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Preprint

THE PLASMA DISTRIBUTION FUNCTION NEAR THE LOSS CONE OF A MIRROR MACHINE

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THE PLASMA DISTRIBUTION FUNCTION NEAR THE LOSS CONE OF A MIRROR MACHINE

by

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ABSTRACT

The particle distribution in phase space near the loss cone of a mirror machine has been determined by an expansion in the ratio of the bounce time between the mirrors to the mean collision time. This determines the density and mean parallel energy of particles escaping from the machine both inside the machine and in the throat of the mirror. For typical reactor parameters the escaping plasma density reaches 10^{11} cm^{-3} in the throat so it will be necessary to re-examine the reflection of plasma waves in the neighbourhood of the mirrors. The mean parallel energy in the throat is typically 1% of the mean perpendicular energy.

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The object of this paper is to clear up an apparent inconsistency in the procedure traditionally used to obtain the plasma distribution function in a mirror machine, maintained in a steady state by particle injection. According to this procedure [1,2], the problem is treated by solving the Fokker Planck equation describing diffusion in velocity space, subject to the boundary condition that the distribution function vanishes at the loss cone (or the loss paraboloid or hyperboloid in cases where the ambipolar electrostatic potential is significant). If this boundary condition were exactly correct, the plasma density would certainly vanish in the throat of the mirror (i.e. where the magnetic field strength reaches its maximum value), since the loss cone is defined in such a way that a particle lying on the loss cone is just reflected at the throat. Since in addition all such particles would possess zero $v_{\parallel}$ (velocity parallel to the field lines) at that point, there would be zero particle flux there, and hence (by continuity) no flux outside the throat either. However this contradicts the principal conclusion of such Fokker Planck calculations, namely that there is a determinate net loss flux of particles out of a mirror machine.

In the context of various proposals for the direct conversion[3] or immediate re-injection [4] of particles escaping into the loss cone, it

is desirable to have a precise knowledge of the distribution function of these escaping particles. Thus a resolution of this apparent inconsistency is of more than merely academic interest. In this note, we show that the traditional procedure gives an accurate approximation to the distribution function everywhere except in the immediate neighbourhood of the loss cone, and we obtain a more accurate solution for this region of velocity space.

For simplicity we consider a one component plasma described by a distribution function f, for which the full Fokker-Planck equation is:

$$\vec{v} \cdot \nabla f + \frac{e}{m} \frac{\vec{v} \times \vec{B}}{c} \cdot \frac{\partial f}{\partial \vec{v}} = - \frac{\partial}{\partial \vec{v}} \vec{F} f + \frac{1}{2} \frac{\partial}{\partial \vec{v}} \frac{\partial}{\partial \vec{v}} : \overleftrightarrow{D} f + S \quad \ldots (1)$$

where $\vec{B}(\vec{r})$ is the confining field, $\vec{F}$ and $\overleftrightarrow{D}$ are the friction and diffusion coefficients and S is the function specifying the rate of injection of particles. Traditionally, the terms on the left are dropped, on the basis that although they are larger than the terms on the right (by a factor $1/\lambda$, where $\lambda \equiv \tau_b/\tau_c$ is the ratio of the bounce time between the mirrors to the mean collision time), they describe an approximately periodic motion in the mirror field $\vec{B}$ so that they vanish on averaging over a whole period. This argument is clearly inapplicable to particles lying in the loss cone. We therefore seek an alternative method of solution which does not involve any averaging over the mirror motion. Such a method is readily found in the case of a 'square well' mirror field, in which the magnetic field strength B_0 is constant over a length L, rising sharply at each end to a strength $B_m = R B_0$. If we assume that f and S have no variation with $\vec{x}_\perp$ (distance perpendicular to the $\vec{B}_0$ direction) or φ (the azimuthal angle in the spherical polar co-ordinate system v, θ, φ used to designate a point in velocity space), (1) becomes:

$$\mu v \frac{\partial f}{\partial z} = \Gamma\left(4\pi f^2 + \frac{1}{2}\frac{\partial^2 g}{\partial v^2}\frac{\partial^2 f}{\partial v^2} + \frac{1}{v^2}\frac{\partial g}{\partial v}\frac{\partial f}{\partial v} + \frac{1}{2v^3}\frac{\partial g}{\partial v}\hat{\mathfrak{L}} f\right) + S \quad \ldots (2)$$

where

$$\mu \equiv \cos\theta \ , \quad \hat{\mathfrak{L}} f \equiv \frac{\partial}{\partial \mu}(1 - \mu^2)\frac{\partial f}{\partial \mu} \ , \quad \Gamma = \frac{4\pi e^4 \ell n \, D}{m^2}$$

(ℓn D being the Coulomb logarithm), $g \equiv \int f(\overline{v}')\ (\overline{v} - \overline{v}')\, d^3 v'$ and we have as usual[1] neglected the terms contributed by the anisotropic part of the Rosenbluth potential g.

