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OPTIMISED MAGNETIC FIELD PROFILES AND A β-LIMITATION IN MINIMUM-B MIRROR REACTORS

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OPTIMISED MAGNETIC FIELD PROFILES AND A β-LIMITATION IN MINIMUM-B MIRROR REACTORS

by

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ABSTRACT

The steady-state collisional distribution function of a mirror-confined plasma which is known from Fokker Planck calculations, implies a unique relationship between the plasma pressure and the magnetic field strength at any point within the plasma. Using this relationship it is possible to deduce the maximum β consistent with macroscopic stability in a mirror machine with given mirror ratio. This β limitation is found to impose a rather mild restriction upon reactor designs. For fields which satisfy this β limitation, the profile of the magnetic field along its axis is restricted but not fully determined, by the minimum B requirement, and the remaining freedom can be used to optimise the magnet design by maximising the ratio of the thermonuclear power produced to the cost of the magnetic field windings. It is found that even when the profile has been optimised in this way, the plasma density and pressure profiles are rather peaked towards the centre of the reactor, and the ratio of the thermonuclear power produced in such an optimised minimum B reactor to the power which would be produced in a reactor of the same dimensions but with a square well profile (if it were stable) is approximately 1/4.

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1. INTRODUCTION

In a previous paper[1] we discussed some of the conflicting requirements which face the designers of a thermonuclear reactor based upon a minimum - B magnetic mirror confinement system. On the one hand, the collisional loss of particles into the loss cone represents a serious energy drain, which must be counteracted by reinjecting a substantial part of the gross output power from the reactor. This expensive activity is minimised by making the mirror ratio as high as possible. On the other hand, plasma stability is only ensured if the vacuum magnetic field (i.e. the magnetic field which would be produced by the magnetic windings if no plasma were present) increases outwards from the magnetic axis at all points lying between the mirrors, and the cost of such a magnet (per unit volume of enclosed plasma) rises rapidly with the mirror ratio R - the cost scaling as R^{ν} for large R, where ν is an index in the range 2-4. It follows from these considerations that there must exist an economically optimum mirror ratio, at which the total capital cost of the reactor per unit of net power output is minimum.

An exact determination of this optimum mirror ratio is not yet possible because of uncertainties about the costs of various essential reactor components, and we do not undertake one here. Rather, we are concerned with a necessary preliminary - the optimisation of the profile of the magnetic field strength between the mirrors. In our previous paper, the discussion on this point was somewhat imprecise, and some preliminary attempts to evaluate the overall cost of a mirror reactor have emphasized the need for greater accuracy in estimates of the magnet cost per unit of thermonuclear power produced, and for further optimisation if possible.

The principal deficiency of our previous paper was the choice of plasma distribution function, which we took as

$$f(\varepsilon,\mu,J) = f(\varepsilon,J)\,\Theta\,(\varepsilon - \mu B_m) \tag{1}$$

where ε is the particle energy, μ and J are adiabatic invariants, B_m is the mirror field strength and Θ is the unit step function. This choice of distribution function has been made by other authors (e.g. Yushmann[2]) and it is made plausible by the curves of the angular distribution in velocity space for large mirror ratios given by Bing and Roberts[3] in their paper on numerical solutions of the Fokker Planck equation for f in a mirror machine with normal mode injection, which are close to the step function in form. On the basis of this distribution, we argued that the plasma density (and hence the thermonuclear power density) would be roughly uniform within the region of plasma confinement, and hence that it was sufficient to minimise the magnet cost per unit of enclosed volume. In fact, however, as is shown by comparison with the work of Ben Daniel and Allis[4] on the Fokker Planck equation, the Bing Roberts curve is incorrect, and results from an inadmissible extrapolation of an empirical analytic expression which fits their results for small mirror ratio. The true distribution function (for normal mode injection) goes smoothly to zero as μ tends to ε/B_m, and the resulting density distribution is appreciably non-uniform, with a consequent reduction in the thermonuclear output.

