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INSTABILITY IN A PERPENDICULAR COLLISIONLESS SHOCK WAVE FOR ARBITRARY ION TEMPERATURES

C. N. LASHMORE-DAVIES

Culham Laboratory
Abingdon Berkshire

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INSTABILITY IN A PERPENDICULAR COLLISIONLESS SHOCK WAVE FOR ARBITRARY ION TEMPERATURES

C.N. LASHMORE-DAVIES

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Two instability mechanisms which could occur under the conditions found in a perpendicular collisionless shock wave are considered. Both mechanisms are due to the negative energy character of the electron Bernstein waves propagating in the direction of the current flow in the shock. The effect of the voltage jump through the shock, and the density and magnetic field gradients at the shock front are considered. Both instability mechanisms occur only within a definite band of k-values. The first mechanism is due to a resonance between the ion acoustic wave and one of the Bernstein harmonics and the second to resonant ions absorbing energy from a negative energy Bernstein mode. The first case requires $T_e \gg T_i$ whereas the second case can occur for arbitrary values of the ratio T_i/T_e, although the maximum effect occurs when this ratio is of order of unity.

U.K.A.E.A. Research Group,
Culham Laboratory,
Abingdon,
Berks

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I. INTRODUCTION

There are two main schools of thought concerning collisionless shock waves[1]. The first of these seeks to explain their properties entirely in terms of ordered non-linear oscillations and the second assumes that the anomalous dissipation at the front is related to plasma turbulence. For the turbulence hypothesis there must be an instability mechanism. In this paper we shall be concerned with such a mechanism.

Sagdeev[1] has suggested an ion wave instability and this idea has been refined by Krall and Book[2,3]. They considered electrostatic waves propagating perpendicularly to the magnetic field and the shock front which were driven unstable by the drifts due to the gradients of density and magnetic field. However, they did not include the effect of the voltage jump at the front and also neglected the part of the dispersion relation which gives rise to Bernstein[4] waves. Gary and Sanderson[5] pointed out that the $\underline{E}_0 \times \underline{B}_0$ drift due to the voltage jump at the front is the dominant one and they obtained an instability with a larger growth rate by allowing for the effect of this drift on the Bernstein waves. The reason for the importance of the Bernstein waves is that in the presence of an $\underline{E}_0 \times \underline{B}_0$ drift Bernstein waves carry negative energy[6] when $0 < \omega < kv_0$, where v_0 is the $-E_{0x}/B_{0z}$ drift velocity. Using this interpretation one expects instability either when the negative energy Bernstein wave comes into resonance with the ion acoustic wave or when it is Landau damped by the ions[6]. This latter mechanism is interesting because it can give rise to instability even when $T_i > T_e$.

Gary and Sanderson[5] considered the first of these instability mechanisms numerically and included the effect of the magnetic field gradient but not the density gradient. Wong[7] has also considered the first of these mechanisms but he neglected all gradients. The present author[6] considered both mechanisms analytically but neglected the effect of density and magnetic field gradients. The aim of this paper is to include the effects of these gradients as well as the voltage jump at the front. It will be found that the inclusion of the density gradient is particularly important.

II. EQUILIBRIUM

We consider the following slab model. The applied magnetic field $\underline{B}_0$ is taken as the z-axis and the gradients of plasma density and applied magnetic field are in the x-direction. There is also a uniform electric field E_{0x} in the x-direction. These gradients are intended as a model of the shock front which propagates in the x-direction.

We take the following zero order magnetic field:

$$\underline{B}_0 = B_{oz}\,(1 + \varepsilon x)\,\hat{\iota}_z \ , \qquad \ldots (1)$$

which gives rise to a guiding centre drift of the electrons in the y-direction:

$$v_B^e = - \varepsilon \frac{v_\perp^2}{2\omega_{ce}} \ , \qquad \ldots (2)$$

where $\omega_{ce} = |e|B_{oz}/m_e$ is the electron cyclotron frequency. We also assume a density profile:

$$n = n_o \ (1 + \varepsilon' x) \ , \qquad \ldots (3)$$

which produces an electron diamagnetic drift in the y-direction:

$$v_d^e = - \varepsilon' \frac{\kappa T_e}{|e| B_o} \ . \qquad \ldots (4)$$

From the Maxwell equation:

$$\underline{\nabla} \times \underline{H} = \underline{J} + \varepsilon_o \frac{\partial \underline{E}}{\partial t} \ , \qquad \ldots (5)$$

we can obtain the following relationship between the zero order electric field E_{ox} and the density and magnetic field gradients:

