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ION CYCLOTRON INSTABILITIES IN HIGH-β PLASMAS

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A B S T R A C T

The complete dispersion equation (including all the electromagnetic terms) for flute like ($k_{\parallel} = 0$) ion cyclotron instabilities in a high beta plasma is solved using both numerical and analytic methods. A new type of ion cyclotron instability is found which is caused by the coupling of an electrostatic ion cyclotron wave with an electromagnetic ion cyclotron wave. The effect of magnetic field inhomogeneity is found to reduce the growth rate of this instability but does not stabilise it.

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I. Introduction

The theory of electrostatic ion cyclotron instabilities has been intensively studied over the last decade and is now virtually complete[1,2,3,4,5]. The electrostatic approximation which is used in all these papers is only strictly valid for β much less than m_e/m_i; however, most mirror reactor calculations assume a β of the order of a ½ or greater for economic reasons. Therefore, a considerable gap in the theory exists for β in the range $m_e/m_i < \beta < 1$. As a result of this discrepancy Callen and Guest[6] have recently considered the effect of some of the electromagnetic terms upon the electrostatic instabilities; their dispersion equation is valid for $\beta < (\frac{m_e}{m_i})^{\frac{1}{2}}$.

In section III of this paper the full dispersion equation including all of the electromagnetic terms is solved numerically for waves with $k_{\parallel} = 0$ for both Maxwellian and mirror distributions. The results for a Maxwellian distribution are presented to enable one to compare the normal electrostatic ion Bernstein modes with the solution of the full dispersion equation. For the mirror distributions a new instability is found, this is caused by a coupling between the electrostatic ion cyclotron wave and an electromagnetic wave. The frequency of the instability is very close indeed to harmonics of the ion cyclotron frequency, and expanding in $\frac{\omega - n\omega_{ci}}{\omega_{ci}}$, analytic expressions for the growth rate are obtained which agree with the numerical solution of the dispersion equation.

The fact that the instability is highly resonant (ie. $\frac{\omega - n\omega_c}{\omega_c} \ll 1$) suggests that it might be possible to stabilise this mode by detuning the resonance. Detuning caused by the variation of the magnetic field strength is discussed in section IV.

II. The Dispersion Equation

Throughout this paper we use the notation of the paper by Harris[1] and take as our starting point the expression for the perturbed charge density (Harris Eq 22) which may be written in the form,

$$f_1 = \frac{-e}{m\omega_c}\left[E_x \frac{\partial f_o}{\partial v_y} I_1 + E_y \frac{\partial f_o}{\partial v_\perp} + E_z \left\{\frac{\partial f_o}{\partial v_z} I_2 + \frac{k}{\omega}\left(v_\perp \frac{\partial f_o}{\partial v_z} - v_z \frac{\partial f_o}{\partial v_\perp}\right) I_1\right\}\right] \quad \ldots (1)$$

where all perturbed quantities are proportional to $\exp i(kx + \omega t)$, the magnetic field is in the z direction and the integrals I_n are given by Harris in the Appendix.

The equilibrium distribution function is the same as that used by Dory, Guest and Harris[2] (henceforth abbreviated to D.G.H.),

$$f_o^{\,j}(v_\perp, v_\parallel) = \frac{1}{\pi^{3/2}\, \alpha_\perp^{2} \alpha_\parallel\, j!} \left(\frac{v_\perp}{\alpha_\perp}\right)^{2j} \exp\left\{-\frac{v_\perp^2}{\alpha_\perp^2} - \frac{v_\parallel^2}{\alpha_\parallel^2}\right\} .$$

The above distribution function is a Maxwellian when $j = 0$, for higher j the distribution function has a positive slope near $v_\perp = 0$ and is similar to the collisional distribution function of a mirror machine (ie. the $j = 1$ distribution approximates the collisional distribution in a mirror machine of ratio 1.5 very closely).

