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STATISTICAL MECHANICS OF GUIDING CENTER PLASMA IN TWO DIMENSIONS

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STATISTICAL MECHANICS OF GUIDING CENTER PLASMA
IN TWO DIMENSIONS

by

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'Fluctuations in a Guiding Center Plasma in Two Dimensions')

A B S T R A C T

Statistical Mechanics is developed for a two dimensional guiding center plasma. Because there is no kinetic energy associated with guiding center motion this development is unconventional. Thermal equilibrium is discussed and an interesting limiting case is noted. Using the random phase assumption a kinetic equation for the density fluctuations is obtained which has thermal equilibrium and its limiting form as the only stationary states. However, despite the phase averaging this kinetic equation is reversible and when disturbed the system oscillates about equilibrium. Similar oscillatory behaviour appears in the microscopic correlation function of the fluctuations and the oscillation frequencies are obtained explicitly, however, these oscillations do not significantly change the macroscopic diffusion coefficient derived earlier by McNamara and Taylor¹.

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CONTENTS

	<u>PAGE</u>
I INTRODUCTION	1
II STATISTICAL MECHANICS OF THERMAL EQUILIBRIUM	3
III DYNAMICS OF THE SPECTRAL DENSITY	5
IV RELAXATION TO THERMAL EQUILIBRIUM	11
V SUMMARY AND DISCUSSION	17
VI ACKNOWLEDGEMENTS	19

I. INTRODUCTION

In a recent paper¹ an analysis was given of the diffusion of a two-dimensional guiding center plasma. This consists of long filaments of charge e/ℓ per unit length, aligned parallel to a large uniform magnetic field \underline{B} and moving with the $\underline{E} \times \underline{B}$ guiding center drift velocity. It was found that in this model diffusion is proportional to $1/B$ and that in thermal equilibrium the diffusion coefficient is similar to the Bohm value ckT/eB .

These results and other unusual features make the two dimensional guiding center model of considerable interest in its own right as well as of importance for the interpretation of the many two dimensional computer simulations now being undertaken. It may also be relevant to the understanding of real plasmas in strong magnetic fields, where charge fluctuations might be expected to have a filamentary character. The guiding center plasma model is also formally identical to the problem of two-dimensional turbulence in an ideal fluid, the charge density being the analogue of the conserved vorticity.

The statistical mechanics of a two dimensional plasma has been considered earlier by Salzberg and Prager² and by May³ who derived an equation of state of the form

$$p = \frac{n}{\ell} \left(kT - \frac{1}{4} \frac{e^2}{\ell} \right)$$

where n is the number of filaments per unit area. The first contribution to the pressure, p , arises from the kinetic energy, and the second from the potential energy of the filaments. As we shall show, the guiding centre model is one in which there is no kinetic energy, so that the pressure is apparently negative. However, the

model also depends on the presence of a magnetic field which so constrains particle motion that compression is only possible if accompanied by a corresponding increase in magnetic energy. Consequently in our model the total pressure must include the magnetic contribution and becomes

$$p = \frac{B^2}{8\pi} - \frac{1}{4} \frac{ne^2}{\ell^2}$$

In reference 1 the spectral distribution for the fluctuating electric field was considered both in thermal equilibrium, when

$$\langle E(k)E^*(k) \rangle = \frac{4 \pi \kappa T}{\ell(1 + k^2 \lambda^2)} \quad \dots (1)$$

(where $\lambda^2 = \kappa T \ell / 4 \pi n e^2$ is the debye length in two dimensions), and for a random distribution of filaments when

$$\langle E(k)E^*(k) \rangle = \frac{16 \pi^2 n e^2}{\ell^2 k^2} \quad \dots (2)$$

The correlation of the electric field was assumed to be described, approximately, by the normal distribution;

$$P\{E(t)\} \propto \exp \left(- \frac{1}{2} \iint \underline{E}(\tau_1) \underline{Q}(\tau_1 - \tau_2) \underline{E}(\tau_2) d\tau_1 d\tau_2 \right)$$

The aim of the present work is to consider the fluctuations in more detail and to develop a kinetic theory for the 2-D guiding center system. This development differs greatly from conventional theories because of the absence of any kinetic energy associated with guiding center motion. Accordingly, in section 2, the relevance of thermal equilibrium for such a system is first discussed. Then in subsequent sections we show that a reasonable and consistent approximation leads to a kinetic equation for the fluctuations and that the distributions (1) and (2) are the only stationary states. We also calculate the correlation function of the fluctuations and the behaviour of the system when disturbed from equilibrium. These calculations encounter

the well known problem of reconciling microscopic reversibility with macroscopic irreversibility and some difficulties remain in describing the relaxation of the system to thermal equilibrium. Finally, we discuss the macroscopic diffusion coefficient.