$$E_{ox} \sim - \frac{B_{oz}^2}{n_o |e| \mu_o} \left(\varepsilon + \frac{\beta}{2} \varepsilon'\right) \ , \qquad \ldots (6)$$

where

$$\beta = 2\mu_o n_o \ \kappa T_e / B_{oz}^2 \ .$$

In order to obtain the unperturbed electron distribution function we must consider the orbit equations of the electrons. These equations are:

$$\ddot{x} = - |e| \frac{E_{ox}}{m_e} - \omega_{ce} \dot{y} (1 + \varepsilon x) \qquad \ldots (7)$$

$$\ddot{y} = \omega_{ce} \dot{x} \ (1 + \varepsilon x) \qquad \ldots (8)$$

$$\ddot{z} = 0 \qquad \ldots (9)$$

The constants of the motion are:

$$v_z \quad , \quad v_x^2 + \left(v_y + \frac{E_o}{B_o}\right)^2 \quad \text{to order } \varepsilon$$

and

$$v_y - \omega_{ce} \, x \left(1 + \frac{\varepsilon x}{2}\right) \ .$$

We may now construct a zero order electron distribution function which satisfies the unperturbed Vlasov equation to order ε . The function we take is similar to that used by Krall and Book[2]:

$$f_o^e = n_o \left\{1 - \varepsilon' \left(\frac{v_y}{\omega_{ce}} - x\right)\right\} \left(\frac{m_e}{2\pi \ \kappa T_e}\right)^{\frac{3}{2}} \exp\left\{- m_e \frac{[v_x^2 + (v_y - v_o)^2 + v_z^2]}{2 \ \kappa T_e}\right\}$$

$$\ldots (10)$$

where

$$v_o = -\frac{E_{ox}}{B_{oz}} \qquad \ldots (11)$$

Following Krall and Book[2] we use the approximate orbit solutions to equations (5)-(7):

$$x(t) = \frac{v_{\perp o}}{\omega_{ce}} \sin(\omega_{ce}\, t + \psi) - \frac{v_{\perp o}}{\omega_{ce}} \sin\psi \qquad \ldots (12)$$

$$y(t) = -\frac{E_o}{B_o}\, t - \varepsilon \frac{v_{\perp o}^2}{2\omega_{ce}}\, t - \frac{v_{\perp o}}{\omega_{ce}} \cos(\omega_{ce} t + \psi) + \frac{v_{\perp o}}{\omega_{ce}} \cos\psi \qquad \ldots (13)$$

$$z(t) = v_{zo} t \qquad \ldots (14)$$

$$v_y(t) = -\frac{E_o}{B_o} - \varepsilon \frac{v_{\perp o}^2}{2\omega_{ce}} + v_{\perp o} \sin(\omega_{ce}\, t + \psi) \qquad \ldots (15)$$

$$v_x(t) = v_{\perp o} \cos(\omega_{ce}\, t + \psi) \qquad \ldots (16)$$

$$v_z(t) = v_{zo} \qquad \ldots (17)$$

where:

$$v_{\perp o}^2 = v_{xo}^2 + v_{yo}^2 \quad , \quad \tan^{-1} \psi = \frac{v_{yo}}{v_{xo}} \quad ,$$

and at $t = 0$ the electron is assumed to be at the origin in configuration space with velocity components v_{xo} , v_{yo} , v_{zo} . The oscillatory terms of order ε have been neglected in the above solutions as have all other terms of higher order in ε .

The treatment of the ions is simplified by the fact that the transit time of the shock is much less than the ion cyclotron frequency (but much greater than the electron cyclotron frequency) and so we may take the ions to be unmagnetized as pointed out by Krall and Book[2]. We therefore take the ion distribution to be Maxwellian.

III. THE DISPERSION RELATION

We start from the linearized Vlasov equation for electrons:

$$\frac{\partial f_1^e}{\partial t} + \underline{v} \cdot \frac{\partial f_1^e}{\partial \underline{r}} - \frac{|e|}{m_e}(\underline{E}_0 + \underline{v} \times \underline{B}_0) \cdot \frac{\partial f_1^e}{\partial \underline{v}} = \frac{|e|}{m_e} \underline{E}_1 \cdot \frac{\partial f_0^e}{\partial \underline{v}}, \qquad \ldots (18)$$

where we have used the electrostatic approximation:

$$\underline{E}_1 = - \underline{\nabla} \varphi . \qquad \ldots (19)$$

Since we shall be concerned with very short wavelength Bernstein modes such that:

