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J. D. LOVE
C. J. H. WATSON

Culham Laboratory
Abingdon Berkshire

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by

J.D. Love[†] and C.J.H. Watson

A B S T R A C T

An expansion procedure is developed which permits the construction of approximate solutions of the vector wave equation for electromagnetic systems with geometrically complicated boundary conditions from solutions of Laplace's equation, provided that the vacuum wavelength of the oscillations substantially exceeds the linear dimensions of the system. In lowest order the field structure is a superposition of oscillating electrostatic and magnetostatic fields: however, it is generally necessary to proceed to higher order to satisfy the boundary conditions and obtain resonant frequencies or dispersion relations. For purpose of illustration, the method is used to derive the resonant frequencies of electromagnetic oscillation of a dielectric sphere, and the results are compared with those obtained from the exact solution derived by Debye.

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[†] Present address:
University of California at San Diego, La Jolla, California, USA

UKAEA Research Group,
Culham Laboratory,
Abingdon,
Berks.

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C O N T E N T S

I. INTRODUCTION

There exists a frequency range — roughly 10-100 megacycles — which fits uncomfortably in between the ranges in which the methods of the radio and microwave engineer are most readily applicable. For reasons of practical convenience, most electromagnetic systems, (e.g. resonators and transmission systems) which operate in this range have linear dimensions L which are substantially smaller than the vacuum wavelength $\lambda = 2\pi/k = 2\pi c/\omega$ of the oscillations. Nevertheless, the A.C. circuit analyses of the radio engineer, which should be applicable in the limit $kL \ll 1$, are unsatisfactory because of the extent to which the inductance and capacitance of these devices are distributed. (Consider, for example, tank circuits and slow wave transmission systems.) Consequently it seems difficult to avoid recourse to the full set of Maxwell's equations for a satisfactory theoretical analysis of these devices, and yet the geometric complexity of the boundaries (which is an almost inevitable feature of systems for which $kL \ll 1$) frequently dictates a co-ordinate system in which Maxwell's equations cannot readily be solved. Thus the question arises whether one can use the smallness of kL to make such problems more tractable.

In the context of the scattering of electromagnetic waves it has long been known that this is possible. As Lord Rayleigh pointed out in 1897[1], a first approximation to the field scattered by an object with dimensions such that $kL \ll 1$ is a superposition of oscillating electrostatic and magnetostatic fields:

$$\underline{E} = -\underline{\nabla}\psi^E e^{i\omega t} : \underline{H} = -\underline{\nabla}\psi^H e^{i\omega t} , \qquad \ldots (1.1)$$

such that the sum of the incident fields and the scattered fields satisfy the appropriate boundary conditions at the object. Rayleigh considered only this approximation, and his analysis was somewhat lacking in mathematical rigour: however, in 1953 Stevenson[3] gave a systematic account of a formal expansion procedure which would in principle yield the scattered field to any desired order in the small parameter $\delta = kL$.

In the present paper we shall show that a similar expansion procedure can also be applied to the boundary value problems which arise in the theory of intermediate frequency electromagnetic resonators and transmission systems. In such systems there is no external prescribed electromagnetic field, and it is necessary to find a 'self-consistent' field. Consequently the ordering is substantially different from that of Rayleigh and Stevenson. In particular, the absence of a prescribed field means that in lowest order there is nothing to determine the relative amplitudes of the various possible electrostatic and magnetostatic solutions, and these can only be determined by going to higher order. Stevenson showed how the higher order terms (in any desired order) could formally be obtained in terms of certain integrals over the Green's function of the Laplacian operator. Such solutions are of little value in most practical applications, and we have preferred instead to find particular integrals of the relevant inhomogeneous differential equations, and hence to obtain solutions in closed form, for the first few orders only. Such solutions can be described as 'quasi-electrostatic'.

Clearly, an expansion procedure of this kind is only useful if there exists a curvilinear co-ordinate system, (q_1, q_2, q_3), appropriate to the problem, in which Laplace's equation can be solved analytically. In the present work, it is assumed that this co-ordinate system is orthogonal and that it possesses a certain degree of symmetry — that the metric coefficients, (h_1, h_2, h_3) are independent of one of the co-ordinates, q_3. These restrictions on the co-ordinate system are not obviously necessary, and in any case they are considerably weaker than the conditions under which Maxwell's equations can be solved exactly by separation of variables; indeed the present work was undertaken with a view to obtaining solutions of electromagnetic problems in toroidal co-ordinates — one of the co-ordinate systems in which it is known that Maxwell's equations do not separate.

In applying this expansion procedure it is necessary to face at the outset a question of tactics. In co-ordinate systems for which Maxwell's equations can be solved exactly there are two alternative procedures for obtaining these solutions. The Hansen[3] procedure consists in constructing solutions of the vector wave equation from solutions of the scalar wave equation. The Bromwich[4] procedure consists in combining Maxwell's equations in such a way as to obtain a scalar partial differential equation for one component of $\underline{E}$ or $\underline{H}$, and then deriving all other field components from it. The Hansen procedure is applicable if, for some solution ψ of the scalar wave equation, the Spence-Wells[5] equation:

$$\text{curl } (\nabla^2 + k^2)\,(\underline{f}\psi) = 0 \,, \qquad \ldots (1.2)$$

is satisfied by some vector function $\underline{f}$ which is independent of ψ. Unfortunately the only known solutions of this equation are $\underline{f}$ = constant vector and $\underline{f} = \underline{r}$, the position vector; and these choices lead to solutions of the vector wave equation in which the fields generally have no simple orientation with respect to the curvilinear co-ordinate system. As regards the Bromwich procedure, it can be shown[6] that it is applicable if and only if the metric coefficients satisfy the conditions:

$$\frac{h_1}{h_2} \equiv g(q_1, q_2) \; : \; h_3 \equiv h_3(q_3) \;, \qquad \ldots (1.3)$$

where g is an arbitrary function of q_1 and q_2 only. This condition is not satisfied in many of the co-ordinate systems (e.g. toroidal co-ordinates) in which Laplace's equation is soluble by separation of variables. The tactical question is therefore whether to use the Laplacian solutions to construct solutions of the scalar wave equation and then apply Hansen's procedure, accepting that the resulting field vectors will not have any simple orientation with respect to the co-ordinate axes, or to use the Laplacian functions to construct solutions of Maxwell's equations directly, accepting that this involves solving vector, rather than scalar, partial differential equations. We have chosen the latter approach.

