.\left.\exp\left(-\frac{2}{3}\frac{\chi_m}{\varepsilon_o}\right)\right]dz\right\}$$

where we have assumed $R = 1 + z^2/L_T^2$, L_T being the magnetic scale length in the mirror region. Completing the integration over z and ignoring the slow variation of ρ gives

$$\delta = \exp\left\{3\cdot 6\ \gamma^{\frac{1}{3}}\left[\frac{-nL_T}{a_i}\left(\frac{T_i}{T_e}\frac{m_e}{m_i}\right)^{\frac{1}{2}}\exp(-\chi_m/3\varepsilon_o)\right]\right\}\cdot \quad \ldots (14)$$

B. Magnetic Confinement of Electrons

Repeating the similar calculation but using Eq.(9) for the equilibrium qualities we find

$$\delta = \exp\left\{\frac{nL_T}{a_i}\left(\frac{T_i}{T_e}\frac{m_e}{m_i}\right)^{\frac{1}{2}}(R_o - 1)^{\frac{1}{2}}\right\} . \quad \ldots (15)$$

One can see from (14) and (15) that the two expressions for the decrement of wave amplitude δ are very similar, and indeed for reactor parameters ($\chi_m/\varepsilon_o \simeq 5$) we may ignore the exponential term inside the square brackets of Eq.(16). Thus we see that a significant reduction in amplitude of the wave will occur in the mirrors if

$$\frac{L_T}{a_i} \gg \left(\frac{T_e}{T_i}\frac{m_i}{m_e}\right)^{\frac{1}{2}} \quad \ldots (16)$$

(n = 1) being the worst case). This is not a difficult condition to satisfy and it should not place any severe restrictions on mirror reactor design.

As mentioned earlier it must be shown that inequality (11) is valid throughout the mirror throat. Using Eq.(8) we find that

$$\frac{1}{k_{\parallel}^2}\frac{dk_{\parallel}}{dz} \simeq \frac{\omega_{po}z^{\frac{1}{2}}}{\omega k_{\perp}L_T^{3/2}}$$

hence as $R\to 1$ and $z\to 0$ $1/k_{\parallel}dk_{\parallel}/kz\to 0$, so the condition for the WKB approximation to be valid is indeed satisfied in the mirror throat.

To summarise the conditions for absorption of plasma waves in a mirror machine containing a collisional plasma have been derived Eq.(16). This condition should be fairly easily satisfied in a mirror reactor. In passing it should be mentioned that the results derived in this paper are not applicable to the present generation of neutral injection mirror machines PHEONIX II and Alice, since in both of these machines the ion distribution is highly peaked in angle and certainly not of the collisional type used here; also and probably more important the plasma is localised near the centre of the mirror.

REFERENCES

1. M.N. ROSENBLUTH and R.F. POST. Phys. Fluids 8, 547, (1965).
2. R.E. AAMODT, and D.L. BOOK. Phys. Fluids 9, 143, (1966).
3. J.B. TAYLOR. Private Communication.
4. H.L. BERK. Research Note.
5. J. KILLEEN and K.D. MARX. Lawrence Radiation Laboratory report UCRL-71662 (1969).
6. C.J.H. WATSON. Private Communication.
7. G.F. BING and J.E. ROBERTS. Phys. Fluids 4, 1039, (1961).