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A FOKKER-PLANCK TREATMENT OF NON-LINEARLY INTERACTING WAVES

by

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ABSTRACT

The statistical properties of non-linearly interacting waves are considered from a physically-motivated viewpoint. A Fokker-Planck wave transport equation is derived. In certain circumstances an H-theorem can be proved.

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C O N T E N T S

A P P E N D I C E S

1. INTRODUCTION

We are concerned with the statistical properties of the set of equations:

$$\frac{\partial A_k}{\partial t} = i\,\omega_k A_k + \varepsilon \sum_j M^{-k}_{j,k-j}\, A_j A_{k-j} \,. \qquad \ldots (1)$$

$A_k(t)$ is the Fourier Transform of a real, centred, stationary random function of position (but not of time) so we impose the properties:

$$\omega_{-k} = -\,\omega_k \qquad \ldots (2)$$

$$A_{-k} = A^*_k \qquad \ldots (3)$$

$$\left(M^{-k}_{j,k-j}\right)^* = M^{k}_{-j,j-k} \qquad \ldots (4)$$

$$A_o = 0 \qquad \ldots (5)$$

$$M^{o}_{j,-j} = 0 \qquad \ldots (6)$$

There is clearly no loss of generality in specifying that M is symmetric in its lower indices.

It proves convenient to work with $E_k(t)$, defined by:

$$A_k(t) \equiv E_k(t)\, e^{i\,\omega_k t} . \qquad \ldots (7)$$

$E_k(t)$ satisfies the equation:

$$\frac{\partial E_k}{\partial t} = \varepsilon \sum_j M^{-k}_{j,k-j}\, e^{i\{-\omega_k + \omega_j + \omega_{k-j}\}t}\, E_j E_{k-j} \qquad \ldots (8)$$

$$= \varepsilon \sum_j \beta^{-k}_{j,k-j}(t)\, E_j E_{k-j} \quad , \text{ say.} \qquad \ldots (9)$$

The spectral function Q_k, the triple correlation R_{kj} and the fourth-order cumulant $S_{kj\ell}$ are defined by the (exact) relations:

$$\langle E_k E_j \rangle \equiv \delta(k+j) Q_k \qquad \ldots (10)$$

$$\langle E_k E_j E_\ell \rangle \equiv \delta(k+j+\ell) R_{kj} \qquad \ldots (11)$$

$$
\begin{aligned}
\langle E_k E_j E_\ell E_m \rangle \equiv\; & \langle E_k E_j \rangle \langle E_\ell E_m \rangle \\
+\; & \langle E_k E_\ell \rangle \langle E_j E_m \rangle \\
+\; & \langle E_k E_m \rangle \langle E_j E_\ell \rangle \\
+\; & (k+j+\ell+m)\, S_{kj\ell} \; .
\end{aligned}
\qquad \ldots (12)
$$

Where:

$$
\begin{aligned}
\delta(k) &= 0 \quad \text{if} \quad k \neq 0 \\
&= 1 \quad \text{if} \quad k = 0 \; .
\end{aligned}
\qquad \ldots (13)
$$

Our purpose is to derive deterministic equations for the 'macroscopic' variables Q_k, R_{kj}, etc. We are immediately faced with some very familiar problems. Firstly, there is an infinite hierarchy of equations. The equation governing the n^{th} cumulant involves the $(n+1)^{th}$ cumulant. Secondly, the equations for the macroscopic variables will distinguish between past and future, whereas in the complete description (in terms of the E_k 's) the equations may be symmetrical with respect to past and future.

We may naively attempt to truncate the hierarchy by neglecting $S_{kj\ell}$. This is called the quasi-normal approximation since the fourth-order moment is related to the spectral function as it would be were E_k joint-normally distributed. The result of this emasculation is:

$$
\frac{\partial Q_k}{\partial t} = \varepsilon \sum_j \beta^k_{j,-k-j} \; R_{kj} + \{k \longleftrightarrow -k\} \qquad \ldots (14)
$$

$$
\frac{\partial R_{kj}}{\partial t} = 2\,\varepsilon\, \beta^{-k}_{-j,k+j}\, Q_j\, Q_{k+j} \qquad \ldots (15)
$$

+ terms obtained by permuting k, j and $-k-j$.