II. EXPANSION PROCEDURE

We consider a medium of uniform conductivity σ, dielectric constant ε_0 and permeability μ. All electromagnetic field quantities are assumed to include the implicit time factor $\exp(i\omega t)$. A complex dielectric constant is defined by:

$$\varepsilon = \varepsilon_0 - \frac{4\pi i \sigma}{\omega} \;. \qquad \ldots (2.1)$$

Thus, Maxwell's equations can be written as:

$$\begin{aligned} \text{curl } \underline{E} &= -ik\mu \underline{H} \\ \text{curl } \underline{H} &= ik\varepsilon \underline{E} \;, \end{aligned} \qquad \ldots (2.2)$$

together with the consequential conditions:

$$\text{div } \underline{E} = \text{div } \underline{H} = 0 \;, \qquad \ldots (2.3)$$

where $\underline{E}$ is the electric field, $\underline{H}$ the magnetic field and $k = \omega/c$ is the wave number in vacuo.

We assume that the field vectors can be expanded in powers of δ in the form:

$$\underline{E} = \sum_{n=0}^{\infty} \underline{E}_n \; : \; \underline{H} = \sum_{n=0}^{\infty} \underline{H}_n \; , \qquad \ldots (2.4)$$

where n denotes the power of δ. Both $\underline{E}$ and $\underline{H}$ are suitably normalised to ensure that no inverse powers of δ are present.

Substituting (2.4) in (2.2) and (2.3):

$$\text{curl } \underline{E}_0 = 0 \; : \; \text{curl } \underline{H}_0 = 0 \qquad \ldots (2.5)$$

$$\text{div } \underline{E}_0 = 0 \; : \; \text{div } \underline{H}_0 = 0 \qquad \ldots (2.6)$$

$$\text{curl } \underline{E}_n = - ik\mu \, \underline{H}_{n-1} \; : \; \text{curl } \underline{H}_n = ik\varepsilon \, \underline{E}_{n-1} \quad (n > 0) \qquad \ldots (2.7)$$

$$\text{div } \underline{E}_n = 0 \; : \; \text{div } \underline{H}_n = 0 \, . \qquad \ldots (2.8)$$

The lowest order solution is given by (2.5) and (2.6):

$$\underline{E}_0 = - \underline{\nabla} \psi_0^E \; : \; \underline{H}_0 = - \underline{\nabla} \psi_0^H \; , \qquad \ldots (2.9)$$

where ψ_0^E and ψ_0^H are solutions of Laplace's equation:

$$\nabla^2 \psi_0^E = \nabla^2 \psi_0^H = 0 \, . \qquad \ldots (2.10)$$

To higher order it is necessary to obtain particular solutions of (2.7) and (2.8).

It will be seen that these equations have the general form:

$$\text{curl } \underline{A} = \underline{B} \qquad \ldots (2.11)$$

$$\text{div } \underline{A} = 0 \qquad \ldots (2.12)$$

$$\text{div } \underline{B} = 0 \, . \qquad \ldots (2.13)$$

A solution of this system in convenient form can be obtained as follows. We first obtain a particular solution of (2.11) such that one component of $\underline{A}$, say A_3, vanishes. The first two components of (2.11) then give:

$$A_1 = \frac{1}{h_1} \int^{q_3} h_1 h_3 B_2 \, dq_3 \qquad \ldots (2.14)$$

$$A_2 = \frac{1}{h_2} \int^{q_3} h_2 h_3 B_1 \, dq_3 \, . \qquad \ldots (2.15)$$

The third component of (2.11) is then satisfied identically provided that (2.13) holds. The general solution of (2.11) is then:

$$\underline{A} = (A_1\,, A_2\,, 0) + \underline{\nabla}\mu\,(q_1\,, q_2\,, q_3)\,, \qquad \ldots\ (2.16)$$

where by equation (2.12):

$$\nabla^2\mu = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\int^{q_3} h_1h_3B_2\,dq_3\right) - \frac{\partial}{\partial q_2}\left(\frac{h_1h_3}{h_2}\int^{q_3} h_2h_3B_1\,dq_3\right)\right\}. \qquad \ldots\ (2.17)$$

This procedure reduces the set of three vector partial differential equations (2.11)-(2.13) to a simple scalar partial differential equation. Its advantages over the conventional procedure (used by Stevenson, for example) for solving (2.11)-(2.13), involving vector and scalar potentials and Green's functions, is that integrals over Green's functions can almost always be performed only after expanding the Green's function as an infinite series. Naturally, this advantage is largely lost if (2.17) can only be solved by means of the Green's function for the Laplace equation. However, in many cases (2.17) can also be solved by finding a particular integral, and the final solution can then be expressed in closed form provided that solutions of Laplace's equation in the co-ordinate system are known. (The above procedure applied to equations (2.7) and (2.8) determines a set of solutions for $\underline{E}_n$ and $\underline{H}_n$ for successive $n > 0$. These solutions are not unique, for we can add linear combinations of solutions of the zeroth order equations (2.5) and (2.6) to each order in δ and still satisfy Maxwell's equations. This indeterminacy is an inevitable consequence of the linearity of Maxwell's equations, and expresses the fact that there is no unique complete set of solutions of a linear system of equations. However, if the co-ordinate system under consideration possesses a symmetry property, this symmetry can be utilised to determine the $\mu(q_1\,, q_2\,, q_3)$ in each order so that the fields possess a simple transformation property under the symmetry group concerned and a unique complete set can then be obtained. It is necessary to use this freedom if, for example, one wishes to derive power series expansions of the spherical harmonics by means of the above procedure.)