Substituting the expression for f_1 (Equation 1) into Maxwells equation and completing the integrals over velocity space gives,

$$\bar{D}\begin{pmatrix} E_z \\ E_y \\ E_x \end{pmatrix} = 0 \qquad \ldots(2)$$

where the dielectric tensor $\bar{D}$ is given by,

$$\bar{D} = \begin{bmatrix} 1 - \left(\frac{\omega}{ck}\right)^2 + \alpha_{11} & 0 & 0 \\ 0 & 1 - \left(\frac{\omega}{ck}\right)^2 + \alpha_{22} & \alpha_{23} \\ 0 & -\alpha_{23} & -\left(\frac{\omega}{ck}\right)^2 + \alpha_{33} \end{bmatrix}$$

$$\alpha_{22} = 2\left(\frac{\omega}{ck}\right)^2 \sum_{i,e} \hat{\omega}_p^{\,2}\left\{ \sum_{n=1}^{\infty} \frac{1}{\hat{\omega}^2 - n^2}\left[\frac{\hat{\omega}^2}{\lambda} D_n^{\,j} - 2(j+1)\, D_n^{\,j+1}\right] - \frac{1}{2\lambda}\left(\delta_{oj} - D_o^{\,j}\right) - \frac{j+1}{\hat{\omega}^2} D_o^{\,j+1}\right\},$$

$$\alpha_{23} = 2i\left(\frac{\omega}{ck}\right)^2 \sum_{i,e} \frac{\hat{\omega}_p^{\,2}}{\hat{\omega}\lambda}\left\{ \sum_{n=1}^{\infty} \frac{\hat{\omega}^2}{\hat{\omega}^2 - n^2}\left[jD_n^{\,j} - (j+1)\, D_n^{\,(j+1)}\right] + \tfrac{1}{2}\left[jD_o^{\,j} - (j+1)\, D_o^{\,j+1}\right]\right\},$$

$$\alpha_{33} = 2\left(\frac{\omega}{ck}\right)^2 \sum_{i,e} \frac{\hat{\omega}_p^{\,2}}{\lambda}\left[\sum_{n=1}^{\infty} \frac{\hat{\omega}^2 D_n^{\,j}}{\hat{\omega}^2 - n^2} + \tfrac{1}{2}\left(D_o^{\,j} - \delta_{oj}\right)\right],$$

$$D_n^{\,j} = -\frac{\alpha_\perp^{\,2}}{2}\int_{-\infty}^{\infty} dv_\parallel \int_{-\infty}^{\infty} dv_\perp\, J_n^{\,2}\left(\frac{k_\perp v_\perp}{\omega_c}\right)\frac{\partial f_o^{\,j}}{\partial v_\perp},$$

$\hat{\omega}_p = \omega_p/\omega_{ci}$, $\hat{\omega} = \omega/\omega_{ci}$ and $\lambda = k_\perp^{\,2}\alpha_\perp^{\,2}/2\omega_c^{\,2}$.

The dispersion equation is

$$D \equiv \det(\bar{\bar{D}}) = 0 \cdot \qquad \ldots(3)$$

In the electrostatic approximation one assumes that α_{23}, α_{11}, α_{22} and $\left(\frac{\omega}{ck}\right)^2$ are small compared with unity and then the dispersion equation 3 reduces to

$$\alpha_{33} - \left(\frac{\omega}{ck}\right)^2 = 0 \, . \qquad \ldots(3a)$$

To simplify equation 3 we make the usual approximation that the electron larmour-radius is small compared with the wavelength ($k_\perp a_e \ll 1$) and then equation 3 can be expressed simply in the form,

$$D \equiv (\alpha_1 + \alpha_2\beta + \beta B)\left[\frac{\alpha_\perp{}^2}{c^2} + \beta\left(\frac{m_e}{m_i} - A\right)\right] + \frac{\beta^2}{2\lambda}(1 - c)^2 = 0, \quad ..(4)$$

where

$$\alpha_1 = 1 - \frac{\hat{\omega}^2}{2\lambda}\left(\frac{\alpha_\perp}{c}\right)^2$$

$$\alpha_2 = \theta_e - \frac{\hat{\omega}^2}{2\lambda}$$

$$A = \frac{1}{\lambda}\left(\sum_{n=1}^{\infty} \frac{2\hat{\omega}^2 D_n{}^j}{\hat{\omega}^2 - n^2} + D_o{}^j - \delta_{oj}\right)$$

$$B = \frac{1}{\lambda}\left\{\sum_{n=1}^{\infty} \frac{\hat{\omega}^2}{\hat{\omega}^2 - n^2}\left[\frac{\hat{\omega}^2}{\lambda} D_n{}^j - 2(j+1)\, D_n{}^{j+1}\right] + \frac{\hat{\omega}^2}{2\lambda}(D_o{}^j - \delta_{oj}) - (j+1)\, D_o{}^{j+1}\right\}$$

$$C = \frac{1}{\lambda}\left\{ \sum_{n=1}^{\infty} \frac{2\hat{\omega}^2}{\hat{\omega}^2 - n^2}\left[jD_n^{(j)} - (j+1)\, D_n^{(j+1)} \right] + jD_o^{(j)} - (j+1)\, D_o^{(j+1)} \right.$$