Notwithstanding the approximation, these equations have the same reversibility properties as the original equation (9). We can obtain an irreversible equation by the following device, borrowed from the quasi-linear theory of plasma instabilities (more generally, it is known in kinetic theory as Bogoliubov's Conjecture). Suppose that on a certain time scale R_{kj} relaxes to become a functional of Q_k, which itself evolves slowly according to a Markoffian description. We select arbitrarily a direction of time and call it positive, then solve equation (15) for $R_{kj}(\{Q_k\}, t = \infty)$ holding $Q_k(t)$ strictly time-independent. If R_{kj} is now inserted in equation (14) the result is:

$$\frac{\partial Q_k}{\partial t} = 4\,\varepsilon^2 \sum_j \delta(-\omega_k + \omega_j + \omega_{k-j})\, M^{-k}_{j,k-j} \left\{ M^{-k*}_{j,k-j}\, Q_j Q_{k-j} + M^{j*}_{k-j,-k}\, Q_k Q_{k-j} + M^{k-j*}_{-k,j}\, Q_k Q_j \right\} . \quad \dots (16)$$

The objections to this line of argument are plain enough:

(a) There is no guarantee that the errors arising from the neglect of the fourth-order cumulant will not accumulate and produce the unphysical behaviour observed in other fields where this approximation has been employed (Orszag, 1966; Betchov, 1966).

(b) The arrow of time has been introduced in an arbitrary fashion.

Many authors have tried to circumvent the above objections (Davidson, 1966; Hasselmann, 1967; Saffman, 1967; Benney and Newell, 1969; Malfliet, 1970). Their treatments are mathematically sophisticated yet strangely lacking in intuitive appeal. A parallel may be found in modern derivations of the Boltzmann equation without using the Stosszahlansatz (Montgomery, 1967). One is reminded of a remark of van Kampen (1962): ".... there cannot be a rigorous mathematical derivation of the macroscopic equations from the microscopic ones. Some additional information or assumption is indispensible. One cannt escape from this fact by any amount of mathematical funambulism".

Let us forget Zermelo and Loschmidt and see how far we can get with elementary, physically-motivated, considerations. The basic idea is due to Peierls (1929).

2. THE FOKKER-PLANCK EQUATION

Introduce $q_k(t)$ by:

$$q_k(t) \equiv E_k E_{-k} = A_k A_{-k} \; . \qquad \ldots (17)$$

We shall give up the attempt to understand microscopic details and suppose that on another level there is a complete description in terms of a function $P(\{f_j\}, t)$. The f_j's are real positive variables with the property $f_{-j} = f_j$. P is the probability that at time t $q_k = f_k$, $q_j = f_j$, $q_\ell = f_\ell$, etc. In certain circumstances P satisfies a Fokker-Planck equation:

$$\frac{\partial P}{\partial t} = \sum_k \frac{\partial}{\partial f_k} \left\{ -\nu_k + \frac{1}{2} \sum_j \frac{\partial}{\partial f_j} D_{kj} \right\} P. \qquad \ldots (18)$$

The dynamical friction coefficient ν_k and the diffusion coefficient D_{kj} are defined in terms of the transition moments as follows:

$$\nu_k \equiv \lim_{T \to 0} \frac{\langle q_k(T) - q_k(0) \rangle}{T} \qquad \ldots (19)$$

$$D_{kj} \equiv \lim_{T \to 0} \frac{\langle (q_k(T) - q_k(0))(q_j(T) - q_j(0)) \rangle}{T} \; . \qquad \ldots (20)$$

The circumstances in which equation (18) is valid are

(a) Each wave interacts ('collides') with many others.

(b) Every such collision can be considered a small perturbation.

(c) The waves are correlated only weakly.

(d) $\tau \gg \omega_k^{-1}$, where τ is a typical interaction time. (This condition will be gone into more fully below.)