In what follows, we show that particular integrals can always be found in those co-ordinate systems for which:

$$\frac{\partial h_1}{\partial q_3} \equiv \frac{\partial h_2}{\partial q_3} \equiv \frac{\partial h_3}{\partial q_3} \equiv 0 \ , \qquad \ldots (2.18)$$

(which includes all co-ordinate systems obtained by translating or rotating a two-dimensional co-ordinate system, in many of which Laplace's equation can be solved by separation) and hence solutions of (2.7) and (2.8) can be obtained at least up to second order in δ. For these systems, (2.14)-(2.16) can be written as:

$$\underline{A} = - h_3 \hat{q}_3 \times \int^{q_3} \underline{B} \, dq_3 + \underline{\nabla}\mu \ , \qquad \ldots (2.19)$$

where $\hat{q}_3 \equiv (0,0,1)$ is a unit vector in the q_3 direction. A further consequence of (2.18) is that solutions of Maxwell's equations in this case can always be written in the form $\underline{E} \equiv \underline{E}_0 (q_1, q_2) \exp(i\alpha q_3)$, $\underline{B} \equiv \underline{B}_0(q_1, q_2) \exp(i\alpha q_3)$ where $\underline{E}_0$ and $\underline{B}_0$ are functions of q_1 and q_2 only and α is a constant. Hence (2.19) and (2.17) reduce to:

$$\underline{A} = - \frac{1}{i\alpha} h_3 \hat{q}_3 \times \underline{B} + \underline{\nabla}\mu \qquad \ldots (2.20)$$

$$\nabla^2\mu = \frac{1}{i\alpha} \operatorname{div} (h_3 \hat{q}_3 \times \underline{B}) \ , \qquad \ldots (2.21)$$

i.e. the particular integral of (2.11) can be written in algebraic form.

The case $\alpha = 0$ requires special consideration, since it might appear from (2.20) that the above solution is not applicable in this case. In fact, however, as we show in Appendix 1, it is always possible to choose a μ satisfying (2.21) such that $\underline{A}$ remains finite in the limit $\alpha \to 0$ (a limit which we can formally approach continuously, even though the physical requirement that fields satisfying Maxwell's equations should be single-valued will in many cases restrict α to integral values), and this is a valid solution of (2.11).

III. THE QUASI-ELECTROSTATIC APPROXIMATION

The quasi-electrostatic approximation (QESA) consists in truncating the series (2.4) after a finite number of terms. For those co-ordinate systems whose metric coefficients satisfy (2.18) we now give the field components correct to second order in δ. Substituting (2.9) in (2.7) with $n = 1$, and using (2.20), a particular solution of (2.7) is:

$$\begin{aligned} \underline{E}_1 &= -\frac{k\mu}{\alpha}\left\{h_3\hat{q}_3 \times \underline{\nabla}\,\psi_0^H - \underline{\nabla}\,\psi_1^E\right\} \\ \underline{H}_1 &= \frac{k\varepsilon}{\alpha}\left\{h_3\hat{q}_3 \times \underline{\nabla}\,\psi_0^E - \underline{\nabla}\,\psi_1\right\}, \end{aligned} \qquad \ldots (3.1)$$

where ψ_1^E and ψ_1^H are scalar functions of position, determined by the conditions (2.8) with $n = 1$:

$$\begin{aligned} \nabla^2\psi_1^E &= \text{curl}\ (h_3\hat{q}_3)\cdot\underline{\nabla}\,\psi_0^H \\ \nabla^2\psi_1^H &= \text{curl}\ (h_3\hat{q}_3)\cdot\underline{\nabla}\,\psi_0^E\ . \end{aligned} \qquad \ldots (3.2)$$

The solution of (3.2) depends upon the type of co-ordinate system under consideration. We here distinguish 'translational' and 'rotational' systems; i.e. co-ordinate systems possessing translational or rotational symmetry with respect to some given axis, taken to lie in the z direction. Clearly for translational systems:

$$h_3 = 1 \quad : \quad \text{curl}\,(h_3\hat{q}_3) = 0, \qquad \ldots (3.3)$$

and for rotational systems:

$$h_3 = \rho = (x^2 + y^2)^{\frac{1}{2}} : \text{curl}\,(h_3\;\hat{q}_3) = 2\,\hat{z}, \qquad \ldots (3.4)$$

where $\hat{z}$ is a unit vector parallel to the axis of rotation. A particular solution of (3.2) for rotational systems is therefore:

$$\psi_1^E = z\,\psi_0^H \quad : \quad \psi_1^H = z\,\psi_0^E, \qquad \ldots (3.5)$$

where z is the distance measured along the axis of rotation, a quantity which is readily evaluated in terms of q_1 and q_2. The solution of (3.2) for translational systems is:

$$\psi_1^E = \psi_1^H = 0\,. \qquad \ldots (3.6)$$

The second order equations are:

$$\begin{aligned} \text{curl}\,\underline{E}_2 &= -\,ik^2\,\frac{\varepsilon\mu}{\alpha}\left\{h_3\hat{q}_3 \times \underline{\nabla}\,\psi_0^E - \underline{\nabla}\,\psi_1^H\right\} \\ \text{curl}\,\underline{H}_2 &= -\,ik^2\,\frac{\varepsilon\mu}{\alpha}\left\{h_3\hat{q}_3 \times \underline{\nabla}\,\psi_0^H - \underline{\nabla}\,\psi_1^E\right\}. \end{aligned} \qquad \ldots (3.7)$$

Applying (2.20) and (2.21) once more, we have:

$$\begin{aligned} \underline{E}_2 &= -\left(\frac{k}{\alpha}\right)^2\varepsilon\mu\left[i\,\alpha h_3\;\psi_0^E\,\hat{q}_3 - h_3^2\underline{\nabla}\,\psi_0^E - h_3\hat{q}_3 \times \underline{\nabla}\,\psi_1^H - \nabla\,\psi_2^E\right] \\ \underline{H}_2 &= -\left(\frac{k}{\alpha}\right)^2\varepsilon\mu\left[i\,\alpha h_3\;\psi_0^H\,\hat{q}_3 - h_3^2\underline{\nabla}\,\psi_0^H - h_3\hat{q}_3 \times \underline{\nabla}\,\psi_1^E - \nabla\,\psi_2^H\right], \end{aligned}$$
$$\ldots (3.8)$$

where ψ_2^E and ψ_2^H are scalar functions determined by the divergence conditions. In order to solve these equations for ψ_2^E and ψ_2^H we again distinguish between the two types of co-ordinate systems.