The dispersion equation (4) is now in a suitable form to be solved numerically and in the next section the numerical and an approximate analytic solution of equation (4) are given.

III. Solution of the Dispersion Equation

In this section we first describe the numerical procedure which was adopted to solve equation (4) and then present the results. Finally the dispersion equation is solved analytically and the growth rate of the unstable modes is derived.

Equation (4) is a quadratic in β and hence once the parameters A,B,C, etc. are known one can readily solve for β. Since A, B and C are infinite series the difficult part of the calculation is obtaining suitable approximations for the series. However, since D_n decreases like $\frac{1}{n}\left(\frac{\lambda}{n^2}\right)^n$, then the terms of the series decrease like $\frac{1}{n^3}\left(\frac{\lambda}{n^2}\right)^n$, it is found to be sufficient to consider only the first five terms to obtain accuracy of 10^{-6}. The two solutions of equation (4) for a Maxwellian distribution $(j = 0)$ are shown as a function ω in figure 1. The modes close to the $\hat{\omega} = n$ lines are electromagnetic cyclotron waves (e.m) and the other modes are the electrostatic Bernstein ion cyclotron modes (e.s). Since for any value of β there are two roots near each cyclotron harmonic then as one would expect the $j = 0$ distribution is stable for all β.

The corresponding results for the $j = 1$ mirror distribution are quite different as one can see from figure 2. The two waves intersect close to $\hat{\omega} = 1$ (ie. $\omega = \omega_{ci}$) and for β greater than

β_c , ω has two complex conjugate roots and hence the plasma is unstable. In figure 3 the growth rate of the mode as a function of β is plotted. The peak growth rate being 0.06 for β of order 1. This instability is very different in character from the D.G.H. electrostatic instabilities which are in fact stable for a $j = 1$ distribution. The instability is caused by an interaction between the positive energy electromagnetic (e.m.) wave and the negative energy electrostatic (e.s.) cyclotron wave; it is characterised by having a frequency very close to harmonics of ω_{ci} and this fact is now used to solve the dispersion equation analytically.

The highly resonant nature of the instability allows one to expand all quantities of equation (3) in $\Delta = (\omega^2 - n^2\omega_{ci}{}^2)/\omega_{ci}{}^2$. Equation (4) then becomes,

$$\left(\frac{C_n{}^2}{2\lambda} - B_nA_n\right)\frac{\beta^2}{\Delta^2} + \left[KB_n - (1+\beta B')\,A_n - (1-C')\,\frac{C_n\beta}{\lambda}\right]\frac{\beta}{\Delta}$$

$$+ K(1+\beta B') + \frac{\beta^2(1-C')^2}{2\lambda} = 0 \qquad \ldots(5)$$

where A_n is the numerator of the resonant term being considered and A' is the sum of the remainder of the terms with $\hat{\omega}$ set equal to n, such that,

$$A(\hat{\omega}) = A' + A_n/\Delta + O(\Delta)$$

and similarly for B and C,

$$K = a_\perp{}^2/c^2 + \beta m_e/m_i \ .$$

Equation (5) is a quadratic in $\frac{\Delta}{\beta}$ and has conjugate complex roots if,

$$b^2 - 4ac \equiv \left\{KB_n + (1+\beta B')\, A_n\right\}^2 + \frac{2}{\lambda}\left\{\beta A_n\,(1-C') - C_n K\right\}\left\{C_n\,(1+\beta B') + \beta\,(1-C')\, B_n\right\} < 0.$$

In figure 3 the growth rate obtained from the solution of equation (5) (dotted line) may be compared with the numerical solution of the exact dispersion equation (4).