ν_k and D_{kj} are calculated using simple perturbation theory in Appendices A and B. The results are:

$$\nu_k = 4\,\varepsilon^2 \sum_j M^{-k}_{j,k-j}\, M^{k}_{-j,j-k}\, f_j\, f_{k-j}\, \delta(-\,\omega_k+\omega_j+\omega_{k-j})$$

$$+\,8\,\varepsilon^2 \sum_j M^{-k}_{j,k-j}\, M^{-j}_{k,j-k}\, f_{-k}\, f_{k-j}\, \delta(-\omega_k+\omega_j+\omega_{k-j}) \quad \ldots (21)$$

and

$$D_{kj} = \delta(k-j)8\,\varepsilon^2 \sum_\ell M^{-k}_{\ell,k-\ell}\, M^{k}_{-\ell,\ell-k} f_{-k} f_\ell f_{k-\ell}\, \delta(-\omega_k+\omega_\ell+\omega_{k-\ell})$$

$$+[1-\delta(k-j)]16\,\varepsilon^2\, M^{-k}_{j,k-j} M^{-j}_{k,j-k}\; f_{-k} f_j f_{k-j}\, \delta(-\omega_k+\omega_j+\omega_{k-j})\,. \quad \ldots (22)$$

It may easily be verified that the following relationship obtains:

$$\sum_j \frac{\partial}{\partial f_j} D_{kj} = 2\,\nu_k\,. \quad \ldots (23)$$

Thus the Fokker-Planck equation (18) reduces to:

$$\frac{\partial P}{\partial t} = \sum_{k,j} \frac{\partial}{\partial f_k}\,\frac{1}{2}\, D_{kj}\,\frac{\partial}{\partial f_j} P\,. \quad \ldots (24)$$

Substituting for D_{kj} from equation (22), we may write equation (24) as:

$$\frac{\partial P}{\partial t} = 4\,\varepsilon^2 \sum_{k,j,\ell} \frac{\partial}{\partial f_k}\, M^{k}_{j\ell} f_k f_j f_\ell\, \delta(k+j+\ell)$$

$$\cdot\, \delta(\omega_k+\omega_j+\omega_\ell)\left\{M^{-k}_{-j,-\ell}\frac{\partial}{\partial f_k} + M^{-j}_{-\ell,-k}\frac{\partial}{\partial f_j} + M^{-\ell}_{-k,-j}\frac{\partial}{\partial f_\ell}\right\} P\,. \quad \ldots (25)$$

We have exploited some symmetry properties and used the convention:

$$M^{-k}_{j,k-j} = \sum_\ell M^{-k}_{j,\ell}\;\; \delta(-k+j+\ell)\,. \quad \ldots (26)$$

It will prove convenient to write expression (25) in the more symmetrical form:

$$\frac{\partial P}{\partial t} = \frac{4}{3}\varepsilon^2 \sum_{kj\ell} \delta(k+j+\ell)\ \delta(\omega_k+\omega_j+\omega_\ell) \left\{ M^k_{j\ell}\frac{\partial}{\partial f_k} + M^j_{\ell k}\frac{\partial}{\partial f_j} + M^\ell_{kj}\frac{\partial}{\partial f_\ell} \right\}$$

$$f_k f_j f_\ell \left\{ M^{-k}_{-j,-\ell}\frac{\partial}{\partial f_k} + M^{-j}_{-\ell,-k}\frac{\partial}{\partial f_j}\ M^{-\ell}_{-k,j}\frac{\partial}{\partial f_\ell} \right\} P . \qquad \ldots (27)$$

We may call equations (25) or (27) the'wave transport equation'. Some of the properties of this equation are investigated in the next two sections.