A more interesting consequence of our previous choice of distribution function was that it led to a pressure distribution which varied significantly only in the immediate neighbourhood of the mirrors, and hence was essentially a function of the strength of the vacuum magnetic field only. This implied the existence of equilibria with all values of β less than unity, since the equation determining pressure balance in the radial direction

$$p_\perp + (\vec{B}_v + \vec{B}_\beta)^2/8\pi = \text{constant} \tag{2}$$

could be regarded simply as an equation determining the magnetisation field $\vec{B}_\beta$, which obviously possessed a solution for all $\beta < 1$, regardless of the profile of the minimum - B vacuum field $\vec{B}_v$. However, the true normal mode Fokker Planck distribution function, which we discuss in Section 2 below, leads to an expression for $p_\perp$ which is a smooth function of the total magnetic field strength B, and equation (2) then becomes a non-trivial consistency condition which possible equilibria must satisfy. Infact, we show in Section 4 below that if the vacuum field is still required (for stability reasons, which are unaffected by the precise choice of f) to be minimum - B, there is a maximum value of β for given mirror ratio, above which no stable equilibrium exists. For values of β less than this, stable solutions for $p_\perp$ and $\vec{B}_\beta$ exist, and are uniquely determined by the profile of the vacuum field: this profile is however arbitrary, and it can be chosen in such a way as to minimise the magnet cost per unit of thermonuclear output. This optimisation is reconsidered in Section 5.

2. THE COLLISIONAL DISTRIBUTION FUNCTION IN A MIRROR MACHINE

In most work on Fokker Planck equations describing the collisional equilibrium distribution functions in a mirror-confined plasma, (for example Petravic et al[5]), the distributions have the 'separated' form

$$f(\vec{v}) = f(v,\theta,\varphi) = U(\varepsilon_0)\ M(u) \tag{3}$$

when $\varepsilon_0 = \frac{1}{2}mv^2$ and $u = \cos\theta = \vec{v}.\vec{B}/vB$. The error introduced by this factorisation of the distribution function is still somewhat uncertain, but the calculations of Bing and Roberts[3], Ben Daniel and Allis[4] and Killeen and Marx[6] indicate that it is not large. In every case, the equation satisfied by M is Legendre's equation:

$$\frac{d}{du}(1 - u^2)\frac{dM}{du} + \Lambda M = 0\ . \tag{4}$$

However the boundary conditions to be applied depend upon the assumption made about the role of the electrostatic potential. In the earliest calculations this was ignored, and consequently M was required to vanish at the loss cone $u = \pm u_c = \pm\left(\frac{R_0 - 1}{R_0}\right)^{\frac{1}{2}}$, where R_0 is the overall mirror ratio. This poses an eigenvalue problem for Λ, of which the solution can be represented with reasonable ($\pm$ 10%) accuracy by $\Lambda = 1/\log_{10}(R_o)$, with a corresponding eigenfunction $M_{R_o}(u)$. More recently, Fowler and Rankin[7] introduced the electrostatic potential in a somewhat inconsistent manner which nevertheless seems to give a reasonable approximation to unseparated solutions, by defining an 'effective' mirror ratio for each species of particle

$$R_e = R_o/(1 \pm e\varphi_m/\varepsilon_o) \tag{5}$$

where φ_m is the total potential difference between the centre of the plasma and the walls. On this basis $M_{R_e}(u)$ is still a solution of Legendre's equation, but it depends parametrically on the particle energy, since the boundary conditions now depend on ε_o.

In most of these papers, the plasma was assumed to be spatially uniform out to the mirrors: however, a paper by Ben Daniel and Allis[4] showed that this was not in fact a serious restriction, and that very similar distribution functions at the centre of the plasma are obtained even if the plasma is confined in a mirror machine with a parbolic mirror profile. The distribution function at other points is obtained by 'continuing' the central distribution function using the adibatic invariance of μ and J, and the constancy of $\varepsilon = \varepsilon_o + e\varphi$. This mathematically convenient state of affairs results from the fact that in a mirror with a smooth magnetic profile, the density falls of monotonically to zero at the mirrors, and with it the net collision frequency, but the loss cone angle widens, and these two effects approximately cancel out.

