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F. A. HAAS

Culham Laboratory
Abingdon Berkshire

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A SHARP-BOUNDARY MODEL OF A HIGH PRESSURE TOKAMAK

by

F.A. Haas

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Abstract

A simple sharp-boundary model for a high pressure Tokamak is described. Its relationship to the recent work of Greene et al[3] is demonstrated.

U.K.A.E.A. Research Group,
Culham Laboratory,
Abingdon,
Berks.

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As a result of recent experimental studies[1] made on Tokamak T3-A it is of great interest to take the theoretical study of magnetohydrodynamic equilibria (and stability) in Tokamaks as far towards the reactor regime as possible. Analyses of magnetohydrodynamic equilibria in Tokamaks have usually been made in terms of the inverse aspect ratio $\varepsilon\ (= r_o/R_o)$, where $\varepsilon \ll 1$. Most studies[2, 3, 4] have been concerned with arbitrary pressure distributions, where $\beta_\theta \sim 1$, β_θ being the ratio of plasma pressure to magnetic pressure due to the poloidal field. In the work of Shafranov[2], and Ware and Haas[4], it is shown that to $0(\varepsilon^2)$ the flux-surfaces are circles with the centres of their cross-sections displaced inwards by a variable distance Δ from the magnetic axis, and where $\Delta \sim \varepsilon$. Very recently, using the same ordering and studying a model in which a sharp-boundary separates plasma and vacuum, Greene et al[3] have carried their calculation to $0(\varepsilon^3)$ and shown that the flux-surfaces are elliptically distorted as well as non-concentric.

Considering specific pressure and current distributions a number of workers[5, 6] have studied models in which $\beta_\theta \sim \varepsilon^{-1}$ and $\Delta \sim 1$. Thus Laval et al[5] have studied a diffuse plasma contained in a torus of elliptical cross-section. Strauss[7] has shown, again for a specific form of pressure distribution, the plasma being contained in a torus of rectangular cross-section, that equilibria with arbitrary β_θ are possible. In both[5, 7] these pieces of work the pressure vanishes at the surrounding conducting wall but the pressure gradient is finite. Thus in the vicinity of the wall there will be a finite current flowing - a feature undesirable in a reactor. Jukes and Haas[6] have described both diffuse and sharp-boundary models for $\beta_\theta \lesssim \varepsilon^{-1}$. For the sharp-boundary case, however, their model has a heavily distorted interface.

In the present note we show the existence of a very simple sharp-boundary Tokamak equilibrium for which $\beta_\theta \sim \varepsilon^{-1}$ and $\Delta \sim 1$. In our model the pressure, which has an essentially parabolic radial dependence, falls to zero at a circular boundary. The note concludes by relating the present work to that of Greene et al[3].

We begin by considering the solution of the equations

$$j \times \underline{B} = \nabla p \qquad j = \nabla \times \underline{B}$$

and

$$\nabla \cdot \underline{B} = 0$$

within an axisymmetric toroidal surface. As is well-known[8] the introduction of a stream function ψ leads to the equation

$$\nabla^2\psi - \frac{\cos\theta}{R_o + r\cos\theta} \cdot \frac{\partial\psi}{\partial r} + \frac{\sin\theta}{r(R_o + r\cos\theta)} \cdot \frac{\partial\psi}{\partial\theta}$$

$$+ II'(\psi) + (R_o + r\cos\theta)^2 p'(\psi) = 0, \qquad (1)$$

where $\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}$,

and where I and p, the current stream function and pressure respectively, are arbitrary functions of ψ. The coordinates

r, θ are local polar coordinates based on the point 0. (see Figure 1). In our particular model we assume the plasma to occupy a toroidal surface with circular cross-section centre 0, the major and minor radii being R_o and r_o respectively. External to the plasma it is assumed that there exists a vacuum magnetic field capable of maintaining the required pressure balance at the surface. We choose I and p to have the forms

$$I = \frac{b\psi}{r_o} \quad \text{and} \quad p = \frac{a}{2R_o^2 r_o^2} (\psi_B^2 - \psi^2), \tag{2}$$

the latter ensuring that the pressure vanishes at the plasma boundary $\psi = \psi_B$. The dimensionless quantities a and b are free parameters. Equation (2) now becomes

$$\nabla^2\psi - \frac{\varepsilon \cos\theta}{1 + \varepsilon r \cos\theta} \cdot \frac{\partial\psi}{\partial r} + \frac{\varepsilon \sin\theta}{r(1 + \varepsilon r \cos\theta)} \cdot \frac{\partial\psi}{\partial\theta} + [b^2 - (1 + \varepsilon r \cos\theta)^2 a]\psi = 0, \tag{3}$$