3. THE H-THEOREM

The H-function is defined as:

$$H \equiv - \int \left\{ df_k \right\} P \ \ell n\ P . \qquad \ldots (28)$$

The integration is with respect to all the variables f_k. Differentiating with respect to time, we obtain:

$$\frac{dH}{dt} = - \int \left\{ df_k \right\} \left(1 + \ln P \right) \frac{\partial P}{\partial t} . \qquad \ldots (29)$$

Using equation (27) and integrating by parts one obtains:

$$\frac{dH}{dt} = \frac{4}{3}\varepsilon^2 P^{-1} \int \left\{ df_k \right\} \sum_{k_j\ell} \delta(k+j+\ell)\ \delta(\omega_k+\omega_j+\omega_\ell)$$

$$\cdot\ f_k f_j f_\ell \left| \left\{ M^k_{j\ell}\frac{\partial}{\partial f_k} + M^j_{\ell k}\frac{\partial}{\partial f_j} + M^\ell_{kj}\frac{\partial}{\partial f_\ell} \right\} P \right|^2 . \qquad \ldots (30)$$

The integrand is non-negative, so $H(t)$ cannot decrease. Equilibrium is attained when the term within the modulus sign is either (a) zero, or (b) a multiple of $(\omega_k + \omega_j + \omega_\ell)$. Two important special cases suggest themselves:-

Case (a)

Let the coefficients M have the property

$$M^k_{j\ell} + M^j_{\ell k} + M^\ell_{kj} = 0 . \qquad \ldots (31)$$

The equilibrium solution of equation (30) is:

$$P_{eqm.} = P\left(\sum_k f_k \right) . \qquad \ldots (32)$$

The physical significance of this result can easily be inferred by going back to equation (1) and noticing that if condition (31) holds then $\sum_k q_k$ is a constant of the motion. Any (positive) function of this conserved quantity is an equilibrium solution.

Case (b)

Let:

$$A_k = P_k + i\,\omega_k\, q_k \; . \qquad \ldots (33)$$

It is shown in Appendix C that equation (1) may be derived from a Hamiltonian, with P_k and q_k as conjugate variables, provided:

$$M^{k}_{j\ell} = \alpha_{kj\ell}\, \omega_k \; , \qquad \ldots (34)$$

where $\alpha_{kj\ell}$ is a function which is unchanged under any permutation of its indices. For this choice of $M^{k}_{j\ell}$, equation (27) is essentially the same as that derived by Peierls (1929) and Prigogine (1962) in connection with the theory of anharmonic crystals. The stationary solution to equation (3) is any (positive) function of the zero-order Hamiltonian:-

$$P_{eqm.} = P\left(\sum_k f_k\right) \; . \qquad \ldots (35)$$

If $P_{eqm.}$ is factorisable it must be the canonical distribution function:

$$P_{eqm.} = \beta^N e^{-\beta \sum_k f_k} \; . \qquad \ldots (36)$$

N is the total number of wave numbers.

This situation is very similar to that in the kinetic theory of dilute gases. The interactions are invoked solely to explain the evolution towards equilibrium: once attained, this equilibrium is specified by the non-interactive part of the Hamiltonian alone.

Notice that in equilibrium there is equipartition of 'energy'. The spectral function is a constant:

$$Q_k \equiv \int f_k\, P \left\{df_j\right\} = \beta^{-1}. \qquad \ldots (37)$$

4. MOMENT EQUATIONS AND RELAXATION TIMES

By multiplying equation (25) by f_α and integrating over all f_k 's we can obtain an equation governing the evolution of the first moment of P — the spectral function Q_α :

$$\frac{\partial Q_\alpha}{\partial t} = 4\,\varepsilon^2 \sum_{j\,\ell} \delta(\alpha + j + \ell)\,\delta(\omega_\alpha + \omega_j + \omega_\ell)$$

$$\cdot\, M^{\alpha}_{j\ell} \left\{ M^{-\alpha}_{-j-\ell}\, Q_j Q_\ell + M^{-j}_{-\ell,-\alpha}\, Q_\alpha Q_\ell + M^{-\ell}_{-\alpha,-j}\, Q_\alpha Q_j \right\} \cdot \qquad \ldots (38)$$

This is identical to equation (16). Equations for higher moments can be obtained in an obvious way.