Normally the left hand side of equation (2) is dropped, and the remaining equation is then solved by 'separation of variables' or, which comes to nearly the same thing[1] , by expanding $f \equiv \sum_j P_{m_j}(\mu) f_j(v)$, and $S = \sum_j P_{m_j}(\mu) S_j(v)$, where the functions $P_{m_j}(\mu)$ are the eigenfunctions of the Legendre operator $\hat{\mathfrak{L}}$ with the boundary conditions $P_{m_j}(\pm \mu_c) = 0$ (μ_c being the cosine of the loss cone angle), and then dropping all but the first term in this expansion. In cases where the ambipolar potential is important, this procedure has been extended by Fowler and Rankin[5] , who allowed the eigenfunctions and eigenvalues of $\hat{\mathfrak{L}}$ to depend parametrically on v, as they clearly must if one is to preserve both the separation of variables approximation and the velocity dependence of the loss paraboloid or hyperboloid. This parametric separation procedure has since been put on a rigorous basis by Brambilla, using parabolic/hyperbolic co-ordinates[6].

For present purposes we shall take this 'parametric separation' procedure one stage further: we suppose that in the neighbourhood of the loss cone the full equation (2) possesses approximate solutions of the form $f = M(\mu, z; v)N(v)$ for $S = M(\mu, z; v)S(v)$, where M depends parametrically on v, and M and N satisfy the 'separated' equations:

$$\frac{\mu v}{\nu}\frac{\partial M}{\partial z} = \hat{\mathfrak{L}} M + \Lambda(v)\, M \quad \ldots (3)$$

$$\Gamma\left(4\pi N^2 + \frac{1}{2}\frac{\partial^2 g}{\partial v^2}\frac{\partial^2 N}{\partial v^2} + \frac{1}{v^2}\frac{\partial g}{\partial v}\frac{\partial N}{\partial v}\right) + S(v) = \nu\,\Lambda(v)N\ , \qquad \ldots\ (4)$$

where:

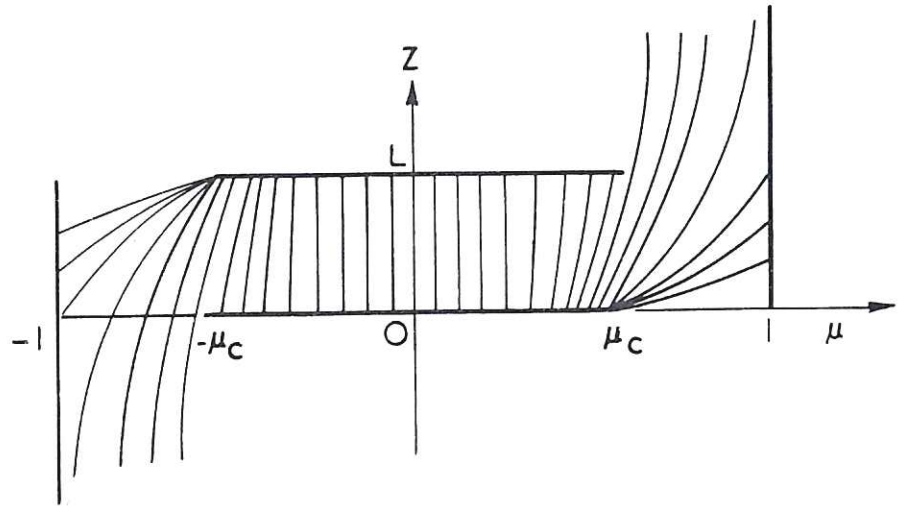
$$\nu(v) \equiv \frac{\Gamma}{2v^3}\frac{\partial g}{\partial v}$$

and Λ is the separation constant, depending parametrically on v. This separation will be justified retrospectively in what follows.