where r is now dimensionless and $r = 1$ represents the plasma boundary. Taking $\varepsilon \ll 1$ we further choose a and b such that $b^2 \sim a \sim 1$ and that $b^2 - a \sim \varepsilon$. Expanding ψ in the form

$$\psi = \psi_o (r, \theta) + \psi_1 (r, \theta) + \dots$$

we can solve equation (3) order-by-order. Choosing ψ_o to be a constant then the leading order equation is trivially satisfied. Setting $\psi_o = \psi_B$ it follows that the pressure is given by

$$p = - \frac{a \psi_o \psi_1}{R_o^2 r_o^2}, \tag{4}$$

the pressure vanishing at $r = 1$ for $\psi_1 = 0$. Since the toroidal magnetic field B_ϕ is given by

$$B_\phi = \frac{I(\psi)}{R_o\ (1 + \varepsilon r \cos\theta)} ,$$

it follows that β (the ratio of plasma pressure to pressure due to the magnetic field) is given by

$$\beta = - \frac{2a}{b^2}\frac{\psi_1}{\psi_o} \sim \varepsilon .$$

The first order equation to be solved is

$$\nabla^2\psi_1 + (b^2 - a)\psi_o - 2\varepsilon a\psi_o r\cos\theta = 0 ,$$

for which the appropriate solution is

$$\psi_1 = -\frac{\psi_o}{4}(1 - r^2)\left[a - b^2 + \varepsilon a r \cos\theta\right] . \qquad (5)$$

By (4) the pressure has an essentially parabolic dependence on r. The positions of the pressure maxima and minima, and hence the magnetic axes are given by the equation

$$3\nu r^2 + 2\mu r - \nu = 0, \qquad (6)$$

where $\nu = \varepsilon a\ (a - b^2)^{-1}$ is taken to be positive and $\mu = \pm 1$. Equations of this form have been obtained by Adam and Mercier[9] and Laval et al.[5] but with ν defined differently. For $\nu < 1$ there is one (outward) magnetic axis corresponding to a pressure maximum. For $\nu > 1$ there are two magnetic axes, the second corresponding to an inward axis with a pressure minimum, which for $p = 0$ at the boundary means a point of negative pressure. Thus we are only concerned with $\nu < 1$, for which the displacement Δ of the magnetic axis is given by

$$\Delta = \frac{1}{3\nu} \{- 1 + (1 + 3\nu^2)^{1/2}\} , \qquad (7)$$

and is a quantity of order one. Writing equation (5) in the form

$$Q = - (1 - r^2) (\nu^{-1} + r \cos\theta) , \qquad (8)$$

then a typical plot of these surfaces is shown in Figure 2.

We now consider the vacuum region. The equation to be solved comprises the first three terms of (3). Expanding the stream-function as before and taking the zero-order solution to be a constant, then to first-order

$$\psi_1^{v} = A \ln r + K (r - r^{-1}) \cos\theta, \qquad (9)$$

where A and K are constants to be determined. We now show that this solution is of sufficient generality to satisfy the pressure balance condition at the boundary. Since the boundary is a circular flux-surface on which the pressure vanishes, this condition becomes

$$B_\phi^2 + B_\theta^2 = B_\phi^{v2} + B_\theta^{v2} . \qquad (10)$$

Now the toroidal field in the vacuum region is given by

$$B_\phi^{v} = \frac{C}{R_o (1 + \varepsilon r \cos\theta)} , \qquad (11)$$

where C is a constant. To leading-order equation (10) gives $B_{\phi o} = B_{\phi o}^{v}$ and hence $C = b\psi_o / r_o$. This implies that to first-order $B_{\phi_1} = B_{\phi 1}^{v}$, which is, in fact, automatically satisfied. To second-order

$$B_{\theta 1}^{2} + 2B_{\phi o} B_{\phi 2} = B_{\theta 1}^{v\,2} + 2B_{\phi o}^{v} B_{\phi 2}^{v} . \tag{12}$$

For p to vanish to second-order at the boundary, $\psi_2 = 0$, and it follows that $B_{\phi 2} = B_{\phi 2}^{v}$, and hence the poloidal field must be continuous at the interface. Since B_θ is given by

$$B_\theta = \frac{1}{R_o\, r_o\, (1 + \varepsilon r \cos\theta)} \frac{\partial \psi}{\partial r} ,$$

it follows that

$$A = \tfrac{1}{2} (a - b^2)\psi_o \quad \text{and} \quad K = \tfrac{1}{4} \varepsilon a \psi_o . \tag{13}$$

Equation (9) can be written in the form

$$Q = 2\nu^{-1} \ln r + (r - r^{-1})\cos\theta , \tag{14}$$

for which a typical plot is shown in Figure 3. We observe that the terms involving $\ln r$ and $r^{-1}\cos\theta$ arise from the currents in the plasma, whilst the $r \cos\theta$ term indicates the presence of an external uniform vertical field, which is given by

$$B_{vert} = \tfrac{1}{4} \frac{\varepsilon a\, \psi_o}{r_o\, R_o} . \tag{15}$$

For a given ν equation (14) exhibits families of open and closed surfaces with a separatrix given by

$$Q_c = 2\nu^{-1} \cosh^{-1} (\nu^{-1}) - 2(\nu^{-2} - 1)^{1/2} .$$

We note that as $\nu \to 1$, $Q_c \to 0$, the value of Q on the boundary.