A. Translational Systems

Here $\psi_1^E = \psi_1^H = 0$, $h_3 = 1$ and the divergence conditions reduce to:

$$\begin{aligned} \nabla^2\psi_2^E &= -\,\alpha^2\psi_0^E \\ \nabla^2\psi_2^H &= -\,\alpha^2\psi_0^H\,. \end{aligned} \qquad \ldots (3.9)$$

Thus, ψ_2^E and ψ_2^H are particular biharmonic functions. The solution of (3.9) is effected by observing that if ψ_o is a solution of Laplace's equation, then ψ_o also satisfies:

$$\nabla^2[(\underline{r}\cdot\underline{\nabla})\,\psi_o] = 0\ , \qquad \ldots (3.10)$$

which can be rearranged as:

$$\nabla^2[(r - z\,\hat{z})\cdot\underline{\nabla}\psi_o] = -\nabla^2\left\{z\,\frac{\partial\psi_o}{\partial z}\right\}\ ,$$

or

$$\nabla^2[\underline{\rho}\cdot\underline{\nabla}\,\psi_o] = 2\,\frac{\partial^2\psi_o}{\partial z^2} = -\,2\,\alpha^2\psi_o. \qquad \ldots (3.11)$$

Consequently, particular solutions of (3.9) are:

$$\psi_2^E = -\frac{1}{2\alpha^2}\,\underline{\rho}\cdot\frac{\partial\psi_o^E}{\partial\underline{\rho}}$$

$$\psi_2^H = -\frac{1}{2\alpha^2}\,\underline{\rho}\cdot\frac{\partial\psi_o^H}{\partial\underline{\rho}}\ , \qquad \ldots (3.12)$$

where the operator $\underline{\rho}\cdot\frac{\partial}{\partial\underline{\rho}}$ can be expressed in terms of the two curvilinear co-ordinates, q_1 and q_2, orthogonal to the z axis.

Hence, correct to second order in δ the electric and magnetic fields for translational co-ordinate systems are:

$$\underline{E} = -\,\underline{\nabla}\psi_o^E - \frac{k\mu}{\alpha}\,\hat{q}_3\times\underline{\nabla}\psi_o^H - \left(\frac{k}{\alpha}\right)^2\varepsilon\,\mu\left[i\,\alpha\psi_o^E\,\hat{q}_3 - \underline{\nabla}\psi_o^E - \underline{\nabla}\psi_2^E\right]$$

$$\underline{H} = -\,\underline{\nabla}\psi^H + \frac{k\,\varepsilon}{\alpha}\,\hat{q}_3\times\underline{\nabla}\psi_o^E - \left(\frac{k}{\alpha}\right)^2\varepsilon\mu\left[i\,\alpha\,\psi_o^H\,\hat{q}_3 - \underline{\nabla}\psi_o^H - \underline{\nabla}\psi_2^H\right]\ , \qquad \ldots (3.13)$$

where ψ_2^E and ψ_2^H are given by (3.12).

B. Rotational Systems

For rotational systems $h_3 = \rho$ and the divergence conditions give:

$$\begin{aligned}\nabla^2\psi_2^E &= -\,2(\rho\,\hat{q}_3\times\hat{z})\cdot\underline{\nabla}\,\psi_o^E - 2z\,\hat{z}\cdot\underline{\nabla}\psi_o^E - (\alpha^2+2)\,\psi_o^E\\ &= -\,2(\underline{r}\cdot\underline{\nabla})\,\psi_o^E - (\alpha^2+2)\,\psi_o^E\\ \nabla^2\psi_2^H &= -\,2(\rho\,\hat{q}_3\times z)\cdot\underline{\nabla}\,\psi_o^H - 2z\,\hat{z}\cdot\underline{\nabla}\,\psi_o^H - (\alpha^2+2)\,\psi_o^H\\ &= -\,2(\underline{r}\cdot\underline{\nabla})\,\psi_o^H - (\alpha^2+2)\,\psi_o^H\ ,\end{aligned} \qquad \ldots (3.14)$$

for which a particular solution is:

$$\psi_2^E = -\frac{1}{2} r^2 \psi_0^E + (\alpha^2 - 1)\chi_0^E$$

$$\psi_2^H = -\frac{1}{2} r^2 \psi_0^H + (\alpha^2 - 1)\chi_0^H \qquad \ldots (3.15)$$

where r is the distance from the origin to the point (q_1, q_2, q_3), and χ_0^E and χ_0^H are particular biharmonic functions satisfying:

$$\nabla^2\chi_0^E = -\psi_0^E \quad : \quad \nabla^2\chi_0^H = -\psi_0^H . \qquad \ldots (3.16)$$

As before, we find particular integrals of this equation in the form:

$$\chi_0^{E,H} = \hat{O}\,\psi_0^{E,H} , \qquad \ldots (3.17)$$

where $\hat{O}$ is a differential operator. Clearly $\hat{O}$ must satisfy the operator relation:

$$\nabla^2\hat{O} = \hat{P} + \hat{Q}\,\nabla^2 , \qquad \ldots (3.18)$$

where $\hat{P}$ is a function of $\partial/\partial q_3$ only, such that $\hat{P}\psi = \psi$, and $\hat{Q}$ is an arbitrary operator. A particular solution of (3.17) (here expressed in cylindrical polar co-ordinates (ρ, φ, z) with $\varphi \equiv q_3$, but readily re-expressed in any other rotational co-ordinate system) is:

$$\chi_0^{E,H} = \hat{O}\,\psi_0^{E,H} = \frac{[(3z^2 - \rho^2)\rho\,\partial/\partial\rho + (z^2 - 3\rho^2)z\,\partial/\partial z + \tfrac{1}{2}(3z^2 - \rho^2)]}{(1 - 4\alpha^2)}\,\psi_0^{E,H} , \qquad \ldots (3.19)$$

since, as is readily confirmed:

$$\nabla^2\hat{O}\,\psi_0^{E,H} = \frac{(1 + 4\partial^2/\partial\varphi^2)}{(1 - 4\alpha^2)}\,\psi_0^{E,H} + \frac{[6(z^2 - \rho^2) + (3z^2 - \rho^2)\rho\partial/\partial\rho]}{(1 - 4\alpha^2)}\,\nabla^2\psi_0^{E,H} = \psi_0^{E,H} . \qquad \ldots (3.20)$$

Other particular integrals may be obtained by adding $\hat{O}$ any operator $\hat{R}$ such that:

$$\nabla^2\hat{R} = \hat{S}\,\nabla^2 , \qquad \ldots (3.21)$$

where $\hat{S}$ is an arbitrary operator. An example is $\hat{R} = \underline{r}\cdot\underline{\nabla}$, for which:

$$\nabla^2\hat{R} = (\hat{R} + 2)\nabla^2 . \qquad \ldots (3.22)$$

This freedom is not of course different from the general freedom to add solutions of the homogeneous equations $\nabla^2\chi_o^{E,H} = 0$; however, it can be used to simplify the solution (3.19) when expressed in other co-ordinate systems.