An interesting problem, and one which leads to the determination of relaxation times, relates to the 'Brownian motion' of waves. If in equation (25) we substitute

$$P = g(f_\alpha, t)\, \exp\left[-\beta \sum_{k \neq \alpha} f_k \right] \qquad \ldots (39)$$

we can study the evolution of a single wave which interacts with a background of waves in thermodynamic equilibrium. We shall confine ourselves here to a discussion of the behaviour of the spectral function. In equation (38) put $Q_j = \beta^{-1}$ for all $j \neq \alpha$:

$$\frac{\partial Q_\alpha}{\partial t} = 4\,\varepsilon^2 \sum_{j\,\ell} \delta(\alpha + j + \ell)\,\delta(\omega_\alpha + \omega_j + \omega_\ell) \left| M^{\alpha}_{j\ell} \right|^2 \left\{ \beta^{-2} - \beta^{-1} Q_\alpha \right\} . \qquad \ldots (40)$$

The solution of equation (40) is:

$$Q_\alpha(t) = Q_\alpha(0)\, e^{-\beta^{-1} \mu_\alpha t} + \beta^{-1} \left\{ 1 - e^{-\beta^{-1} \mu_\alpha t} \right\}, \qquad \ldots (41)$$

where:

$$\mu_\alpha \equiv 4\,\varepsilon^2 \sum_{j\ell} \delta(\alpha + j + \ell)\,\delta(\omega_\alpha + \omega_j + \omega_\ell) \left| M^{\alpha}_{j\ell} \right|^2 . \qquad \ldots (42)$$

We can now redeem a promise made earlier and state that condition (d) for the validity of the Fokker-Planck equation (18) may be reformulated as:

$$\beta \, \omega_k \gg \mu_k \, . \qquad \ldots (43)$$

5. DISCUSSION

If 'some additional assumption is indispensible' it is surely better that it be chosen on grounds of physical plausibility than mathematical convenience. In the present treatment I have tried to expose rather than conceal the physical meaning of the approximations made.

The resemblance between the procedure used to calculate ν_k and D_k and quantum-mechanical time-dependent perturbation theory should be noted. The present problem could in fact be treated by quantum mechanical methods, but little advantage and a good deal of confusion would result from so doing. In calculating the Fokker-Planck coefficients it proved necessary to use the quasi-normality assumption. In this context however such an approximation is quite innocuous. We are merely evaluating a coefficient. Provided we accept the fundamental point — that a complete description is provided by the q_k 's — we are, by the properties of the Fokker-Planck equation and the H-theorem, guaranteed against the occurence of unphysical behaviour.

The relevance of the wave transport equation (25) to problems of interacting waves in plasma physics may be limited. It is obviously not valid if the perturbation expansion does not converge or if the distinct time-scales do not exist. A more subtle source of failure arises if ω_k is replaced by $\omega_k + i\,\gamma_k$, where γ_k is positive in some regions of k-space and negative in others.

The equilibrium solution will correspond to the creation of 'energy' in some regions of k-space and its transport to other regions where it is destroyed. It is open to question whether an H-theorem can be derived for this situation: certainly the equipartition solution is quite irrelevant.

APPENDIX A

CALCULATION OF ν_k

From equation (8) it is easily shown that :

$$\frac{\partial q_k}{\partial t} = 2\,\Re e\,\varepsilon \sum_j M^{-k}_{j,k-j}\, e^{i(-\omega_k+\omega_j+\omega_{k-j})t}\, E_{-k}\, E_j\, E_{k-j} \qquad \ldots (A.1)$$

$$= 2\,\varepsilon^2\,\Re e \sum_j M^{-k}_{j,k-j}\, e^{i(-\omega_k+\omega_j+\omega_{k-j})t}$$

$$\cdot\Big\{E_j(t)E_{k-j}(t)\int_0^t ds \sum_\alpha M^{k}_{\alpha,-k-\alpha}\, e^{i(\omega_k+\omega_j+\omega_{k-j})S}\, E_\alpha(S)E_{-k-\alpha}(S)$$

$$+ 2\,E_{-k}(t)E_{k-j}(t)\int_0^t ds \sum_\alpha M^{-j}_{\alpha,j-\alpha}\, e^{i(-\omega_j+\omega_\alpha+\omega_{j-\alpha})S}\, E\,(S)E_{j-\alpha}(S) \qquad \ldots (A.2)$$