For present purposes it is desirable to have a simple analytic expression for M. To obtain this, we have observed that there exists an asymptotic solution of (4), which is accurate in the limit $R \to \infty$. This is obtained by an iteration procedure: in lowest order one drops the term ΛM altogether to obtain a first approximation to M. In next order this first approximation is used in the term ΛM and one selects the constants of integration in the solution of the remaining equation so as to satisfy the boundary conditions, obtaining

$$M = 1 - \frac{\log(1 - u^2)}{\log(1 - u_c^2)} = 1 + \frac{\log(1 - u^2)}{\log R} . \qquad (6)$$

This asymptotic expression is compared with the exact eigenfunctions in Fig.1 : for $R = 1.5$ and 3.277 the exact eigenfunctions are (as Bing and Roberts show) the integral order Legendre functions $P_2(u)$ and $Q_1(u)$ respectively, which have trivial analytic representations: for $R = 10$, the eigenfunction is derived from the numerical tables of Ben Daniel and Carr[8]. It will be observed that the asymptotic expression agrees with the exact eigenfunctions to within a few per cent even for $R = 1.5$, so that in all practical cases the error involved in using it is comparable with the error due to the separation of the Fokker Planck equation. For smaller values of R, the solution obtained by power series expansion, truncated after two terms:

$$M = 1 - \frac{R}{R-1} u^2 \qquad (7)$$

gives a good approximation.

Most of the results obtained below are not strongly dependent upon the form of $u(\varepsilon_o)$: however, for the purpose of evaluating the role of the electrostatic potential it is necessary to consider particular forms. The electrons we regard as being confined electrostatically, and we neglect the effect of the magnetic field on them altogether, taking a Maxwellian distribution

$$f_- = K_- e^{-\varepsilon_o/T_e} . \qquad (8)$$

Since typically $\varphi_m \sim 2T_e$, this is reasonably accurate except in the immediate neighbourhood of the mirrors. The ion distributions $U(\varepsilon_o)$ obtained numerically (e.g. in (5)) are smoothly varying functions, moderately peaked at (or somewhat below) the energy of injection, and for purposes of estimation only we take

$$f_+ = U(\varepsilon_o)\, M(u) = K_+\, \delta(\varepsilon_o - T_i) M_{R_e}(u) . \qquad (9)$$

3. THE DENSITY AND ELECTROSTATIC POTENTIAL DISTRIBUTIONS

The density distribution along a given flux tube in the mirror machine is determined by applying the condition of quasineutrality to the density distributions derived from the distribution function $f_{\pm}$ considered in the previous section. To obtain the distribution function at points not lying on the mid-plane we 'continue' them using the invariance of ε, μ and J. Thus from (3) and (6)

$$f_+ = U(\varepsilon)\ (1 + \ell n\ (\frac{\mu B_o}{\varepsilon})/\ \ell n\ R_e) \tag{10}$$

where $\varepsilon = \frac{1}{2}\ mv^2 + e\varphi$, $\mu = \frac{1}{2}mv_\perp^2/B$ and $R_e = (B_m/B_o)/(1 - e\varphi_m/\varepsilon)$, φ and B being the local values of the potential and magnetic field strength, φ_m and B_m being their values at the mirrors, and B_o the value of B at the mid plane (at which $\varphi = 0$). Thus the ion density at any point is a function of B,φ, φ_m and R_o:

$$n_i = \int f_+\ d^3 v = \int_{\frac{-e\varphi_m}{R_o - 1}}^{\infty} U_+(\varepsilon) \int_{\frac{\varepsilon - e\varphi_m}{B_m}}^{\frac{\varepsilon - e\varphi}{B}} \frac{(1 + \ell n(\frac{\mu B_o}{\varepsilon})/\ell n R_e)}{(\varepsilon\ -\ \mu B\ -\ e\varphi)^{\frac{1}{2}}}\ Bd\mu\ \ d\varepsilon\ . \tag{11}$$