In conclusion, the full solution of Maxwell's equations for rotational co-ordinate systems, correct to second order in δ is:

$$\underline{E} = -\underline{\nabla}\psi_o^E - \frac{k\mu}{\alpha}\left\{h_3 q_3 \times \underline{\nabla}\psi_o^H - \underline{\nabla}(z\psi_o^H)\right\} - k^2\frac{\varepsilon\mu}{\alpha^2}\left\{i\,\alpha\, h_3\psi_o^E\,\hat{q}_3 \right.$$
$$\left. - h_3^2\,\underline{\nabla}\psi_o^E - h_3\hat{q}_3 \times \underline{\nabla}(z\psi_o^E) + \frac{1}{2}\,\underline{\nabla}\,(r^2\psi_o^E) - (\alpha^2 - 1)\,\underline{\nabla}\,\chi_o^E\right\}, \qquad \ldots (3.23)$$

$$\underline{H} = -\underline{\nabla}\psi_o^H + \frac{k\varepsilon}{\alpha}\left\{h_3\hat{q}_3 \times \underline{\nabla}\psi_o^E - \underline{\nabla}(z\psi_o^E)\right\} - k^2\frac{\varepsilon\mu}{\alpha^2}\left\{i\,\alpha\, h_3\psi_o^H\,\hat{q}_3 \right.$$
$$\left. - h_3^2\,\underline{\nabla}\psi_o^H - h_3\hat{q}_3 \times \underline{\nabla}(z\psi_o^H) + \frac{1}{2}\,\underline{\nabla}\,(r^2\psi_o^H) - (\alpha^2 - 1)\,\underline{\nabla}\,\chi_o^H\right\}, \qquad \ldots (3.24)$$

where χ_o^E and χ_o^H are given by (3.19).

IV. BOUNDARY CONDITIONS

In electromagnetic resonant systems, the effect of the boundary conditions which the electromagnetic field must satisfy (in addition to Maxwell's equations), is to pose an eigenvalue problem for the wave-number k. In electromagnetic transmission structures, the number of boundary conditions is smaller, and they determine a dispersion relation for the structure, relating k to some parameter appearing in the solution of Maxwell's equations. In the majority of simple resonant or transmission structures, the eigensolutions are such that $\delta = kL \sim 1$ and the QESA approach is not applicable. However, if the structure is sufficiently complex, there can exist non-trivial electrostatic and/or magnetostatic solutions of Maxwell's equations for $k = 0$ which nearly satisfy the same boundary conditions, and we may then expect to find eigensolutions for small δ which are given in good approximation by the first few terms in the expansion in δ. To obtain these eigensolutions we apply the normal boundary conditions at any boundary between two media to the QESA expressions for the fields obtained in the preceding section.

In the case of open systems, i.e. systems in which there is a path from the interior of the system to infinity which does not intersect a conducting wall, it is also necessary to impose the Sommerfeld radiation condition at infinity. The imposition of this condition represents a problem, since as we shall see, the QESA representation is valid only for the near field close to the resonant structure, becoming inadequate at distances r such that $kr \sim 1$. However, it is a problem which cannot be bypassed, since by a theorem due to Rellich[7], finite open electromagnetic systems necessarily radiate. The mathematical consequence is that the eigenvalues for k are complex, i.e. the oscillations are damped, unless energy is supplied to the system at a rate equal to the rate at which energy is lost from it by radiation. (It may be remarked that although the finite open resonator must necessarily radiate, certain transmission structures - e.g. the infinite straight helix and dielectric plasma rod - although open, are not finite systems in the sense required by Rellich's theorem, and are non-radiating in a direction perpendicular to the axis of the system, although of course radiation flows in the direction parallel to the axis.)

The possibility of applying the radiation condition to QESA solutions will now be demonstrated using polar co-ordinates, in which the radiation condition takes a particularly simple form. For any other co-ordinate system the radiation condition can be imposed by transforming the asymptotic form of the QESA expression for the field structure at large distances from the system concerned into spherical polar co-ordinates.

The solution of Maxwell's equations in spherical polar co-ordinates, (r, θ, φ), can be simply expressed in terms of the solutions of Helmhotz's equation by Hansen's procedure. If ψ is a solution of:

$$(\nabla^2 + k^2)\,\psi = 0 \,, \qquad \ldots (4.1)$$

where ψ includes the implicit time dependence $\exp(i\,\omega\,t)$, then the electric and magnetic fields are given by linear combinations of the two vector functions:

$$\mathrm{curl}\,(\underline{r}\psi) \;:\; \mathrm{curl}^2(\underline{r}\psi)\,, \qquad \ldots (4.2)$$

where $\underline{r}$ is the position vector. Thus, for $|\underline{r}| \to \infty$ the radiation condition is satisfied if ψ is of the form:

$$\psi = h_n^{(2)}(kr)P_n^m(\cos\theta)\; e^{im\varphi} \qquad \ldots (4.3)$$

where m and n are integers or zero, and $h_n^2(kr)$ is the spherical Hankel function of the second kind, since in this manner ψ represents an outgoing wave. The electromagnetic field is then obtainable from (4.2), and the above choice for ψ ensures that the vector fields likewise satisfy the radiation condition.