II STATISTICAL MECHANICS OF THERMAL EQUILIBRIUM

Before considering the dynamics of the guiding center plasma some remarks on thermal equilibrium for such a system are appropriate. It is, of course, universally accepted that thermal equilibrium is a stationary state of a dissipationless system. However the guiding center plasma is peculiar in that the velocity, rather than the acceleration, is determined by the force and moreover this velocity

$$\underline{v} = c \frac{\underline{E} \times \underline{B}}{B^2} \quad \dots (3)$$

is orthogonal to the force. One consequence of this is that there is nothing corresponding to kinetic energy in this system and this might appear to preclude the application of conventional arguments.

Nevertheless, despite the absence of kinetic energy, the system can still be described in Hamiltonian form. The Cartesian coordinates x_i, y_i (the magnetic field being in the z -direction) are canonically conjugate coordinates and the Hamiltonian is proportional to the electrostatic potential Φ ; in fact $H = \Phi/B$. The potential, and therefore the Hamiltonian, is a function of the coordinates of all the particles in the system:

$$\Phi = \sum_{i \neq j} \left(\frac{e_i}{\ell} \right) \log(\underline{R}_i - \underline{R}_j) \equiv \sum \varphi_i$$

and is a constant of the motion. However the energy φ_i associated with an individual particle is not a constant and any particle can exchange energy with any other. Hence, despite its peculiarities,

the guiding center plasma falls within the scope of the general Gibbs theory of thermal equilibria which leads⁴, via the usual Canonical Ensemble, to the thermal spectrum of Eq.(1). For the formally identical system of two dimensional ideal hydrodynamics, both the Hamiltonian character and the thermodynamic properties were discussed by Onsager⁵ in his study of two dimensional turbulence.

The spectrum (2) corresponds to a completely random distribution of particles, $F\{X_1, X, \dots, X_n\} = \text{constant}$, and it is easily seen from the Liouville equation for guiding center particles

$$\frac{\partial F}{\partial t} + \sum_i \frac{\tilde{E}(\tilde{X}_i) \times \tilde{B}}{B^2} \cdot \frac{\partial F}{\partial \tilde{X}_i} = 0 \quad \dots (4)$$

that this random distribution is also a stationary state of the guiding center system⁶. This becomes less surprising when one observes that if the thermal spectrum (1) is written as

$$\langle |E(k)|^2 \rangle = \frac{16 \pi^2 e^2 n}{\ell^2} \cdot \frac{\lambda^2}{(1 + k^2 \lambda^2)}$$

then the spectrum (2) can be regarded as thermal equilibrium in the limit $T \rightarrow \infty$. This accords with the infinite energy of a random distribution of charged filaments interacting with the coulomb law. In this connection it is noteworthy that if the coulomb interaction were replaced by one which diminished more rapidly with distance, then the completely random state would correspond to thermal equilibrium at some finite temperature T_c . (A random distribution always maximises the entropy and the temperature T_c to which it corresponds is determined by the expectation value of the energy).

III DYNAMICS OF THE SPECTRAL DENSITY

The discussion of thermal equilibrium, outlined above, yields information only about the spectrum at any one instant of time. We turn now to the main topic of the present paper - the derivation, and solution, of an equation for the time development of the spectrum of fluctuations. This will, incidentally, show that the thermal distribution and its limiting form are indeed the only stationary states.

We fix attention on the charge density in a given realisation of the system and for simplicity restrict ourselves to a plasma of identical filaments with a uniform neutralising background.

The density of filaments is

$$\rho(\underline{x}) = \sum_i^N \delta(\underline{x} - \underline{X}_i),$$

$\underline{X}_i(t)$ representing the positions of the N filaments. In terms of the Fourier transform, taken for simplicity over a unit square,

$$\rho(\underline{x}) = \sum_{\underline{k}} \rho_{\underline{k}} e^{i \underline{k} \cdot \underline{x}} \quad \rho_{\underline{k}} = \sum_i e^{-i \underline{k} \cdot \underline{X}_i}$$

and the density $\rho_{\underline{k}}$ satisfies the equation of continuity

$$\frac{\partial \rho_{\underline{k}}}{\partial t} + i \underline{k} \cdot \sum_{\underline{k}'} \underline{v}_{\underline{k}'} \rho_{\underline{k}-\underline{k}'} = 0. \quad \dots (5)$$

The guiding center velocity can be expressed directly in terms of $\rho_{\underline{k}}$ using the equation of motion (3) and the two dimensional Poisson equation which together yield

$$\underline{v}_{\underline{k}} = 4 \pi i \frac{c e (\underline{b} \times \underline{k})}{B \ell k^2} \rho_{\underline{k}}$$

so that

$$\frac{\partial \rho_{\mathbf{k}}}{\partial t} = \alpha \sum_{\mathbf{k}'} \frac{(\underline{\mathbf{b}} \cdot \underline{\mathbf{k}}' \times \underline{\mathbf{k}})}{k'^2} \rho_{\mathbf{k}-\mathbf{k}'} \rho_{\mathbf{k}'} \quad \dots (6)$$

where $\alpha \equiv 4\pi ec/B\ell$ and $\underline{\mathbf{b}}$ is a unit vector in the magnetic field direction.