[The limits of integration in (11) are derived by observing that $\varepsilon - e\varphi \equiv \frac{1}{2}mv^2 \geqslant \frac{1}{2}mv_\perp^2 = \mu B$ and that the angular distribution $M(\mu,\varepsilon)$ vanishes if $R_e \leqslant 1$ (i.e. if $\varepsilon \leqslant \frac{-e\varphi_m}{R_o - 1}$) and if $\frac{\varepsilon}{\mu B_o} > R_e$ (i.e. if $\mu < \frac{\varepsilon - e\varphi_m}{B_m}$)]. Integrating by parts, we obtain

$$n_i = \int_{\frac{-e\varphi_m}{R_o - 1}}^{\infty} d\varepsilon\ \frac{U_+(\varepsilon)}{\ell n\ R_e}\ \sqrt{\varepsilon\ - e\varphi} \left[\ \ell n \left\{ \frac{\sqrt{R} + \sqrt{R - (\varepsilon - e\varphi_m)/(\varepsilon - e\varphi)}}{\sqrt{R} - \sqrt{R - (\varepsilon - e\varphi_m)/(\varepsilon - e\varphi)}} \right\} \right. \tag{12}$$

$$\left. -\ \frac{2}{\sqrt{R}}\ \sqrt{R\ -\ (\ \varepsilon - e\varphi_m)/(\varepsilon\ - e\varphi)}\ \right]$$

where $R \equiv B_m/B$ is the local value of the actual mirror ratio. The electron density is obtained from (8):

$$n_e = n_o e^{e\varphi/T_e} \tag{13}$$

and the electrostatic potential distribution is determined as a function of B, R_o and φ_m by solving the integral equation $n_i = n_e$ using the expressions (12) and (13).

The qualitative form of the profile of φ obtained in this manner is a curve which is almost flat in the central region, rising steeply to the value φ_m in the neighbourhood of the mirror. This can be shown by expanding (12) and (13) in the central region in powers of φ and $(R_o-R)/R_o$. As shown in Appendix 1, this gives an expression of the form

$$n_i = n_e = n_o \left[1 - K\left(\frac{R_o-R}{R_o}\right)/(1 - [K-\tfrac{1}{2}]\, T_e/T_i)\right] \qquad (14)$$

where K is a constant of order unity depending on φ_m and T_i, and

$$\frac{e\varphi}{T_e} = - K\left(\frac{R_o-R}{R_o}\right)/(1 - [K-\tfrac{1}{2}]\, T_e/T_i)\,. \qquad (15)$$

Since the ratio $T_e/T_i \ll 1$ under mirror reactor conditions, (14) and (15) show that the main fall off in plasma density occurs in a region where the influence of φ can be neglected, except insofar as the value of φ_m affects the effective mirror ratio R_e. Thus for present purposes it is adequate to neglect φ but retain φ_m in the expression (10) for the ion distribution function.

A further approximation, which is rather less precise but nevertheless adequate, is to replace the energy-dependent effective mirror ratio R_e in (10) by its energy average:

$$\bar{R}_e \equiv R_o/(1 - e\varphi_m/T_i) \qquad (16)$$

where T_i is a suitably chosen average energy. Since n_i only depends logarithmically on R_e and $U_+(\varepsilon)$ is in any case moderately peaked in energy at (or somewhat below) the injection energy[(5)], this approximation only introduces a rather small error, and it greatly simplifies the algebra.

4. THE PLASMA AND MAGNETIC PRESSURE PROFILES

The plasma pressure tensor is (in good approximation) diagonal, and as shown in[1], it is the perpendicular component which is of primary importance in a mirror machine in which the ratio of the length to the transverse dimension is reasonably large (as is the case in most plausible reactor designs). To calculate $p_\perp$ we proceed as in the previous section, introducing an additional μB into the μ integration, obtaining

$$p_\perp = \tfrac{1}{2}\int f_+ v_\perp^2 d^3 v = \int_{\frac{-e\varphi_m}{R_o - 1}}^{\infty} \frac{U_+(\varepsilon)(\varepsilon - e\varphi)^{\frac{3}{2}}}{2\ \ell n R_e} \left[\ell n \left\{ \frac{\sqrt{R} + \sqrt{R - (\varepsilon - e\varphi_m)/(\varepsilon - e\varphi)}}{\sqrt{R} - \sqrt{R - (\varepsilon - e\varphi_m)/(\varepsilon - e\varphi)}} \right\} \right.$$

$$\left. - \frac{2}{\sqrt{R}}\left\{R - (\varepsilon - e\varphi_m)/(\varepsilon - e\varphi)\right\} + \frac{1}{3R\sqrt{R}}\left\{R - (\varepsilon - e\varphi_m)/(\varepsilon - e\varphi)\right\}^{\frac{3}{2}} \right] d\varepsilon \, . \tag{17}$$