The expression for the growth rate and equation (5) may be simplified considerably in two limits:

a. High Beta $\beta \gg a_\perp^2/c^2$.

In this case terms with K in are neglected compared with those containing β then,

$$b^2 - 4ac \equiv (1+\beta B')^2\, A_n^{\,2} + \frac{2}{\lambda}\,\beta A_n\,(1-C')[C_n\,(1+\beta B') + \beta\,(1-C')\, B_n] .$$

For a Maxwellian this function is always positive so that the plasma is always stable; however, for mirror distributions A_n passes through zero as we vary λ and then $b^2 - 4ac$ can be negative. The minimum value is easily deduced and is approximately given by

$$\mathrm{Min}(b^2 - 4ac) = -\left.\frac{\beta^2(1-C')^2}{\lambda^2}\,\frac{[C_n(1+\beta B') + \beta(1-C')\,B_n]^2}{(1+\beta B')^2}\right|_{\lambda=\lambda_{min}},$$

and the peak growth rate is given approximately by

$$\frac{\gamma}{\omega_{ci}} = \left.\frac{C_n(1+\beta B') + \beta(1-C')\,B_n}{2n(1+\beta B')(1-C')}\right|_{\lambda=\lambda_{min}}.$$

This expression is in agreement with the growth rate curve (figure 3) which was obtained numerically by a solution of the full dispersion equation.

b. Low Beta $\beta \ll \alpha_\perp^2/c^2$.

In this case the terms in β are neglected in preference to those containing $\alpha_\perp^2/c^2$ and we find by a similar analysis that the peak growth rate is

$$\frac{\gamma}{\omega_{ci}} = \frac{C_n\beta}{2n\sqrt{2\lambda K}} \circ$$

IV. Magnetic Field Inhomogeneity

In this section the variation in the magnetic field strength along the field lines is taken into account. The dispersion equation for this case is,

$$\int_{-L}^{L} D(s)ds = 0, \qquad \ldots(6)$$

where D is given by equation (4). Equation (6) is obtained in a similar manner to equation 9.12 in the paper by Mikhailovskii[7] and more recently by Baldwin et al[5]. In a mirror machine most of the parameters in equation (4) will be a function of s; however, here we are only interested in the variation of ω_{ci} as a function of s and to simplify the analysis the discussion is restricted to this case.

Now in section III it was shown that the instability was extremely resonant (ie. $\hat{\omega}^2 - n^2 \ll 1$) so that one may replace the infinite series by a resonant term and a term independent of $\hat{\omega}$. The same approximation is used in this section and equation (6) then becomes

$$\int \left[K(1+\beta B') + \frac{\beta^2(1-C')^2}{2\lambda} + \frac{\beta}{\Delta}\left\{ KB_n - (1+\beta B')\,A_n - (1-C')\,\frac{C_n\beta}{\lambda} \right\} + \frac{\beta^2}{\Delta^2}\left(\frac{C_n^{\,2}}{2\lambda} - A_nB_n \right) \right] ds = 0, \qquad \ldots(7)$$

where $\Delta \simeq 2n^2 \left\{ \frac{\omega}{n\omega_{co}} - 1 - \frac{s^2}{L^2}\,(R-1) \right\}$ and ω_{ci} has been assumed parabolic in s ie. $\omega_{ci} = \omega_{co}\,(1 + (R-1)s^2/L^2)$.

The only s dependence in equation (4) occurs in Δ and two integrals involving Δ may be evaluated in the usual manner

$$\int_{-L}^{L} \frac{ds}{\Delta} = \frac{1}{2n^2}\int_{-L}^{L} \frac{ds}{\delta\omega - \frac{s^2}{L^2}(R-1)} = \begin{cases} \dfrac{L}{4n^2\delta\omega^{1/2}(R-1)^{1/2}}\left[2\pi i + \log_e\left(\dfrac{1+\delta\omega^{1/2}/(R-1)^{1/2}}{1-\delta\omega^{1/2}/(R-1)^{1/2}} \right) \right], & 0<\delta\omega<R-1 \\[2ex] \dfrac{L}{4n^2\delta\omega^{1/2}(R-1)^{1/2}} \log_e \left(\dfrac{1+(\frac{R-1}{\delta\omega})^{1/2}}{1-(\frac{R-1}{\delta\omega})^{1/2}} \right), & \delta\omega>R-1 \end{cases} \qquad \ldots(8)$$