We now show that if we were to solve this problem by expansion in δ, and were to proceed to all orders in δ , then provided we started with the correct linear combination of the two independent solutions of Laplace's equation, we would derive precisely the same field components. Conversely by expanding (4.3) as a power series in kr, substituting in (4.2) and selecting the lowest order terms in the series expansions for the field components (i.e. those obtainable from the gradient of solutions of Laplace's equation) one can obtain the correct linear combination of potentials to start the QESA procedure, if the resultant fields continued into the radiation zone are to satisfy the radiation condition. To demonstrate this in detail we use the power series expansion for the $h_n^{(2)}$'s:

$$h_n^{(2)}(kr) = \left(\frac{kr}{2}\right)^n \sum_{p=o}^{\infty} \frac{(-)^p \pi^{\frac{1}{2}} (\frac{kr}{2})^p}{p!\ 2\Gamma(n+p+\frac{3}{2})} - \left(-\frac{2}{kr}\right)^{n+1} \sum_{p=o}^{\infty} \frac{(-)^p \pi^{\frac{1}{2}} (\frac{kr}{2})^p}{p!\ 2\Gamma(-n+p+\frac{1}{2})} \qquad \ldots (4.4)$$

Clearly we can use any non-vanishing component of the electric and magnetic fields given by (4.2) to determine this linear combination of potentials, and for convenience we use the radial components, which are of the form:

$$\frac{1}{r}\, h_n^{(2)}(kr)\, P_n^m(\cos\theta)\, e^{im\varphi} . \qquad \ldots (4.5)$$

The Laplacian part of (4.5), (i.e. the part which can be expressed as the gradient of a solution of Laplace's equation), is:

$$\left[r^{n-1} \left(\frac{k}{2}\right)^n \frac{\pi^{\frac{1}{2}}}{2\,\Gamma(n+\frac{3}{2})} + i(-)^n \frac{1}{r^{n+2}} \left(\frac{2}{k}\right)^{n+1} \frac{\pi^{\frac{1}{2}}}{2\,\Gamma(-n+\frac{1}{2})} \right] P_n^m(\cos\theta)\, e^{im\varphi}. \qquad \ldots (4.6)$$

(These terms are in fact derived from the terms with $p = o$ in (4.4).)
The corresponding Laplacian potential is obtained by integrating (4.6):

$$\left[\frac{1}{n} \left(\frac{kr}{2}\right)^n \frac{\pi^{\frac{1}{2}}}{2\,\Gamma(n+\frac{3}{2})} + \frac{i}{(n+1)} \left(-\frac{2}{kr}\right)^{n+1} \frac{\pi^{\frac{1}{2}}}{2\,\Gamma(-\frac{1}{2}+n)} \right] P_n^m(\cos\theta)\, e^{im\varphi} \qquad \ldots (4.7)$$

Thus, it will be seen that in order to construct QESA solutions, such that if one were to continue the expansion to all orders in k, they would satisfy the radiation condition, it is necessary to take a mixture of the two linearly independent Laplacian solutions weighted in the proportion:

$$i(-)^{n+1}\, 2^{2n+1}\, n\Gamma(n+\tfrac{3}{2}) \; : \; (n+1)\Gamma(-n+\tfrac{1}{2})k^{2n+1} .$$

It should be noted at this point that the correct linear combination of Laplacian parts (4.7) for the incorporation of the radiation condition in QESA theory cannot be deduced from (4.5) by taking the limit $k = 0$ in the power series expansions obtained by substituting (4.4) in (4.5). However, it is a valid procedure to take this limit in each of the two series (real and imaginary radial variance) separately, provided, of course, that each series is suitably normalized in k. The fact that the starting point for QESA theory must include terms of different order in k need not, however, interfere with the execution of the expansion procedure, since one can expand each part separately to any order and then take appropriate linear combinations of the two solutions.

In the above argument we considered a co-ordinate system in which the wave equation can be solved and the radiation condition applied exactly, and consequently the procedure described is in that context of formal interest only. However, we are actually interested in co-ordinate systems for which the wave equation is not soluble exactly and for which it is impracticable to go beyond a finite number of stages in QESA theory. We therefore ask if under these circumstances the above method is relevant. The answer to this may be seen by imagining that we were not capable of solving the scalar wave equation in spherical polar co-ordinates, and that instead we had laboriously constructed field components derived from the terms with $p = 1,2, \cdots, n$ in (4.5) by the QESA method. Clearly the field components would be a good approximation to the true field components provided that the finite series in p converged sufficiently rapidly, i.e. at points for which $kr \ll 1$. Analogous considerations will apply in any co-ordinate system - i.e. for certain regions of space, and for suitably restricted values of k, the QESA series converges so rapidly that it is unnecessary to proceed beyond the first few terms. The regions of space for which the finite QESA series will converge in this way is at best a small central region close to the electromagnetic system, and at sufficiently large distances from the system the full series is required to give an accurate representation of the fields. However, if one is only interested in the near field, it is sufficient to take only a finite number of terms in the QESA series, for as the above argument shows, the same expansion taken to all orders would accurately represent the distant field and (with the correct choice of Laplacian parts) would describe an out-going wave satisfying the radiation condition at infinity.

Consequently, if all the boundary conditions, except the radiation condition at infinity, arise at points for which the QESA representation is valid, then there is no need to proceed to higher order in the expansion since, as the above argument shows, the correct choice of the ratio of the Laplacian parts automatically ensures that if one were to take the expansions to all orders in k, the resulting solution would describe an outgoing wave at infinity. Naturally, this approach is only applicable to those systems for which the QESA fields are good approximations at all the finite boundaries of the system. Whether this condition is satisfied or not depends on the system itself. For example, a solid perfectly conducting sphere is a system which does not meet the condition since, as is clear from the analysis given by Stratton[8], all the natural electromagnetic modes of oscillation of this system are such that even the near field requires many terms in the expansion to represent it. On the other hand, a plasma sphere with dielectric coefficient less than zero does meet the condition, as we shall see below. Although it is always possible to determine retrospectively whether this condition is satisfied (by showing that subsequent terms in the expansion make a negligible contribution at the boundaries) it does not seem to be possible to specify in advance whether a given system will be amenable to this procedure.

V. APPLICATIONS

In subsequent papers we shall apply QESA theory to various open and closed slow wave systems with toroidal geometry. However, for the purpose of demonstrating the effectiveness of QESA theory it is more convenient to consider a simple system, the dielectric sphere resonator, for which exact solutions of Maxwell's equations can be obtained. This is the only system of which we are aware, which both possesses spherical symmetry and admits a QESA solution. This is unfortunate, since the existing literature on the subject is rather unsatisfactory.

Debye[9] investigated in detail the oscillations of spheres of uniform dielectric constant ε and resistivity ρ, surrounded by vacuum. Such a system satisfies the requirements of Rellich's Theorem, and consequently all natural modes of oscillation necessarily radiate. However, in the case $\varepsilon \gg 1$, Debye claimed to have found solutions of the dispersion relation of the system for which k (and thus ω) were real. This apparent violation of Rellich's theorem is resolved by observing that Debye made an approximation for the fields external to the sphere which is inadequate. We shall therefore repeat Debye's calculation, showing how the imaginary part of k can be obtained. This case ($\varepsilon \gg 1$) is one in which QESA theory is not applicable inside the sphere, though it is outside. In the opposite limit, however, ($\varepsilon \ll 1$) there exist solutions in which the QESA expressions are valid both inside and immediately outside the sphere.