This shows that $p_\perp$ depends upon B through the "local effective mirror ratio" $R_\ell \equiv R(\varepsilon - e\varphi)/(\varepsilon - e\varphi_m)$. If, as before, we neglect the terms in $e\varphi$ and simplify the terms in $e\varphi_m$, replacing R_ℓ by its energy averaged value $\bar{R}_\ell = R/(1 - e\varphi_m/\bar{\varepsilon})$ and we define $\bar{R}_e = R_o/(1 - e\varphi_m/\bar{\varepsilon})$ we obtain

$$p_\perp(B) = p_{\perp o}\left[\ell n \left\{ \frac{1 + (1 - 1/\bar{R}_\ell)^{\frac{1}{2}}}{1 - (1 - 1/\bar{R}_\ell)^{\frac{1}{2}}} \right\} - 2(1 - 1/\bar{R}_\ell)^{\frac{1}{2}} + \tfrac{1}{3}(1 - 1/\bar{R}_\ell)^{\frac{3}{2}} \right] / \gamma \tag{18}$$

where

$$\gamma = \ell n \left\{ \frac{1 + (1 - 1/\bar{R}_e)^{\frac{1}{2}}}{1 - (1 - 1/\bar{R}_e)^{\frac{1}{2}}} \right\} - 2(1 - 1/\bar{R}_e)^{\frac{1}{2}} + \tfrac{1}{3}(1 - 1/\bar{R}_e)^{\frac{3}{2}} \, .$$

The density and pressure profiles as a function of B obtained in this approximation are shown in Figure 2 for a number of mirror ratios R_o.

Equation (18) gives the relationship between $p_\perp$ and B determined by the form of the diffusional equilibrium distribution function. However there is a second relationship between $p_\perp$ and B, resulting from the requirement that the plasma be in magnetohydrodynamic equilibrium with the magnetic field. As shown in[1], for long mirror machines this equilibrium condition is approximately

$$p_\perp + \frac{B^2}{8\pi} = \frac{B_v^2}{8\pi} \tag{19}$$

where $p_\perp$ and B are the values on the magnetic axis, and B_v is the strength of the vacuum magnetic field just outside the plasma. Eliminating $p_\perp$ between these two equations one obtains a unique relationship between the profiles of B and B_v, and we shall now show that above a certain critical value of $\beta \equiv \frac{8\pi p_{\perp o}}{B_{vo}^2}$, the profile of B_v corresponding to any acceptable B profile violates the plasma stability conditions.

It is implicit in the whole of the above analysis that B is a monotonically increasing function of z, the distance along field lines from the mid-plane, since otherwise the distribution function could not be obtained by 'continuing' the mid-plane diffusional distribution function. Now it is readily shown that $p_\perp$ as given by equation (17) is a monotonically decreasing function of B (and hence z), vanishing at $B = B_m$. Thus in the neighbourhood of B_m at least, $p_\perp + B^2/8\pi$ is an increasing function of B. In the neighbourhood of the mid-plane however $p_\perp + \frac{B^2}{8\pi}$ may be an increasing or decreasing function of B (and hence z), depending on the magnitude of β. For sufficiently small β it is obviously increasing, and hence (by (19)) B_v^2 increases monotonically with z - i.e. the vacuum field has a single minimum at the mid-plane. Conversely, for sufficiently large β, since $p_\perp$ decreases linearly with B near $B = B_o$, $p_\perp + \frac{B^2}{8\pi}$ decreases with B and hence B_v^2 must have two symmetrically placed minima away from the mid-plane. We show in Appendix 2 that such a field configuration is necessarily unstable, since stability requires that B_v should possess an absolute minimum at the mid-plane. It follows that there exists a β limitation set by the requirement that

$$\frac{d}{dB}\left(p_\perp + \frac{B^2}{8\pi}\right)\Bigg|_{B=B_o} > 0 \tag{20}$$

Performing the differentiation, we obtain

$$- \frac{1}{8\pi} \frac{\beta}{\gamma} \frac{B_o}{(1-\beta)} \left(1 + \frac{1}{2R_e}\right)\left(1 - \frac{1}{R_e}\right)^{\frac{1}{2}} + \frac{2B_o}{8\pi} > 0$$

i.e.