$$\int \frac{ds}{\Delta^2} = \begin{cases} \dfrac{L}{8n^4\delta\omega^{3/2}(R-1)^{1/2}}\left[\dfrac{2\delta\omega^{1/2}(R-1)^{1/2}}{\delta\omega-(R-1)} + \pi i + \log_e \left\{ \dfrac{1+(\frac{\delta\omega}{R-1})^{1/2}}{1-(\frac{\delta\omega}{R-1})^{1/2}} \right\} \right], & 0<\delta\omega<R-1 \\[2ex] \dfrac{L}{8n^4\delta\omega^{3/2}(R-1)^{1/2}}\left[\dfrac{2\delta\omega^{1/2}(R-1)^{1/2}}{\delta\omega-(R-1)} + \log_e \left\{ \dfrac{\delta\omega^{1/2}+(R-1)^{1/2}}{\delta\omega^{1/2}-(R-1)^{1/2}} \right\} \right], & \delta\omega > R-1 \end{cases}$$

where $\delta\omega = \frac{\omega}{n\omega_{co}} - 1$.

<u>Case 1</u> $\delta\omega \gg R-1$

The dispersion relation is identical to that of uniform magnetic field and hence the growth rate is the same as for the uniform magnetic field case (Section III).

Case 2 $\delta\omega \ll R-1$

The dispersion relation for this case is

$$K(1+\beta B') + \frac{\beta^2}{2\lambda}(1-C')^2 + \frac{i\pi B}{4n^2\delta\omega^{1/2}(R-1)^{1/2}}\left[KB_n - (1+\beta B')A_n - \frac{(1-C')C_n\beta}{\lambda}\right] + \frac{i\beta^2\pi}{8n^4\delta\omega^{3/2}(R-1)^{1/2}}(C_n^2/2\lambda - A_nB_n) = 0 \, .$$

The third term in this equation may be neglected if

$$\beta C_n/\delta\omega > 1$$ (this condition is usually satisfied)

then

$$\frac{\mathrm{Im}\delta\omega}{\omega_{co}} = \left[\frac{\pi\beta^2(C_n^2/2\lambda - A_nB_n)(R-1)^{-1/2}}{8n^4[K(1+\beta B') + \beta^2(1-C')^2/2\lambda]}\right]^{2/3} \, .$$

One can get a rough estimate of $\mathrm{Im}\delta\omega$ for $\beta = 1.0$, by substituting for C_n, A_n, B_n etc. from the computed values for D_n^1 etc.
For $R = 10$, $\mathrm{Im}\delta\omega = .01$ as opposed to .053 for $R = 1$ (Uniform magnetic field). Thus although the growth rate is reduced by magnetic field inhomogeneity the mode is not completely stabilised.

V. Conclusion

The complete electromagnetic dispersion equation for flute like ($k_{\parallel} = 0$) ion cyclotron modes in a high beta plasma has been solved using both Maxwellian and mirror particle distribution. The electromagnetic dispersion equation is found to contain an extra set of ion cyclotron waves as well as the usual electrostatic ion Bernstein modes. For mirror machine particle distributions the electromagnetic cyclotron waves interact with the ion Bernstein modes and at high densities the plasma is unstable. This new type of instability has a fairly small growth rate ($\gamma = 0.05\ \omega_{ci}$ for $\beta = 1$) and is highly resonant ($\frac{\omega - n\omega_{ci}}{\omega_{ci}} \ll 1$). Although the growth rate of the instability is reduced by inhomogeneity of the magnetic field the mode is still unstable.