Eq.(6) is the fundamental equation of motion for the density fluctuations. It describes the rate of change of the complex wave amplitude $\rho_{\mathbf{k}}$ due to non-linear interaction with other waves. It must be noted however, that the present problem differs essentially from the usual non-linear wave-wave interaction in plasmas. The conventional situation is one in which otherwise independent waves interact with one another through non-linear processes, whereas in the present problem there is no meaningful linear approximation. In the guiding center system the non-linear coupling not only causes transitions between waves; it is responsible for their very existence.

The equation of motion (6) exhibits the connection between this problem and the motion of a two dimensional ideal fluid. The charge density ρ is analogous to the conserved vorticity ψ and except for the constant α the relation between ρ and $\underline{\mathbf{y}}$ is exactly that between ψ and $\underline{\mathbf{y}}$.

The spectral function for density fluctuations is defined as

$$q_{\mathbf{k}} \equiv \langle \rho_{-\mathbf{k}} \rho_{\mathbf{k}} \rangle$$

where the angular bracket denotes an ensemble average, and is related to the electric field spectrum by

$$\langle E_{-\mathbf{k}} E_{\mathbf{k}} \rangle = 16\pi^2 \left(\frac{e}{\ell} \right)^2 \frac{q_{\mathbf{k}}}{k^2}.$$

To develop an equation for $q_{\mathbf{k}}(t)$ we differentiate twice with respect to time and consider

$$\ddot{q}_k = \langle \rho_k \ddot{\rho}_{-k} + \dot{\rho}_k \dot{\rho}_{-k} \rangle + c.c.$$

The fact that we must work with the second time derivative rather than the first is one of the consequences of the absence of linear waves in the present problem.

Using Eq.(6) for the time derivatives of the ρ_k , one obtains for \ddot{q}_k the equation

$$\begin{aligned} \ddot{q}_k = 2 \operatorname{Re} \alpha^2 \left\langle \sum_{k', k''} b \cdot \frac{\underline{k}' \times \underline{k}}{k'^2} \left\{ b \cdot \frac{\underline{k}'' \times \underline{k}'}{k''^2} \rho_{k''} \rho_{k' - k''} \rho_{k - k'} \rho_{-k} \right. \right. \\ \left. \left. + b \cdot \frac{\underline{k}'' \times (\underline{k} - \underline{k}')}{k''^2} \rho_{k''} \rho_{k - k' - k''} \rho_{k'} \rho_{-k} - b \cdot \frac{\underline{k}'' \times \underline{k}}{k''^2} \rho_{k''} \rho_{k - k'} \rho_{k'} \rho_{-k - k''} \right\} \right\rangle \end{aligned} \quad \dots (7)$$

So far this equation is exact, however in taking the ensemble averages we shall now adopt the random-phase assumption:- that only terms involving products of the form $\rho_k \rho_{-k}$ survive the averaging process and that $\langle \rho_k \rho_{-k} \rho_{k'} \rho_{-k'} \rangle$ may be factored as

$$\langle \rho_k \rho_{-k} \rangle \langle \rho_{k'} \rho_{-k'} \rangle = q_k q_{k'}.$$

In connection with the random phase approximation we note that for any spatially homogeneous system

$$\langle \rho_k \rho_{k'} \rangle = q_k \delta(\underline{k} + \underline{k}')$$

If the particles are also distributed at random then

$$\langle \rho_k \rho_{k'} \rangle = N \delta(\underline{k} + \underline{k}')$$

and

$$\begin{aligned} \langle \rho_{k_1} \rho_{k_2} \rho_{k_3} \rho_{k_4} \rangle &= N^2 \delta(k_1 + k_2) \delta(k_3 + k_4) + \text{perms} \\ &\quad + N \delta(k_1 + k_2 + k_3 + k_4) \\ &= \langle \rho_{k_1} \rho_{k_2} \rangle \langle \rho_{k_3} \rho_{k_4} \rangle + \text{perms.} + 0(1/N), \end{aligned}$$

so that the random-phase approximation is exact for a random distribution of particles in the limit $N \rightarrow \infty$. Furthermore if the correlation between quartets of particles is adequately approximated by the product of pair correlations, as might be the case if all correlations are small, the random phase result is again obtained. Despite its frequent use, it is difficult to justify the random phase approximation beyond this level and in what follows we shall explore its consequences.