$$\beta < 1/\left\{1 + (1 - 1/\bar{R}_e)^{\frac{1}{2}}(1 + 1/2\bar{R}_e)/2\gamma\right\} . \tag{21}$$

For large $\bar{R}_e$ (say > 5) (21) can be written approximately as

$$\beta < 1/\left\{1 + 1/(2 \ \ell n \ 4 \ \bar{R}_e - 10/3)\right\} . \tag{22}$$

For example, for $\bar{R}_e = 5$, $\beta < 0.73$. For values of $\bar{R}_e$ in the range 1.5 - 5 equation (21) should be used: for smaller values, the above analysis is repeated with M given by (7) instead of (6), which gives $\beta < (R_e-1)/(7/4 \ R_e-1)$. The resulting β limitation is represented graphically in Fig.3.

5. THE OPTIMUM VACUUM FIELD PROFILE

Provided that β lies below the stability limit described in the preceding section, the profile $B_v(z)$ is arbitrary and can be chosen to optimise the reactor design. In this paper we regard its overall mirror ratio as a fixed parameter, and we are concerned only with the optimisation of its profile. As in our previous paper[1] we perform this optimisation by minimising the cost/power ratio $\int B^\nu \, ds / \int n^2 dv$, where ν is an index which determines the manner in which the cost of the windings of the magnetic field scales with field strength and $\int ds$ is taken over the winding surface (considered to coincide with the outer plasma flux surface) and $\int dv$ is taken over the volume occupied by plasma. Our present optimisation is more precise than our previous calculation in three interrelated respects - we allow the vacuum magnetic well depth to exceed the minimum required for stability by an arbitrary amount, if this permits a decrease in the cost/power ratio, we ensure that the magnetic field profile flattens out at the mirrors as it should, and we take into account the variation of the plasma density n along the machine (an unfavourable effect which did not appear in our previous calculation because of the different choice of distribution function).

As in our previous calculation, the equations determining the vacuum field profile b (normalised to unity at the mid-plane) are (cf eqns. (35) and (36) of[1])

$$b(c'^2 + (\frac{b'}{b}) - 2\frac{b''}{b}) = \delta \cosh c \tag{23}$$

$$2\, bc'' = - \delta \sinh c \tag{24}$$

where primes signify differentiation with respect to z (in dimensionless units) and δ is the radial well-depth, taken in our previous calculation to be $\kappa\beta$ (where κ is a constant of order unity) but here allowed to be any arbitrary function of z (or c) which is everywhere greater than β. It is convenient to rewrite these equations with c as the independent variable:

$$\frac{d\ b^{\frac{1}{2}}}{dc^2} - \tfrac{1}{4}b^{\frac{1}{2}} = - \frac{\delta}{4\ b^{\frac{1}{2}}\ c'^2}\left(\cosh c - \frac{\sinh c}{b}\ \frac{db}{dc}\right) \tag{25}$$

$$b\,\frac{dc'^2}{dc} = - \delta \sinh c\ . \tag{26}$$

In the limit $\delta \to 0$, (25) gives immediately the profile

$$b = \cosh^2 c/2 = \frac{1 + \cosh c}{2} = \frac{1 + \cosh\,(c'_0\ z)}{2} \tag{27}$$

obtained before. It will be observed that this profile increases indefinitely with z, so even in the limit $\beta \to 0$, it is necessary to exceed the minimum radial well depth for stability somewhere in order to produce a profile with finite mirror ratio. Where and how this should be done is part of the optimisation problem. However, a more serious part of the problem results from the fact that since the density n is a decreasing function of B through equation (15) and hence (through (18) and (19)) a decreasing function of b, it is no longer adequate simply to use the volume of the mirror machine as a measure of the thermonuclear power output. Rather, one should take the smaller quantity $\int n^2\, dv$, and the profile b(z) should attempt to maximise this as far as possible. If this were the only requirement, one would be led to a profile with a long uniform field section in the middle and sharply rising ends, a profile which we shall see can indeed by obtained from eqns. (25) and (26) with a suitable choice of $\delta(c)$. However we shall also see that this leads to an exponential increase in the cost of the magnetic field, essentially because of the greatly increased area of the magnetic fishtails at the two ends of the machine which are needed to maintain the minimum-B property. Thus in the optimum design only a rather short uniform-field section can be included, and an approximate analytic estimate of its length can be obtained.