Integrating and averaging:

$$\langle\Delta q_k(T)\rangle = 2\,\varepsilon^2\Re e \sum_{j\,\alpha} M^{-k}_{j,k-j}\, M^{k}_{\alpha,-k-\alpha}\int_0^T ds \int_0^S ds'$$

$$\cdot\,\langle E_j(S)E_{k-j}(S)E_\alpha(S')E_{-k-\alpha}(S')\rangle e^{i(-\omega_k+\omega_j+\omega_{k-j})\,S+i(\,\omega_k+\omega_\alpha+\omega_{-k-\alpha})S'}$$

$$+ 4\,\varepsilon^2\Re e \sum_{j\,\alpha} M^{-k}_{j,k-j} M^{-j}_{\alpha,j-\alpha}\int_0^T ds\int_0^S ds'\,\langle E_{-k}(S)E_{k-j}(S)E_\alpha(S')E_{j-\alpha}(S)\rangle$$

$$\cdot\; e^{i(-\omega_k+\omega_j+\omega_{k-j})\,S\,+\,i(-\omega_j+\omega_\alpha+\omega_{j-\alpha})S'}. \qquad \ldots (A.3)$$

If we require ν_k to order ε^2 only we may replace the slowly varying E's in equation (A.3) by their zero order (constant) values. If we also drop the fourth cumulant (see the main text for a discussion) we obtain:-

$$\langle\Delta\, q_k(T)\rangle =$$

$$4\,\varepsilon^2\,\Re e \sum_j M^{-k}_{j,k-j} M^{k}_{-j,j-k} Q_{,}(T) Q_{k-j}(T)\int_0^T ds\int_0^S ds'\, e^{i(-\omega_k+\omega_j+\omega_{k-j})(S-S')}$$

$$+ 8\,\varepsilon^2\Re e \sum_j M^{-k}_{j,k-j} M^{-j}_{k,j-k} Q_{-k}(T) Q_{k-j}(T)\int_0^T ds\int_0^S ds'\, e^{i(-\omega_k+\omega_j+\omega_{k-j})(S-S')}\,. \qquad \ldots (A.4)$$

If $T \gg \omega_k^{-1}$ and $T \ll$ (relaxation time) the double integrals behave like $T\,\delta(-\omega_k + \omega_j + \omega_{k-j})$ plus a term which disappears owing to antisymmetry. Dividing throughout by T we finally obtain:

$$\nu_k = 4\,\varepsilon^2 \sum_j M^{-k}_{j,k-j}\, M^{k}_{-j,j-k}\, f_j f_{k-j}\, \delta(-\omega_k + \omega_j + \omega_{k-j})$$

$$+ 8\,\varepsilon^2 \sum_j M^{-k}_{j,k-j}\, M^{-j}_{k,j-k}\, f_{-k}\, f_{k-j}\, \delta(-\omega_k + \omega_j + \omega_{k-j})\,. \quad \ldots \text{(A.5)}$$

APPENDIX B

CALCULATION OF D_{kj}

As this calculation is so similar to that of Appendix A it will be given in outline only. From equation (A.1) we easily obtain:

$$\langle \Delta q_k(T) \Delta q_j(T) \rangle = \varepsilon^2 \sum_{\ell, m} M^{-k}_{\ell, k-\ell} \, M^{-j}_{m, j-m} \int_0^T ds \int_0^T ds'$$

$$\cdot \langle E_{-k}(S) E_\ell(S) E_{k-\ell}(S) E_{-j}(S') E_m(S') E_{j-m}(S') \rangle$$

$$\cdot \, e^{i(-\omega_k + \omega_\ell + \omega_{k-\ell})S + i(-\omega_j + \omega_m + \omega_{j-m})S'} . \quad \ldots \text{(B.1)}$$

The six-fold moment is broken up thus:

$$\langle E_{-k} E_\ell E_{k-\ell} E_{-j} E_m E_{j-m} \rangle$$

$$= \langle E_{-k} E_\ell \rangle \langle E_{k-\ell} E_{-j} \rangle \langle E_m E_{j-m} \rangle \quad \ldots \text{(B.2)}$$

+ fourteen other terms obtained by permuting the indices.