Using the random-phase approximation Eq.(7) reduces to

$$\ddot{q}_0 = 2 \alpha^2 \sum_{\underline{k}_1, \underline{k}_2} \delta(\underline{k}_0 + \underline{k}_1 + \underline{k}_2) \frac{(\underline{b} \cdot \underline{k}_1 \times \underline{k}_0)^2}{k_1^2} \cdot \left\{ \left(\frac{1}{k_0^2} - \frac{1}{k_1^2} \right) q_0 q_1 + \left(\frac{1}{k_1^2} - \frac{1}{k_2^2} \right) q_1 q_2 + \left(\frac{1}{k_2^2} - \frac{1}{k_0^2} \right) q_2 q_0 \right\} \dots (8)$$

where we have replaced the three vectors $\underline{k}, \underline{k}', \underline{k}''$ by $\underline{k}_0, \underline{k}_1, \underline{k}_2$, and for brevity have labelled the q by the index of \underline{k} , i.e.

$$q_1 \equiv q(\underline{k}_1) \equiv q_{\underline{k}_1}.$$

Eq. (8) is the equation of motion for the spectral function $q_{\underline{k}}(t)$. It can be put in a more convenient form by noting that the factor $\delta(\underline{k}_0 + \underline{k}_1 + \underline{k}_2) \cdot (\underline{b} \cdot \underline{k}_1 \times \underline{k}_0)^2$ is symmetric⁷ under any permutation of $(\underline{k}_0, \underline{k}_1, \underline{k}_2)$, while the term in braces is antisymmetric in \underline{k}_1 and \underline{k}_2 . On interchanging the dummy indices $\underline{k}_1, \underline{k}_2$ and introducing the notation

$$A_{012} = 2\alpha^2 \delta(\underline{k}_0 + \underline{k}_1 + \underline{k}_2) (\underline{b} \cdot \underline{k}_1 \times \underline{k}_0)^2$$

$$B_{ij} = \left(\frac{1}{k_i^2} - \frac{1}{k_j^2} \right).$$

Eq.(8) becomes

$$\ddot{q}_0 = \sum_{1,2} \frac{1}{2} A_{012} B_{12} \left(B_{12} q_1 q_2 + B_{20} q_2 q_0 + B_{01} q_0 q_1 \right) \dots (9)$$

or

$$\frac{\ddot{q}_0}{q_0} = \sum_{1,2} \frac{1}{2} A_{012} \frac{B_{12}}{q_0} \left(\frac{B_{12}}{q_0} + \frac{B_{20}}{q_1} + \frac{B_{01}}{q_2} \right) q_0 q_1 q_2 \dots (10)$$

(a) Thermal Equilibrium

We can now establish that the thermal equilibrium spectrum

Eq.(1) and its limiting form Eq.(2) are the only stationary solutions of (10). To show this we first sum over k_0 to obtain

$$\sum_{k_0} \frac{\ddot{q}_0}{q_0} = \sum_{0,1,2} \frac{1}{6} A_{012} \left(\frac{B_{12}}{q_0} + \frac{B_{20}}{q_1} + \frac{B_{01}}{q_2} \right)^2 q_0 q_1 q_2 \dots (11)$$

Since A_{012} and the q are both positive definite, this establishes that

$$\sum_{k_0} \frac{\ddot{q}_0}{q_0} \geq 0 \dots (12)$$

equality holding if and only if

$$\left(\frac{B_{12}}{q_0} + \frac{B_{20}}{q_1} + \frac{B_{01}}{q_2} \right) = 0$$

for all k 's such that $(\underline{k}_0 + \underline{k}_1 + \underline{k}_2) = 0$.

This functional equation can be solved by setting $q_k = \frac{k^2}{\psi(k^2)}$, (clearly q_k can only depend on $|\underline{k}|$ in an isotropic system), then $\psi(k^2)$ must satisfy

$$(k_2^2 - k_1^2) \psi(k_0^2) + (k_0^2 - k_2^2) \psi(k_1^2) + (k_1^2 - k_0^2) \psi(k_2^2) = 0.$$