A somewhat simplified version of Debye's argument is given in Stratton[8] (pp.554-60): in brief, the field vectors are constructed from solutions of the scalar wave equation in spherical polar coordinates by Hansen's procedure:

$$\begin{aligned} \underline{E} &= \underline{r} \times \underline{\nabla}\, \varphi^{E} \\ \underline{H} &= \frac{i}{k}\, \mathrm{curl}\ \underline{E} \end{aligned} \qquad \text{or} \qquad \begin{aligned} \underline{H} &= \underline{r} \times \underline{\nabla}\, \varphi^{H} \\ \underline{E} &= -\frac{i}{\varepsilon k}\, \mathrm{curl}\ \underline{H} \end{aligned} \quad , \qquad \ldots (5.1)$$

where within the sphere:

$$\varphi^{E} = \varphi^{Ei} = E_o^j\, j_n(k_1 r)\, P_n^m(\cos\theta)\, e^{im\varphi} \qquad \ldots (5.2)$$

$$\varphi^{H} = \varphi^{Hi} = H_o^i\, j_n(k_1 r)\, P_n^m(\cos\theta)\, e^{im\varphi} \qquad \ldots (5.3)$$

($k_1 = k\,\varepsilon^{\frac{1}{2}}$) and outside the sphere the spherical Bessel functions are replaced by the spherical Hankel functions $h_n^{(2)}(kr)$:

$$\varphi^{E_o} = E_o^o\, h_n^{(2)}(kr)\, P_n^m\, (\cos\theta)\, e^{im\varphi} \qquad \ldots (5.4)$$

$$\varphi^{H_o} = H_o^o\, h_n^{(2)}(kr)\, P_n^m\, (\cos\theta)\, e^{im\varphi} . \qquad \ldots (5.5)$$

The continuity of the tangential electric and magnetic fields at the surface of the sphere ($r = a$) leads to the dispersion relations:

$$\frac{[k_1 a\ j_n(k_1 a)\ ]'}{j_n(k_1 a)} = \frac{[ka\ h_n^{(2)}\ (ka)]'}{h_n^{(2)}(ka)} \qquad \ldots (5.6)$$

$$\frac{[k_1 a\ j_n(k_1 a)]'}{k_1^2\ j_n\ (k_1 a)} = \frac{[ka\ h_n^{(2)}\ (ka)]'}{k^2\ h_n^{(2)}\ (ka)} , \qquad \ldots (5.7)$$

where dash denotes differentiation with respect to $k_1 a$ on the left-hand sides, and with respect to ka on the right-hand sides of (5.6) and (5.7). Note that (5.6) and (5.7) hold for $n = \pm 1, \pm 2, \ldots$;

the case $n = 0$ is unphysical since all the field components vanish.

For $\varepsilon \gg 1$, we seek solutions with $ka \ll 1$; $k_1 a = \varepsilon^{\frac{1}{2}} ka = 0(1)$. We can therefore expand the Hankel functions for small ka. Upon rearranging (5.7) and retaining only lowest order terms in ka of the real and imaginary parts respectively, we have:

$$\frac{[k_1 a\ j_n(k_1 a)]'}{j_n(k_1 a)} = -\varepsilon\left\{i\left(\frac{ka}{2}\right)^{2n+1}\frac{2\pi(n+1)}{(2n+1)[\Gamma(n+\frac{1}{2})]^2} + n\right\}. \qquad \dots (5.8)$$

For $\varepsilon \gg 1$ a first approximation to the solution of (5.8) is:

$$j_n(k_1 a) = 0\ , \qquad \dots (5.9)$$

which has an infinity of roots which we shall denote by $x_{n,p}$ for $p = 1, 2, \cdots$. A second approximation is obtained by putting:

$$k_1 a = x_{n,p} + \delta \quad : \left|\frac{\delta}{x_{n,p}}\right| \ll 1\ . \qquad \dots (5.10)$$

We note that the smallest value of $x_{u,p}$ for any n or p is greater than 3. To lowest order in δ we find:

$$\delta = \frac{-x_{n,p}}{(1+\varepsilon n)}\left[1 - \frac{2\pi i\varepsilon}{(1+\varepsilon n)}\left(\frac{x_{n,p}}{2\varepsilon^{\frac{1}{2}}}\right)^{2n+1}\frac{(n+1)}{(2n+1)[\Gamma(n+\frac{1}{2})]^2}\right]. \qquad \dots (5.11)$$

Thus, in this approximation the dispersion relation for the magnetic modes is:

$$\frac{a\omega}{c} = x_{n,p}\left[\frac{1}{\varepsilon^{\frac{1}{2}}} + \frac{2\pi i}{\varepsilon^{\frac{1}{2}}}\left(\frac{x_{n,p}}{2\varepsilon^{\frac{1}{2}}}\right)^{2n+1}\frac{(n+1)}{(2n+1)[\Gamma(n+\frac{1}{2})]^2}\right]. \qquad \dots (5.12)$$

By analogous considerations to (5.6) we find for the electric modes the dispersion relation:

$$\frac{a\omega}{c} = x_{n,p}\left[\frac{1}{\varepsilon^{\frac{1}{2}}}\frac{n}{(n+1)} + \frac{2\pi i}{\varepsilon^{\frac{1}{2}}}\left(\frac{x_{n,p}}{2\varepsilon^{\frac{1}{2}}}\right)^{2n+1}\frac{1}{(2n+1)[\Gamma(n+\frac{1}{2})]^2}\right]. \qquad \dots (5.13)$$

It will be seen that the imaginary parts of ω required by Rellich's Theorem are indeed present, but are very small, being of order $\left(\frac{1}{\varepsilon}\right)^{n+1}$ or less.