To solve equation (3) we use the fact that in the central region, $|\mu| \ll |\mu_c|$, the solution must approach that which is obtained by the traditional procedure: $M = M_0 \equiv P_{m_1}(\mu)$. Thus we write $M = M_0 + M_1$, where M_1 also satisfies equation (3) but is small compared with M_0 in the central region. We then expand (3) in the neighbourhood of $\mu = \mu_c$, writing $\mu = \mu_c + x$, $t = z/L$ and $D = \frac{1-\mu_c^2}{\mu_c}\frac{\nu L}{v}$, obtaining

$$D\frac{\partial^2 M_1}{\partial x^2} = \frac{\partial M_1}{\partial t} + O(D^{\frac{1}{2}})M_1 \qquad \ldots\ (5)$$

(To demonstrate that the neglected terms are indeed smaller by a factor $D^{\frac{1}{2}}$, it is necessary to make use of the solution of equation (5) discussed below.) Equation (5) is just the equation of heat transport in one dimension. The boundary conditions which are appropriate in the present context may be inferred from the boundary conditions which the solution of equation (3) must satisfy. These are somewhat asymmetric in μ and z and can best be described in relation to Fig.1.



For $|\mu| < \mu_c$, particles are reflected at the surfaces $z = 0$ and L. Thus $M(0,\mu) = M(L,-\mu)$. For $\mu > \mu_0$, $M(z,\mu) = 0$ for $z < 0$, since there are no particles entering the machine from the outside in the positive z direction. Likewise for $\mu < -\mu_c$, $M(z,\mu) = 0$ for $z > L$. Finally, for $|\mu| > \mu_c$, $M(\mu,z)$ must remain bounded for all z. Contours of constant M consistent with these boundary conditions are illustrated in Fig.1 and it will be seen that M is symmetric with respect to inversion through the point $\mu = 0$, $z = L/2$, so that it is sufficient to consider the region $\mu > 0$. The corresponding boundary conditions upon $M_1(x,t)$ in the neighbourhood of $x = 0$ are clearly

$$M_1(x,t) = 0 \quad , \quad x > 0 \quad , \quad t \leq 0 \qquad \ldots (6)$$

and continuity of:

$$M \equiv M_0 + M_1 \quad \text{and} \quad \frac{\partial M}{\partial x} \quad \text{at} \quad x = 0 \; . \qquad \ldots (7)$$

The continuity condition (7) is of paramount importance, for although M_0 is continuous in μ, vanishing for $\mu \geq \mu_c$, $\frac{\partial M_0}{\partial x}$ is discontinuous in μ , being equal to:

$$-\frac{\Lambda}{(1-\mu_c^2)} \int_0^{\mu_c} M_0(\mu)\, d\mu$$

for $x = 0-$ and vanishing for $x > 0$. Thus by equation (7) the slope $\frac{\partial M_1}{\partial x}$ must likewise have a discontinuity at $x = 0$: i.e. regarding equation (5) as analogous to the equation for the temperature distribution $M_1(x,t)$ in a bar, there is a source of heat at $x = 0$ during the time interval $0 < t < 1$. Clearly the problem is not symmetrical about $x = 0$, since otherwise half the heat flow would be in the negative x direction, and for $x \to -\mu_c$, the solution for M_1 would (in order of magnitude) be $\frac{1}{2} M_0$, contrary to our assumption that $M_1 \ll M_0$ in the central region. Thus we are obliged to take $\left|\frac{\partial M_1}{\partial x}\right| \ll \left|\frac{\partial M_0}{\partial x}\right|$ for $x < 0$, and the boundary condition (7) becomes:

$$H(t) \equiv \left.\frac{\partial M_0}{\partial x}\right|_{x=0-} = \left.\frac{\partial M_1}{\partial x}\right|_{x=0+} \qquad \ldots (8)$$

i.e., the problem is analogous to heat flow in an initially cold semi-infinite bar with a source of heat of magnitude $H(t)$ in the time interval $0 < t < 1$. The solution to this problem is well-known (see for example Carslaw and Jaeger, pp.75-6)[7]

$$M_1 = -2 \left.\frac{\partial M_0}{\partial x}\right|_{0-} (Dt)^{\frac{1}{2}} \text{ierfc}\left(x/2\sqrt{Dt}\right); 0 < t \leq 1 \qquad \ldots (9)$$