Writing $\underline{k}_2 = -(\underline{k}_0 + \underline{k}_1)$,

$$\begin{aligned} (k_0^2 + 2 k_0 k_1 \cos \theta) \psi(k_0^2) - (k_1^2 + 2 k_0 k_1 \cos \theta) \psi(k_1^2) \\ + (k_1^2 - k_0^2) \psi(k_0^2 + k_1^2 + 2 k_0 k_1 \cos \theta) = 0, \end{aligned}$$

and if ψ is expanded about $\cos \theta = 0$, a comparison of the coefficients of $(\cos \theta)^n$ shows that the only analytic solution must be of the form $\psi \equiv A + B k^2$. When the constants A, B, are appropriately identified this unique solution corresponds to

$$q_k = \frac{n \lambda^2 k^2}{(1 + \lambda^2 k^2)} \equiv Q_k \quad \dots (13)$$

which confirms that the thermal equilibrium spectrum and its limiting form are the only stationary solutions of Eq. (10). As noted by Onsager⁵, for finite systems in which k is bounded below, negative temperatures are possible. This would correspond to a negative value for λ^2 .

(b) Two Invariants

There are two significant invariants associated with Eqs.(9) or (10), for using Eq. (9) we can write,

$$\sum_{k_0} \frac{\ddot{q}_0}{k_0^2} = \sum_{0,1,2} \frac{1}{6} A_{012} \left(\frac{B_{12}}{k_0^2} + \frac{B_{20}}{k_1^2} + \frac{B_{01}}{k_2^2} \right) (B_{12} q_1 q_2 + B_{20} q_2 q_0 + B_{01} q_0 q_1) \quad \dots (14)$$

and

$$\sum_{k_0} \ddot{q}_0 = \sum_{0,1,2} \frac{1}{6} A_{012} (B_{12} + B_{20} + B_{01}) (B_{12} q_1 q_2 + B_{20} q_2 q_0 + B_{01} q_0 q_1) \quad \dots (15)$$

and the right sides of both these equations are identically zero by the definition of the coefficients B_{ij} . The first result (14) represents conservation of energy. The second (15) corresponds to the exact invariance of the quantity

$$\iint \rho^2(\underline{x}) dx dy$$

which follows from the Liouville equation (4). It is of interest that these exact invariants are still preserved after the Random Phase Approximation has been introduced.

IV RELAXATION TO THERMAL EQUILIBRIUM

In this section we investigate how the q_k 's vary if the system is displaced slightly from thermal equilibrium. To discuss this we consider again the equation of motion, Eq. (10) writing

$$q_k = Q_k + \delta q_k$$

and expanding to lowest order in δq_k . The term in square brackets vanishes for $q_k = Q_k$, so that

$$\frac{\ddot{\delta q}_0}{Q_0} = - \sum_{1,2} \frac{1}{2} A_{012} \frac{B_{12}}{Q_0} \left(\frac{B_{12}}{Q_0^2} \delta q_0 + \frac{B_{20}}{Q_1^2} \delta q_1 + \frac{B_{01}}{Q_2^2} \delta q_2 \right) Q_0 Q_1 Q_2 \dots \quad (16)$$

Eq.(16) constitutes a set of coupled linear equations, with constant coefficients, for $\delta q_k(t)$, so that the solutions will be made up of normal modes each varying with time as e^{ist} . If we consider a single mode with eigen-frequency s_λ , multiply (16) by $[\delta q_k^\lambda]^*$ and sum over k we obtain

$$s_\lambda^2 \sum_k \frac{|\delta q_k^\lambda|^2}{Q_k^2} = \sum_{0,1,2} \frac{1}{6} A_{012} Q_0 Q_1 Q_2 \left(\frac{B_{12}}{Q_0^2} \delta q_0 + \frac{B_{20}}{Q_1^2} \delta q_1 + \frac{B_{01}}{Q_2^2} \delta q_2 \right)^2 \dots \quad (17)$$

which indicates that s_λ^2 is positive definite and that all the normal modes are purely oscillatory. [The only non-oscillatory mode is merely a displacement to a neighbouring thermal equilibrium $[\delta q_k \propto Q_k]$]. Determining the actual frequencies of the normal modes is difficult, but they must be of the form

$$\omega_k = \frac{an^{\frac{1}{2}}}{\lambda} f(\lambda k) = \frac{c\kappa T}{eB} \cdot \frac{1}{n_D^{\frac{1}{2}} \lambda} f(\lambda k) \dots \quad (18)$$

where $n_D = n\lambda^2$ is the number of particles per debye square.

The oscillatory character of the normal modes is unusual, but this need not itself prevent a form of relaxation to thermal

equilibrium. For a disturbance expressed in terms of oscillating modes would tend to zero as $t \rightarrow \infty$ if the eigenvalues were continuous and the conditions of the Riemann-Lesbesque lemma were satisfied⁸. It must be recognised, however, that equation (16) is fully time reversible, despite the introduction of random phase in its derivation. True irreversibility appears only after some degree of coarse-graining has been introduced and we shall return to this question after the auto-correlation function has been considered.