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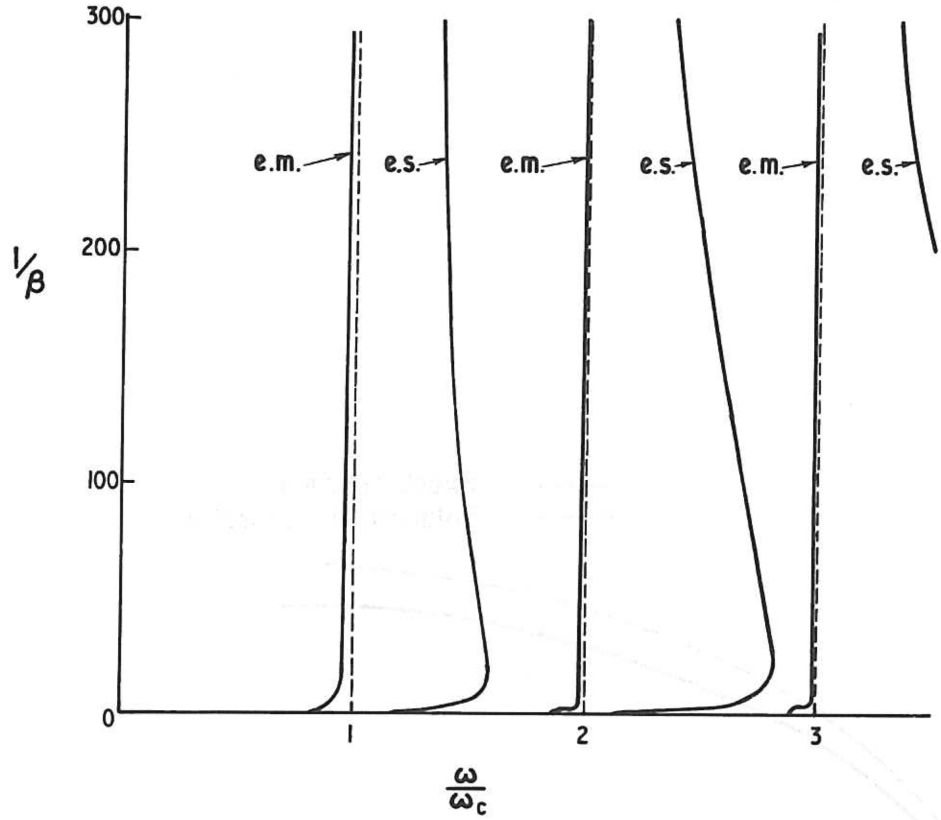

Fig. 1 A typical plot of $\frac{1}{\beta}$ against $\frac{\omega}{\omega_c}$ for a Maxwellian distribution with $\lambda = 1.0$.