We have demonstrated the existence of a simple sharp-boundary, $\beta \sim \varepsilon$ (or $\beta_\theta \sim \varepsilon^{-1}$), $\Delta \sim 1$ equilibrium. We now show that it is compatible with the recent work of Greene et al.[3], in which they consider a sharp boundary, $\beta \sim \varepsilon^2$, $\Delta \sim \varepsilon$ equilibrium. In the present note the poloidal fluxes have been evaluated to $O(\varepsilon)$. If we now take $\beta \sim \varepsilon^2$ (i.e. $\nu \sim \varepsilon$) then the fluxes will contain terms up to $O(\varepsilon^2)$. They will not, of course, be correct to $O(\varepsilon^2)$. We wish to show, however, that quantities derived from these fluxes do agree with the appropriate terms evaluated in the work of Greene et al.[3]. Expressing the poloidal flux within the plasma as

$$\bar{\underline{\psi}} = (1 - r^2)(1 + \nu r \cos\theta) ,$$

and taking $\nu \sim \varepsilon$ ($\beta \sim \varepsilon^2$), it follows that the flux-surfaces are circles displaced outwards by an amount $\frac{1}{2}\nu\bar{\underline{\psi}}$ from the origin 0. Alternatively, denoting the displacement inwards from the magnetic axis by $\Delta(r)$, we have

$$\Delta(r) = \Delta(1) + \tfrac{1}{2}\nu(r^2 - 1) ,$$

where $\Delta(1)$ is the distance of the magnetic axis from the origin. The latter equation can be written as

$$\Delta(r) = \Delta(1) + \tfrac{1}{2}\Delta'(1)(r^2 - 1) . \qquad (16)$$

Expressing the poloidal field in the vacuum as

$$\bar{\underline{\psi}}^V = 2 \ln r + \nu(r - r^{-1})\cos\theta ,$$

and taking $\nu \sim \varepsilon$, the flux-surfaces become circles displaced inwards by an amount $\frac{\nu}{2}\,(\exp(\bar{\Psi}^{\nu}) - 1)$. The displacement inwards from the magnetic axis is also given by (16). Now equation (47) of Greene et al.[3] is

$$\Delta(r) = \Delta_a + \tfrac{1}{4}\,(r^2 - a^2)\left(\frac{2\Delta'_a}{a} - \frac{1}{R}\right) + \frac{r^2}{2R}\,\ln\,(r/a)\ , \qquad (17)$$

where r is the radius of a flux-surface, and $r = a$ is the boundary. For $\Delta_a \sim \Delta'_a \sim 1$, the last two terms are negligible. Then taking $\nu \sim \varepsilon$, r as defined in the present note and r as defined by Greene et al.[3], agree to $O(\varepsilon)$. Thus setting $a = 1$ we find that equation (16) agrees with the appropriate terms in (17).

As a further check we consider equation (42) of Greene et al., that is

$$\Delta'(r) = \frac{1}{R\,r\,f^{(1)2}(r)}\int_0^r\left(f^{(1)2}(r) - \frac{2rp'(r)}{B_o^{\,2}}\right)r\,dr\ , \qquad (18)$$

where $B_o \equiv B_{\phi o}$ and $f^{(1)} \equiv \dfrac{B_{\theta 1}}{B_{\phi o}}$. Since we are concerned with $p \sim \varepsilon$ the first term in the integrand can be neglected. Taking $\nu \sim \varepsilon$, and using the flux obtained earlier in this note to evaluate p and B_θ, $\Delta'(r)$ within the plasma is given by $\Delta'(r) = \nu r$, from which equation (16) is obtained. If we evaluate (18) for the vacuum region, we again recover equation (16).

Finally, we consider the externally imposed vertical field. From equation (80) of Greene et al.,

$$\frac{B_{\perp ext}}{B_o} = -\frac{a\, f_a^{(1)}}{2R}\left(\ln\left(\frac{8R}{a}\right) - \frac{3}{2} + \frac{R\Delta'_a}{a}\right), \qquad (19)$$

where $f_a^{(1)} = \frac{B_\theta^{(1)}(a)}{B_o}$. For $\Delta'_a \sim 1$ only the last term need be considered. From equation (15),

$$\frac{B_{vert}}{B_{\phi o}} = \frac{1}{4}\,\frac{\varepsilon\, a\, \psi_o}{r_o R_o\, B_{\phi o}} ,$$

which can be written as

$$\frac{B_{vert}}{B_{\phi o}} = \frac{\Delta'(1)}{2}\;\frac{B_{\theta 1}(1)}{B_{\phi o}} .$$

Setting $a = 1$ in equation (19) and noting that the difference in sign is due to B_θ being taken in the opposite direction by Greene et al., we see that the results are in agreement.