V THE AUTO CORRELATION FUNCTION

So far we have examined only the time development of the fluctuating spectrum itself. In order to calculate the transport coefficients one needs to consider the correlation of the electric field at two different times. This is directly related to the auto-correlation of the density fluctuations which in a steady state is a function only of the separation of the two times,

$$S_k(\tau) \equiv \langle \rho_{-k}(t) \rho_k(t - \tau) \rangle.$$

To obtain an equation for the time development of $S_k(\tau)$ we again consider the second derivative, this time with respect to τ , obtaining

$$\begin{aligned} \frac{d^2 S_k}{d\tau^2} = \alpha^2 \left\langle \rho_{-k} \sum_{k', k''} \frac{\underline{b} \cdot \underline{k}' \times \underline{k}}{k'^2} \left[\frac{\underline{b} \cdot \underline{k}'' \times \underline{k}'}{k''^2} \rho_{k''} \rho_{k' - k''} \rho_{k - k'} \right. \right. \\ \left. \left. + \frac{\underline{b} \cdot \underline{k}'' \times (\underline{k} - \underline{k}')}{k''^2} \rho_{k''} \rho_{k - k'} - k'' \rho_{k'} \right] \right\rangle \dots (19) \end{aligned}$$

where the quantity in square brackets is taken at $(t - \tau)$ but ρ_{-k} is taken at t . We again invoke the random-phase approximation and so reduce (19) to

$$\frac{d^2 S_k}{d\tau^2} = \alpha^2 \sum_{k'} \frac{(\underline{b} \cdot \underline{k}' \times \underline{k})^2}{k'^2} \left\{ \left[\frac{1}{k^2} - \frac{1}{k'^2} \right] q_{\underline{k}'} + \left[\frac{1}{|\underline{k}-\underline{k}'|^2} - \frac{1}{k^2} \right] q_{\underline{k}-\underline{k}'} \right\} S_k \dots (20)$$

or using the symmetry properties of the coefficients and the notation of section 3,

$$\frac{d^2 S_k(\tau)}{d\tau^2} = \sum_{1,2} \frac{1}{4} A_{012} B_{12} (B_{01} q_1 + B_{20} q_2) S_k(\tau). \dots (21)$$

It should be noted that the scope of the random-phase assumption has been extended in this section to a situation in which one of the four ρ_k in each term is at a different time to the others. The random-phase assumption is still exact, even in this extended sense, for a random distribution of non-interacting particles. In other cases it should be equally as valid as our earlier application so long as the time difference $(t-t')$ is small enough. It cannot be relied on at larger $(t-t')$ where our results may need modification.

Returning to Eq.(21) we note that in thermal equilibrium

$$(B_{01} q_1 + B_{20} q_2) q_0 + B_{12} q_1 q_2 = 0$$

so that (21) becomes

$$\frac{d^2 S_k}{d\tau^2} = - \frac{1}{4} \left[\sum A_{012} B_{12}^2 \frac{q_1 q_2}{q_0} \right] S_k \dots (22)$$

The coefficient on the right hand side is negative definite and $S_k(\tau)$ must therefore be an oscillating function of τ . In fact, since $S_k(0) = q_k$ it must be given by

$$S_k(\tau) = q_k \cos \Omega_k \tau \dots (23)$$

For the guiding plasma, therefore, the behaviour of the correlation function of the charge fluctuations is extremely simple; each mode oscillates independently at a given frequency Ω_k .

This oscillation frequency Ω_k can be found explicitly by introducing the thermal equilibrium values of q_k into Eq.(20) and replacing the sum over k' by an integral (using the substitution $[2\pi]^2 \sum \rightarrow \int d\mathbf{k}$). Then

$$\Omega_k^2 = - \frac{\alpha^2}{(2\pi)^2} \int d^2 \ell \frac{(\mathbf{\ell} \times \mathbf{k})^2}{\ell^2} \left[\left(\frac{1}{k^2} - \frac{1}{\ell^2} \right) q_\ell - \left(\frac{1}{k^2} - \frac{1}{(\mathbf{k}-\mathbf{\ell})^2} \right) q_{\mathbf{k}-\mathbf{\ell}} \right] \dots (24)$$