$$k^2 a_e^2 \gg 1 ,$$

where k is the wave number and a_e is the electron Larmor radius, equation (19) should be a good approximation even when β is not small[8]. For such short wavelength oscillations it should be a good approximation to neglect the x-variation (i.e. $k/\varepsilon' \sim k/\varepsilon \gg 1$) of the perturbed quantities and assume a variation of the form :

$$\exp i(ky - \omega t) .$$

With the aid of equations (12) - (17) for the orbits and the method of orbit integration we obtain the perturbed electron charge density:

$$\rho_1^e = - \frac{e^2 n_0}{\kappa T_e} \varphi - \frac{e^2 n_0}{\kappa T_e} \varphi \, (\omega - kv_0 - kv_d^e) \sum_{n=-\infty}^{n=\infty} \mathcal{I}_n , \qquad \ldots (20)$$

where

$$\mathcal{I}_n = \int_0^\infty e^{-m_e v_\perp^2 / 2 \kappa T_e} \frac{J_n^2\left(\frac{kv_\perp}{\omega_{ce}}\right) v_\perp \, dv_\perp}{\left[k\left(v_0 - \frac{\varepsilon v_\perp^2}{2\omega_{ce}}\right) - \omega + n \, \omega_{ce}\right]} \qquad \ldots (21)$$

and we have neglected a term $\sim (k^2 a_e^2)^{-1}$ since we shall always assume

$$k^2 a_e^2 \gg 1 .$$

Now:

$$\left| \frac{\langle v_B^e \rangle}{v_0} \right| \sim \beta$$

and so for $\beta \ll 1$ the drift due to the magnetic field can be neglected in comparison with that due to the electric field. This simplifies the analysis enormously and enables a clear physical interpretation to be made of the resulting dispersion relation. The effect of neglecting the drift due to the magnetic field gradient will be considered later. However, with the neglect of this drift we obtain the following expression for the perturbed electron charge density:

$$\rho_1^e = -\frac{n_o e^2}{\kappa T_e}\varphi + \frac{n_o e^2}{\kappa T_e}\varphi \sum_{n=-\infty}^{n=\infty} \frac{(\omega - kv_o - kv_d^e)}{(\omega - kv_o - n\,\omega_{ce})}\beta_n \qquad \ldots (22)$$

where:

$$\beta_n = \exp(-k^2 a_e^2)\, I_n(k^2 a_e^2) \qquad \ldots (23)$$

and I_n is the n^{th} order modified Bessel function of the first kind.

Using the fact that the ions can be treated as unmagnetized and as having a Maxwellian distribution we can easily obtain the perturbed ion charge density.

$$\rho_1^i = -\frac{n_o e^2}{\kappa T_i}\varphi\left(1 + \xi_i Z(\xi_i)\right) \qquad \ldots (24)$$

where $(\kappa T_i/m_i)^{\frac{1}{2}} = v_{T_i}$ is the ion thermal velocity and $\xi_i = \omega/\sqrt{2}\,kv_{T_i}$. Substituting expressions (22) and (24) into Poisson's equation we obtain the dispersion relation

$$1 + \frac{\omega_{pi}^2}{k^2 v_{T_i}^2}\left(1 + \xi_i Z(\xi_i)\right) + \frac{\omega_{pe}^2}{k^2 v_{T_e}^2} - \frac{\omega_{pe}^2}{k^2 v_{T_e}^2}\sum_{n=-\infty}^{n=\infty}\frac{(\omega - kv_o - kv_d^e)}{(\omega - kv_o - n\,\omega_{ce})}\beta_n = 0$$

$$\ldots (25)$$

IV. STABILITY ANALYSIS

Before attempting to obtain unstable solutions of equation (25) we will first of all consider the small signal energy. This is important because it will indicate the conditions required for instability.

The small signal energy of an electrostatic wave is given by:

$$\mathcal{E} = \tfrac{1}{4}\,\varepsilon_0 \left|E_1\right|^2 \frac{\partial}{\partial\omega}\Big(\omega\, \varepsilon_\ell(\omega,k)\Big)_{\varepsilon_\ell=0} \qquad \ldots (26)$$

where ε_ℓ is the longitudinal dielectric constant of the plasma. In all the cases of instability considered below the electron Bernstein waves will play a fundamental role. The reason for this (as already mentioned) is that under certain conditions these waves may have negative small signal energy. In order to discover these conditions we consider the electron part of ε_ℓ :

$$\omega\,\frac{\partial\varepsilon_{\ell e}}{\partial\omega} \approx (kv_0 + n\,\omega_{ce})\beta_n\,\frac{\omega_{pe}^2}{v_{Te}^2}\;\frac{(-\,k\,v_d^e + n\,\omega_{ce})}{(\omega - kv_0 - n\omega_{ce})^2} \qquad \ldots (27)$$

where we have assumed that:

$$\omega - kv_0 \approx n\omega_{ce} \qquad \ldots (28)$$

and therefore take only one term in the expression for $\varepsilon_{\ell e}$. For a perpendicular collisionless shock wave the density and magnetic field gradients are of the same sign and almost equal[9] , i.e.