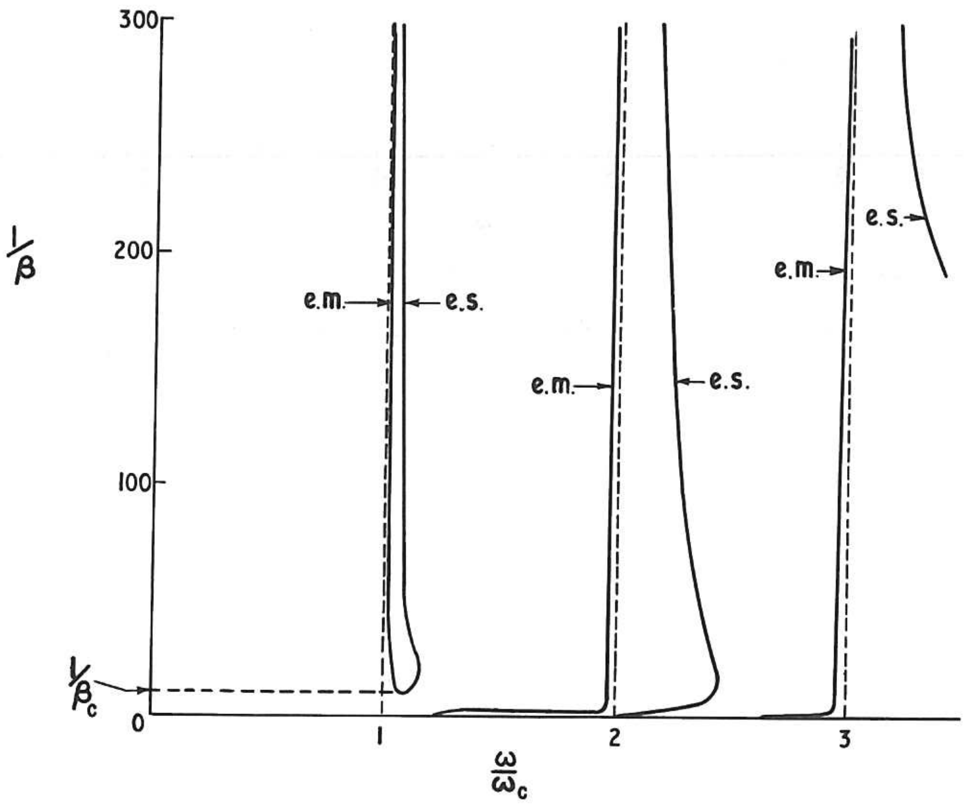

Fig. 2 A typical plot of $\frac{1}{\beta}$ against $\frac{\omega}{\omega_c}$ for the mirror distribution with $j = 1$ and $\lambda = 1.4$. For $\beta > \beta_c$ the plasma is unstable.

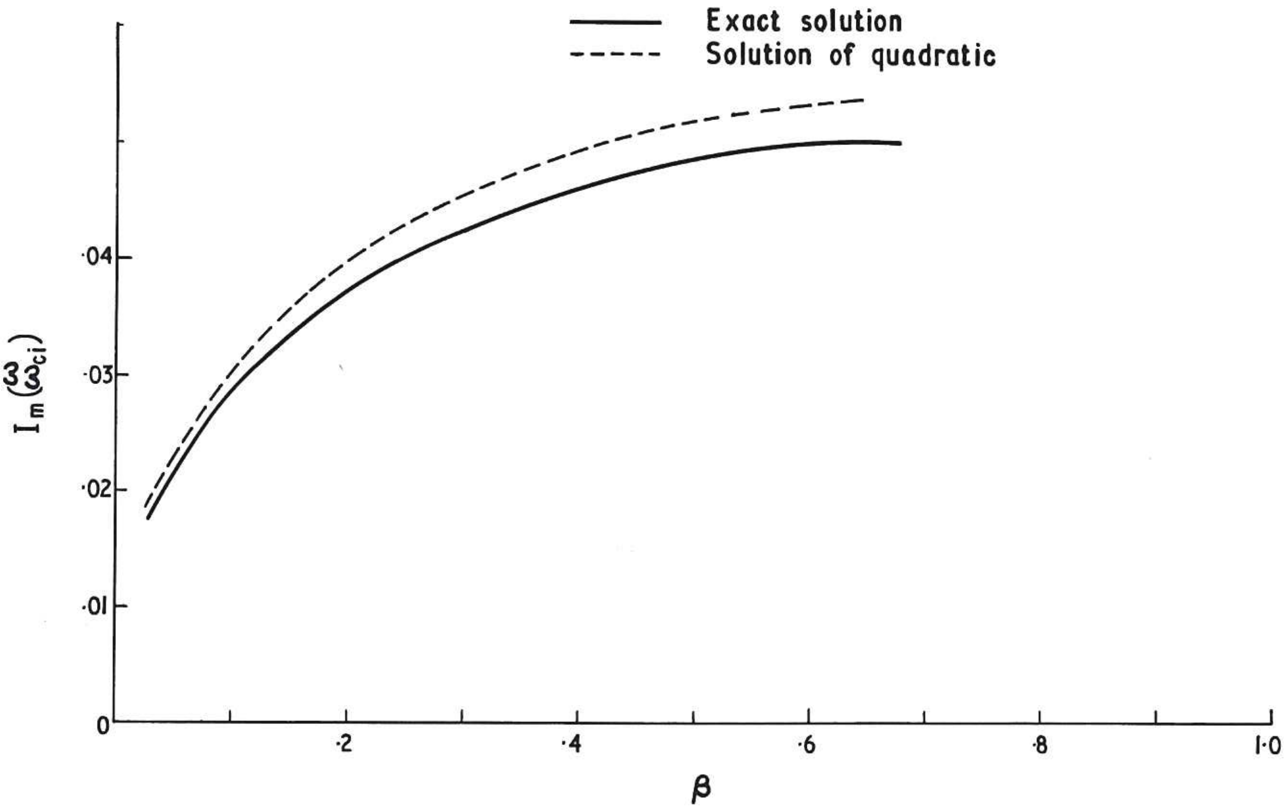

Fig. 3 Growth rate $\mathrm{Im}(\frac{\omega}{\omega_{ci}})$ against β for the j = 1 mirror distribution.
The dotted line is the analytic solution, the continuous line is the exact solution.