Since the separate terms in this integral are individually divergent, a good deal of care is needed in its evaluation⁹. This difficulty may be circumvented by introducing a cut off in q_k at large k and using $\mathbf{k} - \mathbf{\ell}$ as the variable of integration in the second term. The integral then becomes

$$\begin{aligned} \Omega_k^2 &= - \frac{\alpha^2}{(2\pi)^2} \int d\ell \ell \int d\theta \sin^2 \theta \ell^2 k^2 \left(\frac{1}{\ell^2} - \frac{1}{\ell^2 + k^2 - 2k\ell \cos\theta} \right) \\ &\quad \left(\frac{1}{k^2} - \frac{1}{\ell^2} \right) \frac{n \lambda^2 \ell^2}{1 + \lambda^2 \ell^2} \dots (25) \\ &= \frac{n\alpha^2}{\lambda^2} F(\lambda^2 k^2) \end{aligned}$$

where

$$F(x) = \frac{1}{8\pi} \left[\frac{(1+x)^2}{x} \log(1+x) - \left(1 + \frac{3}{2}x\right) \right], \dots (26)$$

a result which is independent of the cut-off introduced to secure convergence. For small x , $F(x) \rightarrow \frac{1}{24\pi} x^2$, hence in the long wave length limit the frequency becomes

$$\Omega_k^2 \rightarrow \frac{n \alpha^2 \lambda^2}{24\pi} k^4 = \frac{1}{6} \left(\frac{\kappa T c^2}{\ell B^2} \right) k^4 \dots (27)$$

For large x , $F(x) \rightarrow \frac{1}{8\pi} x \log x$, so that at short wave lengths

$$\Omega_k^2 \rightarrow \frac{n\alpha^2}{\lambda^2} \frac{1}{8\pi} \lambda^2 k^2 \log \lambda^2 k^2 = 2\pi n \frac{e^2 c^2}{\ell^2 B^2} k^2 \log(\lambda^2 k^2) \dots (28)$$

For the guiding center model, therefore, there is a remarkably complete description of the correlation function $S_k(\tau)$. As already

noted $S_k(\tau)$ is purely oscillatory and this behaviour reflects the inherent reversibility of Eq.(20). Since these results are based on the random phase approximation we expect (26) to give a good description of S_k only for short times. Its long time behaviour may be modified; in particular it appears that the correlation function will decay with time only when irreversibility is introduced by some form of 'coarse graining'.

VI THE DIFFUSION COEFFICIENT

The diffusion coefficient can be introduced by observing that the coarse-grained density $\bar{\rho}$ is described by a Fokker-Planck equation

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial \underline{x}} \cdot \frac{1}{2} \frac{\overline{\Delta \underline{x} \Delta \underline{x}}}{\partial \underline{x}} \bar{\rho} = 0 \quad \dots (29)$$

where $\overline{\Delta \underline{x} \Delta \underline{x}}$ denotes the macroscopic rate of change of the square of the displacement. This leads to the well known representation of the diffusion coefficient, in a uniform isotropic system, as

$$D = \frac{1}{2} \left(\frac{c}{B} \right)^2 \left\langle \underline{b} \times \underline{E}(\underline{x}, t) \cdot \int_0^\infty \underline{b} \times \underline{E}(\underline{x}[t+\tau], t+\tau) d\tau \right\rangle \quad \dots (30)$$

In this equation \underline{E} is integrated along each particle trajectory so that $\underline{E} \cdot \underline{E}$ appearing here differs from the autocorrelation functions computed in section (5). In terms of the Fourier components,

$$D = 2 \left(\frac{ce}{B\ell} \right)^2 \iiint k dk d\theta \left\langle \rho_k(t) \rho_{-k}(t+\tau) \exp(i \underline{k} \cdot \underline{\Delta x}(\tau)) \right\rangle \quad \dots (31)$$

In order to introduce macroscopic irreversibility, essential to the calculation of a diffusion coefficient, we consider not the diffusion of the field particles themselves but of a group of independent test particles. Then the average of $\rho_k \rho_k^*$ can be separated from that of Δx and

$$D = \frac{4\pi e^2 c^2}{B^2 \ell^2} \int_0^\infty \frac{dk}{k} S_k(\tau) \langle \exp(i \underline{k} \cdot \Delta \underline{x}(\tau)) \rangle d\tau \quad \dots (32)$$

Furthermore, since we assume that test particles do obey a diffusion equation

$$\langle \exp(i \underline{k} \cdot \Delta \underline{x}(\tau)) \rangle = \exp(-k^2 D \tau)$$

Introducing this, and the value for $S_k(\tau)$ found in the preceding section, we obtain an implicit equation for the diffusion coefficient

$$D = \frac{4\pi e^2 c^2}{B^2 \ell^2} \int \frac{dk}{k} q_k \frac{k^2 D}{(k^4 D^2 + \Omega_k^2)} \quad \dots (33)$$