Many of these terms can be combined using symmetry: others give vanishing contributions. The survivors are:

$$2 Q_{-k} Q_\ell Q_{k-\ell} \; \delta(k - j) \; \delta(\ell + m)$$

$$+ 4 \; Q_{-k} Q_{-j} Q_{k-j} \;\; \delta(k - m) \; \delta(j - \ell) . \quad \ldots \text{(B.3)}$$

Upon substitution of (B.3) into (B.1) and evaluation of the integrals we obtain, for $k = j$:

$$D_{kk} = 8 \; \varepsilon^2 \sum_\ell M^{-k}_{\ell, k-\ell} M^{k}_{-\ell, \ell-k} f_{-k} f_\ell f_{k-\ell} \; \delta(-\omega_k + \omega_\ell + \omega_{k-\ell}) . \quad \ldots \text{(B.4)}$$

If $k \neq j$:

$$D_{kj} = 16 \; \varepsilon^2 \; M^{-k}_{j, k-j} \; M^{-j}_{k, j-k} \; f_{-k} \; f_{-j} \; f_{k-j} \; \delta(-\omega_k + \omega_j + \omega_{k-j}) . \quad \ldots \text{(B.5)}$$

APPENDIX C

Equation (1) can be derived from a Hamiltonian in a simple way provided M satisfies certain conditions. Let:

$$A_k \equiv \rho_k + i\,\omega_k\,q_k\,, \qquad \dots\ (C.1)$$

and

$$M^k_{j\ell} \equiv A^k_{j\ell} + i\,B^k_{j\ell}\,. \qquad \dots\ (C.2)$$

The equations of motion of ρ_k and q_k are:

$$\frac{\partial \rho_k}{\partial t} = -\,\omega_k\,q_k + \sum_{j\ell} A^k_{j\ell}\left(\rho_j\rho_\ell - \omega_j\,\omega_\ell\,q_j\,q_\ell\right) - \sum_{j\ell} B^k_{j\ell}\left(\omega_j\,q_j\,\rho_\ell + \omega_\ell\,q_\ell\,\rho_j\right)\,, \qquad \dots\ (C.3)$$

$$\frac{\partial q_k}{\partial t} = \rho_k + \omega_k^{-1}\sum_{j\ell} A^k_{j\ell}\left(\omega_j q_j \rho_\ell + \omega_\ell q_\ell \rho_j\right) + \omega_k^{-1}\sum_{j\ell} B^k_{j\ell}\left(\rho_j\rho_\ell - \omega_j\omega_\ell q_j q_\ell\right)\,. \qquad \dots\ (C.4)$$

The Hamiltonian is:

$$H = \frac{1}{2}\sum_k\left(\rho_k^2 + \omega_k^2\,q_k^2\right) + \varepsilon\sum_{kj\ell}\left\{\frac{1}{3}\lambda_{kj\ell}\,\omega_k\,\omega_j\,\omega_\ell\,q_k\,q_j\,q_\ell + \frac{1}{3}\mu_{kj\ell}\,\rho_k\,\rho_j\,\rho_\ell - \mu_{kj\ell}\,\omega_k\,\omega_j\,q_k\,q_j\,\rho_\ell - \lambda_{kj\ell}\omega_k q_k \rho_j \rho_\ell\right\}. \qquad \dots\ (C.5)$$

λ and μ are symmetrical in all their indices and are related to A and B by:

$$A^k_{j\ell} = \lambda_{kj\ell}\,\omega_k\,, \qquad \dots\ (C.6)$$

$$B^k_{j\ell} = \mu_{kj\ell}\,\omega_k\,. \qquad \dots\ (C.7)$$

It must therefore be possible to express M as:-

$$M^k_{j\ell} = \left\{ \lambda_{kj\ell} + i\ \mu_{kj\ell} \right\} \omega_k . \qquad \ldots \text{(C.8)}$$

Note that the zero-order Hamiltonian is:

$$H_o = \frac{1}{2} \sum_k \left(\rho^2 + \omega_k^2\ q_k^2 \right) \qquad \ldots \text{(C.9)}$$

$$= \frac{1}{2} \sum_k A_k A_{-k} . \qquad \ldots \text{(C.10)}$$

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