$$\varepsilon/\varepsilon' \approx 1. \qquad \ldots (29)$$

For the form of the gradients we have taken we then have:

$$v_0 > 0 \quad \text{and} \quad v_d^e < 0 \qquad \ldots (30)$$

From equation (27) we then obtain the following:

$$\underline{n > 0} \qquad \omega\,\frac{\partial\varepsilon_{\ell e}}{\partial\omega} > 0$$

$$\underline{n < 0} \qquad \omega\,\frac{\partial\varepsilon_{\ell e}}{\partial\omega} < 0\ ,$$

provided

$$k\, v_0 > |n|\omega_{ce} \qquad \ldots (31)$$

and:

$$k|v_d^e| < |n|\omega_{ce} \qquad \ldots (32)$$

Equation (31) is the condition obtained by considering only the effect of the voltage jump and neglecting the density gradient. Equation (31) gives the condition on the frequency $0 < \omega < kv_0$. For $k > |n|\omega_{ce}/v_0$ the n^{th} Bernstein harmonic has negative energy. However, when the effect of the density gradient is allowed for we see that as k increases the energy of the wave becomes positive again, i.e. there is now a band of k inside which the Bernstein harmonics (of negative n number) have negative energy:

$$|n|\omega_{ce}/v_0 < k < |n|\omega_{ce}/|v_d^e| \qquad \ldots (33)$$

The significance of equation (33) lies in the fact that a negative energy wave is a potentially active wave. If there is a positive energy wave with which it can come into frequency resonance or a sink to which it can lose its energy, instability will result. Thus equation (33) gives the range of k inside which we can look for instability[10]. We now obtain solutions to the dispersion relation, first for the case of cold ions and secondly for the case of warm ions.

For cold ions the dispersion equation can be written:

$$(\omega^2 - k^2c_s^2)(\omega - kv_0 + |n|\omega_{ce}) = \omega^2(\omega - kv_0 + k|v_d^e|)\beta_n \qquad \ldots (34)$$

where we have assumed:

$$\omega - k\, v_0 \approx - |n|\omega_{ce} \quad , \quad k^2 \lambda_{de}^2 \ll 1 \; ,$$

and

$$k^2 a_e^2 \gg 1,$$

where $\lambda_{de}^2 = v_{T_e}^2/\omega_{pe}^2$ is the electron Debye length. The condition for instability is the resonance condition

$$kc_s = kv_o - |n|\omega_{ce} \, . \qquad \ldots (35)$$

Using a perturbation analysis on equation (34) we look for solutions:

$$\omega = kc_s + \delta\omega \, , \qquad \ldots (36)$$

and obtain:

$$(\delta\omega)^2 = -\frac{kc_s}{2}\,(|n|\omega_{ce} - k|v_d^e|)\,\beta_n \, . \qquad \ldots (37)$$

We therefore have instability provided

$$|n|\omega_{ce} > k|v_d^e| \, .$$

This is just condition (32) again which states that when it does not hold, the slow Bernstein wave (i.e. the mode for which $\omega < kv_o$) is a positive energy wave and so will not 'intersect' the ion acoustic branch to produce instability. From equation (37) we obtain the growth rate:

$$\frac{\gamma}{\omega_{ce}} = \frac{|n|^{\frac{1}{2}}}{(8\pi)^{\frac{1}{4}}}\left(\frac{m_e}{m_i}\right)^{\frac{1}{4}}\left(1 - \frac{k\,|v_d^e|}{|n|\omega_{ce}}\right)^{\frac{1}{2}} . \qquad \ldots (38)$$

From the resonance condition (35) we see that:

$$v_o > c_s \, , \qquad \ldots (39)$$

and

$$k\,a_e = |n|/(v_o/v_{T_e} - c_s/v_{T_e}) , \qquad \ldots (40)$$

where the value of k obtained from equation (40) must lie in the range given by equation (33). The condition for the validity of this solution is:

$$k\,a_e \gg \frac{|n|^{\frac{1}{2}}}{(8\,\pi)^{\frac{1}{4}}}\left(\frac{m_i}{m_e}\right)^{\frac{1}{4}}\left(1 - \frac{k|v_d^e|}{|n|\;\omega_{ce}}\right)^{\frac{1}{2}} \qquad \ldots (41)$$

The maximum growth rate calculated by Krall and Book[3] was:

$$\left(\frac{\gamma}{\omega_{ce}}\right)_{max} \approx \left(\frac{m_e}{m_i}\right)^{\frac{1}{2}} . \qquad \ldots (42)$$

This is appreciably smaller than the growth rates given by equation (38). At this point it is worth mentioning the effect of the drift due to the magnetic field gradient. Since this drift depends on the perpendicular velocity distribution it reduces the sharpness of the resonance condition, given by equation (35). For $\beta_e \sim 0.4$ Gary and Sanderson[4] have shown that the growth rate is reduced by a factor of two approximately.