In the absence of the Ω_k^2 term this equation could be solved immediately for D^2 as

$$D^2 = \frac{1}{4\pi} \frac{c^2}{B^2} \int \frac{dk}{k} \langle E^2(k) \rangle \quad \dots (34)$$

the expression obtained in reference 1. This diverges at small k which suggests that the major contribution to (34) comes from the small k region and that little error will be introduced if Ω_k^2 is replaced by its small k approximation $\Omega_k^2 = \beta^2 k^4$. Then we obtain

$$D^2 = \frac{1}{4\pi} \frac{c^2}{B^2} \int \frac{dk}{k} \langle E^2(k) \rangle - \beta^2$$

which does not differ significantly from (34) itself. If we identify the lower limit of integration k_m with $2\pi/L$, L^2 being the area of the total system, then for large L/λ

$$D = \frac{c\kappa T}{eB} n_D^{-\frac{1}{2}} \left[\frac{1}{4\pi} \log \frac{L}{2\pi\lambda} \right]^{\frac{1}{2}} \quad \dots (35)$$

The logarithm may also be written as $\frac{1}{2} \log \left(\frac{Ne^2}{\pi \ell n T} \right)$ where N is the total number of filaments in the system.

If the particles do, in fact, diffuse, then a better approximation to S_k should include the effects of this diffusion. This would presumably result in S_k itself decaying as $e^{-k^2 Dt}$, at least

for small k . If such a factor were included it would alter the calculated diffusion coefficient only by a factor $2^{-\frac{1}{2}}$.

VII SUMMARY AND DISCUSSION

Although it can be described in Hamiltonian form, the guiding center system is unusual in that the Hamiltonian contains no kinetic energy term. Consequently, statistical mechanics for this system differs considerably from the conventional description of plasma and introduces many novel features.

The dynamics of the system can be described exactly in terms of interacting density waves; however even this is unconventional in that the waves concerned have a non zero frequency only by virtue of their interaction with other waves - truly a strong interaction or bootstrap situation!

Despite its peculiarities, thermal equilibrium for our system can be described in the usual Gibbs ensemble theory and the stationary equilibrium spectrum calculated. The completely random distribution is also a stationary state which can properly be regarded as the $T \rightarrow \infty$ limit of thermal equilibrium. Its interest lies in the fact that it can be treated 'exactly' and in recognising that if the filaments interacted otherwise than with the Coulomb law, this random state could represent thermal equilibrium at a finite temperature.

The statistical theory of the guiding center plasma involves only the amplitudes of the density waves and has been developed through the random phase assumption. This leads to a kinetic equation for the waves which is conventional in that it describes 'collisions' between waves but is distinctly unconventional in that

it involves the second time derivative and is time-reversible. Thus although the random phase assumption discards phase information and leads to a kinetic equation it does not itself introduce irreversibility. This may occasion some surprise although it has been noted earlier in connection with a model system¹⁰. In this connection it may be significant that there is one situation in which the random phase approximation is exact.

The kinetic equation derived here, in addition to possessing appropriate invariants, has the property that thermal equilibrium, and its limiting form, are the only stationary states. However when disturbed from equilibrium the wave amplitudes oscillate about thermal equilibrium instead of relaxing toward it. This behaviour is, of course, a reflection of the reversibility of the kinetic equation. To obtain irreversible behaviour it would be necessary to introduce a measure of 'coarse-graining' in addition to random phase.

A similar oscillating behaviour is seen in the correlation function of the fluctuations. Indeed a remarkably complete description of the correlation function has been obtained, at least within the random phase approximation, in which each mode oscillates at a calculated frequency. A diffusion coefficient can be defined for a cloud of independent test particles, and the oscillating behaviour of the correlation function is found to have negligible effect. This coefficient is dependent on the size of the system because it is dominated by long wave length fluctuations.

Once irreversibility has been introduced it raises the well known difficulty of reconciling it with the reversibility of the basic dynamics. In our case, this difficulty is very clearly illustrated because our kinetic equation itself is reversible. It

seems that this equation, although obtained by discarding phase information, is still a microscopic one and the microscopic correlation function for example does indeed oscillate. However, on a coarse-grained, macroscopic, scale the system is irreversible and the macroscopic correlation $\bar{S}_k(\tau)$ would be damped.

We have mentioned earlier that the guiding center model may be relevant to real plasmas in strong magnetic fields, particularly when charge imbalance has arisen during plasma formation or as a result of an unstable phase. However, the theory developed here is probably of most interest in connection with computer simulations and with studies of statistical mechanics. This interest must be heightened by the detail which can be obtained from the theory and its freedom from arbitrary parameters. As it is both detailed and explicit the guiding center model should help to illuminate the relation between microscopic reversibility and macroscopic irreversibility and in this connection comparison with a reliable computer calculation of the correlation function would be of great interest.

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