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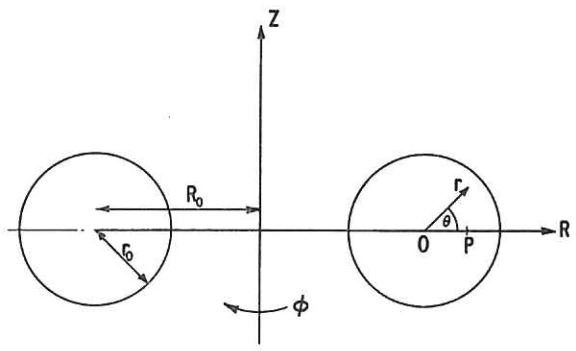

Fig. 1 Coordinate systems

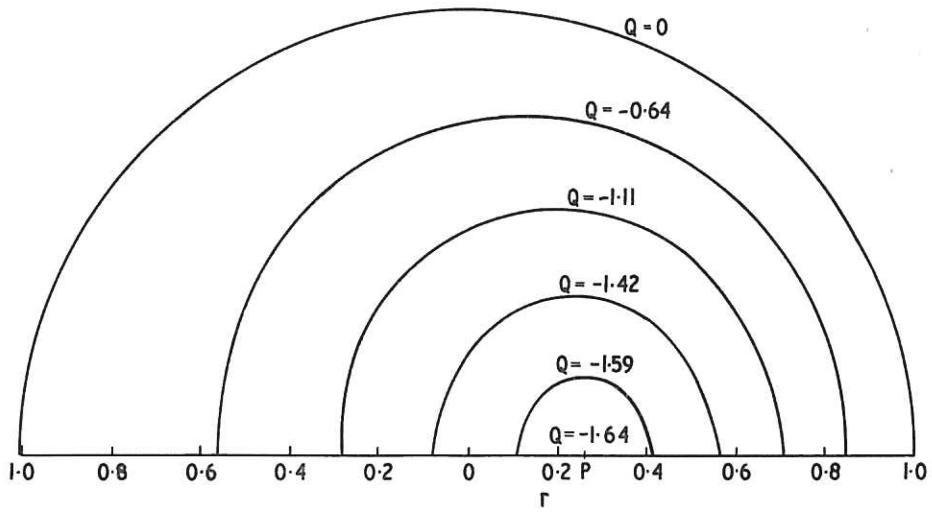

Fig. 2 Flux surfaces in the plasma for $\nu = ^2/_3$
The magnetic axis P is displaced $\Delta = 0.25$ from the centre O of the plasma

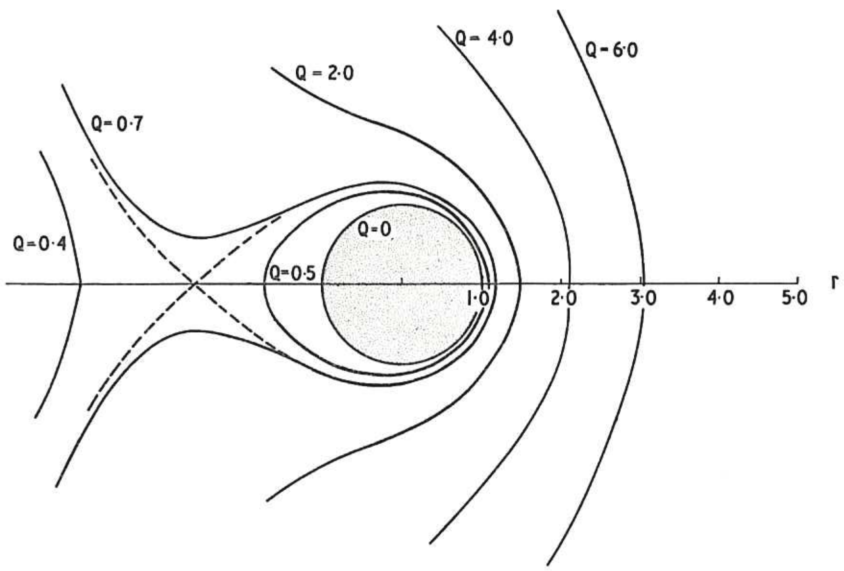

Fig. 3 Flux surfaces in the vacuum in the vicinity of the plasma for $\nu = ^2/_3$
The Q-value for the separatrix (the broken curve) is $Q_C = 0.65$