When the effect of warm ions is considered we obtain a new instability[6] which can occur under conditions ($T_i \approx T_e$) where previous authors[2,3,5,7] predict stability. This instability is due to resonant ions absorbing energy from a negative energy Bernstein mode thus causing it to grow. We expect the maximum effect for frequencies such that:

$$\omega \approx kv_{T_i} \qquad \ldots (43)$$

Taking $\xi_i = 1$ the dispersion relation can be written in the form:[11]

$$\left\{\frac{T_i}{T_e} + e^{-1}(-1 + i\sqrt{\pi})\right\}(\omega - kv_o + |n|\omega_{ce}) = \beta_n \frac{T_i}{T_e}(\omega - kv_o + k|v_d^e|) \quad \ldots (44)$$

where we have assumed:

$$\omega \approx k\, v_o - |n|\omega_{ce} \,,$$

where

$$n < 0,$$

and

$$k^2 \lambda_{de}^2 \ll 1.$$

For k in the band of values given by equations (31) and (32) the slow Bernstein wave has negative energy and we get instability as can be seen from equation (44). The growth rate is given by:

$$\frac{\gamma}{\omega_{ce}} = \frac{|n|}{\sqrt{2}} \; \frac{e^{1}}{\pi + \left(e^{1}\dfrac{T_i}{T_e} - 1\right)^2} \; \frac{T_i}{T_e}\frac{1}{ka_e}\left(1 - \frac{k|v_d^e|}{|n|\omega_{ce}}\right). \qquad \ldots (45)$$

For the validity of this solution we require (approximately):

$$k^2 a_e^2 \gg \left(\frac{T_i}{T_e}\right)^{\frac{1}{2}}\left(\frac{m_i}{m_e}\right)^{\frac{1}{2}}.$$

The maximum growth rate of this instability occurs for $T_i \approx T_e$. The larger T_i the larger the range of unstable k-values. For $T_i = T_e$ the unstable k-values are in the vicinity of:

$$k\, a_e \approx |n| / [v_0/v_{T_e} - (2\, m_e/m_i)^{\frac{1}{2}}] \qquad \ldots (46)$$

Using equation (45) and taking $v_0/v_{T_e} = 0 \cdot 1$ we obtain:

$$\frac{\gamma}{\omega_{ce}} = 0 \cdot 02 \; .$$

The significant point about this instability is that it can occur for arbitrary values of T_i and in particular for $T_i \gtrsim T_e$.

V. CONCLUSIONS

We have considered some high frequency ($\omega \gg \omega_{ci}$) electrostatic instabilities which may occur in a collisionless perpendicular shock wave. In all cases the instability was due to the presence of a negative energy Bernstein wave. Instability was produced either when the negative energy Bernstein wave came into frequency resonance with the ion acoustic wave or when resonant ions absorbed energy from the negative energy Bernstein wave. The former case gave rise to the larger growth rates but required $T_i \ll T_e$. The latter case could occur for arbitrary T_i although the maximum growth rate was for $T_i \approx T_e$.

In a shock wave the density and magnetic field gradients are in the same direction and almost equal. Consequently the drifts due to the magnetic field and density gradients are in the opposite direction to the $-E_{ox}/B_{oz}$ drift. As a result only half of the Bernstein harmonics could have negative energy and so only these gave rise to instability. Also, the negative energy property was only satisfied within a band of k-values so that instability (of the type discussed in this paper) only occurred for k-values within this band.

An interesting point about the resonant ion instability is that in contrast to the corresponding case for a positive energy wave, the ions continually gain energy as the negative energy wave grows in amplitude.

Finally, since the instabilities discussed in this paper have growth rates similar to or greater than those discussed by Krall and Book they may well be connected with the anomalous resistivity, observed in a collisionless shock wave.

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10. This only refers, of course, to the instability mechanism considered in this paper.

11. We have taken $Z(1) \approx - e^{-1}(1 + e^{1}) + i\sqrt{\pi}\, e^{-1}$.