To substantiate these qualitative remarks, our procedure is in two parts: we first consider an analytic model in which the machine consists of a flat central section of half length ℓ, enclosed by two mirror sections, each of length a, and we seek solutions of (25) and (26) for this case which have continuous values of b, b', c and c' at the joints. In the central section we select $\delta(c) = \delta_o \operatorname{sech}^2 c$ so that b is constant: in the mirror sections we take $\delta = 0$, a choice which makes the profile rise as sharply as possible (it will be confirmed that the right hand side of (25) is negative, at least for solutions similar to the $\delta = 0$ solution) and has the further merit of being analytically soluble. Since the mirror fields do not reach a maximum in this case, we simply cut them off when b has reached the required vacuum mirror ratio R. We optimise this model profile with respect to the ratio ℓ/a. We then describe some numerical solutions of (25) and (26) based on continuous functions $\delta(c)$ and show that it is possible to obtain acceptable profiles which are close to the model profile in form. We have attempted, with a range of trial functions $\delta(c)$, to improve upon the profiles approximated by the model profile and (as one would expect from the manner in which it was constructed) have been unable to do so. Therefore believe that the model profile is close to the optimum profile for a minimum-B reactor.

The calculation of the model profile proceeds as follows. We first obtain the profile of $\delta(c)$ for which $b = 1$, $b' = 0$. Inserting these in (25) and (26), one obtains two equations for c' and δ, which on eliminating δ give $\frac{d\,\ell n\, c'^2}{dc} = -\tanh c = -\frac{d}{dc}\ell n\,(\cosh c)$ and hence

$$c'^2 = \delta_0 \operatorname{sech} c \tag{28}$$

where δ_0 is a constant. Hence by (25) $\delta = \delta_0 \operatorname{sech}^2 c$ and (from (25)), c_ℓ the value of c at the end of the central section, is related to its half length ℓ by

$$\delta_0^{\frac{1}{2}}\, \ell \int_0^{c_\ell} \cosh^{\frac{1}{2}} c\, dc. \tag{29}$$

In the mirror section $\delta = 0$, and the solution which satisfies the required continuity conditions at the joint is

$$b = \frac{1 + \cosh\,(c - c_\ell)}{2} \tag{30}$$

$$c' = c'_\ell = \text{constant}\ . \tag{31}$$

The length of the mirror section a is given by

$$a = \int_\ell^{\ell+a} dz = \int_{c_\ell}^{c_m} \frac{dc}{c'} = \frac{c_m - c_\ell}{c'_\ell} \tag{32}$$

and from (30) $b_m = R_v = \frac{1}{2}\,(1 + \cosh\,(c_m - c_\ell))$ so

$$c_m - c_\ell = \cosh^{-1}\,(2R_v - 1)\ . \tag{33}$$

Combining (28) and (29) we have

$$\ell\, c'_\ell = \operatorname{sech}^{\frac{1}{2}} c_\ell \int_0^{c_\ell} \cosh^{\frac{1}{2}} c\, dc \tag{34}$$

and hence

$$\frac{a}{\ell} = \frac{\cosh^{-1}(2R_v - 1)}{\operatorname{sech}^{\frac{1}{2}} c_\ell \int_0^{c_\ell} \cosh^{\frac{1}{2}} c\, dc}\ . \tag{35}$$

This equation shows that even regardless of expense, there is a limit to the length of the flat section (obtained by taking $c_\ell \to \infty$)

$$\frac{a}{\ell} > \frac{\cosh^{-1}(2R_v - 1)}{2} \quad . \tag{36}$$

We now derive the cost/power ratio M for the model profile; proceeding as in our previous paper (i.e. assuming that R_v is large enough that $e^{c_m} \gg 1$):

$$M \simeq \int_0^{\ell+a} b^{\nu-\frac{1}{2}} e^{c/2} \, dz \Big/ \int_0^{\ell+a} n^2(b)/n^2(o) \, dz/b$$

$$= \frac{1}{\delta_0^{\frac{1}{2}}} \left[\int_0^{c_\ell} e^{c/2} \cosh^{\frac{1}{2}} c \, dc + \cosh^{\frac{1}{2}} c_\ell \int_{c_\ell}^{c_m} \left(\frac{1 + \cosh(c - c_\ell)}{2}\right)^{\nu-\frac{1}{2}} e^{c/2} dc \right] / (\ell + \alpha a)$$

where α

$$\alpha = \frac{1}{a} \int_\ell^{\ell+a} \left(\frac{n}{n_o}\right)^2 \frac{dz}{b} \quad .$$