$$M_1 = -2 \left.\frac{\partial M_0}{\partial x}\right|_{0-} (Dt)^{\frac{1}{2}} \left[\text{ierfc}\left(x/2\sqrt{Dt}\right) - \left(1-\frac{1}{t}\right)^{\frac{1}{2}} \text{ierfc}\left(x/2\sqrt{D(t-1)}\right)\right]; t > 1 \qquad \ldots (10)$$

where $\text{ierfc}(y) \equiv \frac{1}{\sqrt{\pi}} e^{-y^2} - y \,\text{erfc}(y)$. It follows that M_1 is very small for $x \gg \sqrt{Dt}$, showing that in the neighbourhood of the mirror machine only a rather small part of the loss cone, close to $\mu = \mu_c$, is significantly populated with particles. The parametric dependence of M_1 on v is of the qualitative form:

$$M_1 \sim e^{\frac{x^2 v^4 L}{\lambda \bar{v}^4 z}}$$

where λ is the small parameter defined earlier, so when it is acted on by the operators $\frac{\partial}{\partial v}$ and $\frac{\partial^2}{\partial v^2}$ appearing in the full Fokker-Planck equation, they bring down terms of order x^2 and x^4, as compared with the terms of order 1 brought down by the operator $\hat{\mathcal{L}}$. This justifies the 'parametric separation' procedure for small x used to derive equations (3) and (4).

We now consider a number of applications .

1. <u>The density and mean velocity of the escaping plasma</u>

We evaluate these quantities first of all at the end of the flat central field region (i.e. just before the particles reach the mirror) and then re-evaluate them in the throat of the mirror. At the end of the central region $z = L$ so $t = 1$ and the solution (9) applies. Thus the distribution function of the particles escaping at $z = L$ is

$$f = -2\,N(v)\left.\frac{\partial M_o}{\partial x}\right|_{o-} D^{\frac{1}{2}}\ \mathrm{ierfc}\left(x/2D^{\frac{1}{2}}\right) \qquad \ldots (11)$$

so this density n_{esc} is given by

$$n_{esc} = -\,2\left.\frac{\partial M_o}{\partial x}\right|_{o-}\int_0^{\infty} N(v)\int_{\mu_c}^{1} D^{\frac{1}{2}}\ \mathrm{ierfc}\left(x/2D^{\frac{1}{2}}\right)\,d\mu\ v^2\,dv$$

$$\simeq -\,4\left.\frac{\partial M_o}{\partial x}\right|_{o-} n_o\,\bar{D}\int_0^{\infty}\mathrm{ierfc}\,(y)\,dy$$

$$= -\left.\frac{\partial M_o}{\partial x}\right|_{o-} n_o\,\bar{D}\ .$$

As we have shown elsewhere[9], a good analytic approximation to M_o for $R \gtrsim 3$ (i.e. $\mu_c \sim 1$) is:

$$M_o = \left(2 - \frac{1\cdot 2}{\ell nR}\right)^{-1}\left(1 - \frac{\ell n(1-\mu^2)}{\ell n(1-\mu_c^2)}\right) \qquad \ldots (12)$$

and this gives $\left.-\frac{\partial M_o}{\partial x}\right|_{o-} = R/(\ell nR - 0\cdot 6)$. Thus:

$$n_{esc}/n_o \simeq \bar{D}\,R/(\ell nR - 0\cdot 6)\ . \qquad \ldots (13)$$

Similarly, the flux of particles escaping at $x = L$ is:

$$F = -\,2\left.\frac{\partial M_o}{\partial x}\right|_{o-}\int_0^{\infty} N(v)\int_{\mu_c}^{1} D^{\frac{1}{2}}\ \mathrm{ierfc}\left(x/2\,D^{\frac{1}{2}}\right)\mu\,d\mu\ v^3 dv$$

$$\simeq \left(\frac{R}{\ell nR - 0\cdot 6}\right) n_o\ \overline{Dv}$$

$$= n_{esc}\ \overline{Dv}/\bar{D} \equiv n_{esc}\ \bar{v}_{esc}\ . \qquad \ldots (14)$$