The existence of solutions of the dispersion relation (5.7) for $\varepsilon < 0$ was not discussed by Debye, presumably because media with negative dielectric coefficients were then unknown. However, if (5.7) is expanded for small ka and k_1a, and only terms of lowest order in k in the real and imaginary parts are retained, one obtains:

$$\left(\frac{ka}{2}\right)^{2n+1} = i(-)^{n+1} \frac{\Gamma(n+\frac{3}{2})}{\Gamma(-n+\frac{1}{2})} \left[\frac{n(\varepsilon+1)+1}{(n+1)(\varepsilon-1)}\right], \quad \ldots (5.14)$$

where $n = 1, 2, \cdots$, and $\varepsilon \neq 1$. If for ε we substitute the expression for the dielectric coefficient of a cold uniform plasma, $\varepsilon = 1 - \omega_p^2/\omega^2$ (where ω_p is the plasma frequency), equation (5.14) can be solved to obtain weakly-damped oscillations with frequencies:

$$\omega = \omega_p \frac{1}{(2+\frac{1}{n})^{\frac{1}{2}}} \left[1 + \frac{i(-)^n}{2n} \left\{\frac{\omega_p\ a/c}{2(2+\frac{1}{n})^{\frac{1}{2}}}\right\}^{2n+1} \frac{\Gamma(-n+\frac{1}{2})}{\Gamma(n+\frac{3}{2})}\right]. \quad \ldots (5.15)$$

The real part of this expression coincides with the expression given by Gildenburg and Kondratev[10] for the resonant frequencies of a plasma sphere in an external high frequency field. It is readily confirmed that at these frequencies, which are close to the frequencies at which the numerator of (5.14) vanishes ($n[\varepsilon+1]+1=0$), the expansion leading to (5.14) is justified. It follows that this case ought to be tractable by QESA procedures. This is readily demonstrated as follows: in lowest order, since the mode is of transverse magnetic (TM) type, the solutions of equations (2.5) and (2.6) are:

$$\underline{H}_0 = 0$$

$$\underline{E}_0 = -\underline{\nabla}\,\psi_0^E = -\underline{\nabla}\left[\beta(r^n + \alpha\, r^{-n-1})\, P_n^m(\cos\theta)\, e^{im\varphi}\right], \quad \ldots (5.16)$$

where α is a constant determining the relative proportions of the two linearly independent solutions of Laplace's equation, and β is a normalising constant. Inside the sphere, the requirement that $\underline{E}_0$ should be well-behaved at $r = 0$ implies $\alpha = 0$; outside the sphere the radiation condition at infinity is satisfied if (c.f.(4.7)):

$$\alpha = -\left(\frac{2i}{k}\right)^{2n+1} \frac{\Gamma(n+\frac{3}{2})\, n}{\Gamma(-n+\frac{1}{2})(n+1)} \quad . \qquad \ldots (5.17)$$

Since $\underline{H}_0 \equiv 0$, there is no correction to $\underline{E}$ in next order, and we can use the fields (5.16) directly to obtain the approximate eigenmodes. Applying the electromagnetic boundary conditions at the surface of the sphere (of radius a) we have:

$$\begin{aligned} \psi_o^E\Big|_{r=a-o} &= \psi_o^E\Big|_{r=a+o} \\ \varepsilon \frac{\partial \psi_o^E}{\partial r}\Big|_{r=a-o} &= \frac{\partial \psi_o^E}{\partial r}\Big|_{r=a+o} \end{aligned} \quad , \qquad \ldots (5.18)$$

whence

$$\left(\frac{ka}{2}\right)^{2n+1} = i(-)^{n+1} \frac{\Gamma(n+\frac{3}{2})}{\Gamma(-n+\frac{1}{2})} \left[\frac{n(\varepsilon+1)+1}{(n+1)(\varepsilon-1)}\right] . \qquad \ldots (5.19)$$

in exact agreement with (5.14).

APPENDIX 1

To show the existence of a solution of (2.21) for μ such that the fields given by (2.20) remain finite in the limit $\alpha \to 0$, we observe that in the limit $\alpha = 0$, the problem (2.2) can be solved in another manner, since it can be shown (see for example reference 6) that for $\alpha = 0$ the third component equation of the vector wave equation:

$$\text{grad div } \underline{A} - \text{curl curl } \underline{A} + k^3 \underline{A} = 0 \ , \qquad \dots (A.1)$$

decouples from the other two component equations for any co-ordinate system satisfying (2.18), even if it does not satisfy the Bromwich conditions (1.3). Since the curl of any solution of this equation is also a solution, it follows that for $\alpha = 0$ there exist solutions of the problem (2.2) with either $E_3 \equiv 0,\ H_3 \not\equiv 0$ or $H_3 \equiv 0,\ E_3 \not\equiv 0$, the general solution being a superposition of these. Thus, one or other of the equations (2.7) is of the form curl $\underline{A} = \underline{B}$ with $B_3 = 0$, and as is well known, this problem can be solved by seeking a solution in which $\underline{A} \equiv A_3 \hat{q}_3$ and introducing a 'stream function' $\chi \equiv h_3 A_3(q_1, q_2)$ such that:

$$\text{curl } (A_3 q_3) = \text{curl}\left(\frac{\chi}{h_3} q_3\right) = -\frac{q_3 \times \underline{\nabla}\chi}{h_3} = (B_1, B_2, 0) \ , \qquad \dots (A.2)$$

hence that:

$$\underline{\nabla}\chi = h_3 \hat{q}_3 \times \underline{B} \ . \qquad \dots (A.3)$$

It is readily confirmed that the choice:

$$\mu = \frac{\chi \, e^{i \alpha q_3}}{i\alpha} \ , \qquad \dots (A.4)$$

(where $\chi(q_1, q_2)$ satisfies (A.3) at once ensures the existence of $\underline{A}$

(as given by (2.20)) and satisfies (2.21) in the limit $\alpha \to 0$, since:

$$\nabla^2 \mu = \operatorname{div}\left(\frac{h_3 q_3 \times \underline{B}\, e^{i\alpha q_3}}{i\alpha}\right) + \frac{i\alpha\chi\, e^{i\alpha q_3}}{h_3^2} \qquad \ldots \text{(A.5)}$$

It will be seen that this choice of μ is precisely that which cancels out the components A_1 and A_2 in (2.20), as required by the stream function approach. A consequence of this is that whichever member of the pair of equations (2.17) is not covered by the above discussion is also solved by (2.20) even in the limit $\alpha \to 0$ since for it $\hat{q}_3 \times \underline{B} \to 0$.

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