To obtain the density of escaping plasma in the throat of the mirror (where $B = B_m = R\,B_o$) we 'continue' the distribution function (11) using the conservation of the particle energy ε and its magnetic moment P :

$$\varepsilon = v^2$$

$$P = v_\perp^2/B\ .$$

Thus in terms of ε and P, f has the form

$$f(\varepsilon,P) = -2\left.\frac{\partial M_o}{\partial x}\right|_{o-} N(\varepsilon^{\frac{1}{2}})\, D^{\frac{1}{2}}(\varepsilon)\ \text{ierfc}\left[\left\{\left(1-\frac{PB_o}{\varepsilon}\right)^{\frac{1}{2}}-\mu_c\right\}\ 2D^{\frac{1}{2}}\right]$$

so at $B = B_m$

$$n_{esc} = -\left.\frac{\partial M_o}{\partial x}\right|_o \int_o^\infty N\, D^{\frac{1}{2}} \int_0^{\varepsilon/B_m} \text{ierfc}\,\frac{d\,P\,B_m}{(\varepsilon - P\,B_m)^{\frac{1}{2}}}\,d\varepsilon$$

$$\simeq -4\left.\frac{\partial M_o}{\partial x}\right|_o \int_0^\infty N(v) D^{\frac{3}{4}}\, v^2\, dv (R\mu_c)^{\frac{1}{2}} \int_0^\infty \text{ierfc}(y)\, dy/y^{\frac{1}{2}}$$

$$= \frac{4}{3\sqrt{\pi}}\,\Gamma\left(\tfrac{1}{4}\right)\frac{R^{\frac{3}{2}}}{(\ell nR - 0{\cdot}6)}\, n_o\, \overline{D^{\frac{3}{4}}}\ .$$

By conservation of flux, the mean parallel velocity of the particles escaping through the throat

$$\bar{v}_{throat} = \frac{3\sqrt{\pi}}{4\,\Gamma\left(\frac{1}{4}\right)}\,\frac{\overline{Dv}\ R^{\frac{1}{2}}}{\overline{D^{\frac{3}{4}}}} \simeq \frac{3}{8}\, D^{\frac{1}{4}}\,\bar{v}\, R^{\frac{1}{2}}\ .$$

For a typical thermonuclear plasma ($n \sim 10^{14}$ cm^{-3}, $n\tau \sim 3\ 10^{13}$ sec cm^{-3}, $\bar{v} \sim 5\ 10^8$ cm^{-1}sec, L $\sim$ 1000 cm, R$\sim$10) $\bar{D} \sim 10^{-6}$, so the density of escaping plasma before it reaches the throat is 5×10^8 cm^{-3}, rising to 10^{11} cm^{-3} in the throat, where the spread in parallel energy of typical escaping particles is about 1% of their mean (perpendicular) energy. However it is worth pointing out that for escaping particles with an energy well below the average there is a much greater spread, since D scales as $1/v^4$.

2. Reflection of plasma waves at the mirrors

The problem of plasma wave reflection at the magnetic mirrors is important for determining the stability of mirror plasmas. The plasma wave reflection coefficient may be determined, once the electron density profile in the neighbourhood of the throat is known. The electron density profile is obtained from the ion density profile by use of the quasi neutral approximation ($n_i = n_e$).

Previous reflection coefficient calculations [8,9] have assumed that the electron and ion densities vanish at the mirror throat or at least that the density in the throat is low enough ($\omega_{pe} < \omega \simeq n\omega_{ci}$) for waves to become evanescent beyond the throat. Now for the usual mirror reactor parameters ($n \sim 10^{14}$, B = 30 kG, L = 1000 cm, E_i = 100 keV and R = 10), using section 1 the density in the mirror throat will be 10^{11} and ω_{pe} will be greater than $n\,\omega_{ci}$ for several harmonics (values of n).

This means that future reflection calculations will have to be extended into the region beyond the throat until a low density is reached where the wave is completely absorbed.

3. Comparison of distribution function formula with numerical experiment

We are only aware of one numerical (Monte Carlo) calculation of the velocity distribution in the loss-cone (Englert).[10] In this paper, the author was primarily interested in the effect of collisions in the loss-cone upon the containment time; however one can deduce from his figures the nature of the velocity distribution in the loss-cone. The parameter D in the above theory is in Englert's notation equivalent to $\ell/R\lambda$ and values of D between 6.0 and 10^{-4} are considered.

Since the present paper relies upon an expansion in $D^{\frac{1}{2}}$, a comparison between our theory and his numerical results is only relevant for small values of D. We have compared the theoretical value of the half width of the distribution of escaping plasma (i.e. of order D from equation 11) with the curves of Englert's Fig.5, and we find satisfactory agreement for D between 10^{-1} and 10^{-4}.

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