Neglecting terms of order e^{c_ℓ} compared with terms in e^{c_m} we have

$$M \simeq \frac{\cosh^{\frac{1}{2}} c_\ell}{\delta_0^{\frac{1}{2}} (\ell + \alpha a)} \frac{e^{\nu c_m}}{4^{\nu-\frac{1}{2}} \nu} = \frac{2(R_v - \frac{1}{2})^\nu e^{\nu c_\ell}}{\nu(\ell + \alpha a) \, c_\ell'}$$

$$= \frac{2(R_v - \frac{1}{2})^\nu e^{\nu c_\ell}}{\nu[\operatorname{sech}^{\frac{1}{2}} c_\ell \int_0^{c_\ell} \cosh^{\frac{1}{2}} c \, dc + \alpha \cosh^{-1}(2R_v - 1)]} \quad . \tag{37}$$

To minimise this with respect to c_ℓ we note that for large c_ℓ it increases exponentially and hence the optimum must be at some small value of c_ℓ, for which $\operatorname{sech}^{\frac{1}{2}} c_\ell \int_0^{c_\ell} \cosh^{\frac{1}{2}} c \, dc \simeq c_\ell$. Thus near the optimum

$$M = \frac{2(R_v - \frac{1}{2})^\nu e^x}{x + A}$$

when $x = \nu c_\ell$ and $A = \nu \alpha \cosh^{-1}(2R_v - 1)$, an expression which possesses a minimum (for positive x) at $x = 1 - A$ if $A < 1$ and at $x = 0$ otherwise. Thus the minimum value of M is

$$\begin{aligned} M_{min} &= 2(R_v - \tfrac{1}{2})^\nu e^{(1-A)} && A < 1 \\ &= 2(R_v - \tfrac{1}{2})^\nu / A && A > 1 \; . \end{aligned} \tag{38}$$

It will be seen that the optimum value of

$$\frac{\ell}{a} = \frac{1 - A}{\nu \cosh^{-1}(2R_v - 1)} \qquad A < 1$$

$$= 0 \qquad A > 1$$

is always small, and a flat section is only advantageous at all if A<1.

The quantity A can be evaluated in terms of

$$F \equiv \overline{\left(\frac{n}{n_o}\right)^2} = \int_{\ell}^{\ell+a} \left(\frac{n}{n_o}\right)^2 \frac{dz}{b} \Big/ \int_{\ell}^{\ell+a} \frac{dz}{b} \tag{39}$$

(which is a measure of the effectiveness with which the plasma is packed into the mirror sections):

$$A = \frac{F\,\nu \cosh^{-1}(2R_v - 1)}{a} \int_{\ell}^{\ell+a} \frac{dz}{b} \simeq 2\nu F\ . \tag{40}$$

Since certainly $\nu \geqslant 2$, it follows that A< 1 only if $F < \frac{1}{4}$ i.e. if the density profile fall-off leads to a very unfavourable total reaction rate. Numerical values of the filling parameter F (and hence of A) are obtained from the numerical profiles considered below, and it is shown that $F \simeq \frac{1}{4}$, so the optimum profile has essentially no flat section.

The above analytical model has two defects: the magnetic field does not satisfy the radial minimum-B stability criterion at the inner ends of the mirror sections, and its profile does not turn over at the mirrors, as it should. We therefore solved equation (23) and (24) numerically, using a number of different analytic expressions for δ, in order to obtain fully acceptable profiles. By varying the form of δ we sought to obtain profiles with better values of the filling parameter F. The most satisfactory profiles were obtained with

$$\delta = \frac{\delta_o}{b^2}\left[1 + G(1 + \tanh H\,(z - z_m))\right] \tag{41}$$

where δ_o, z_m, G and H are constants chosen so that the field profile is acceptable and F is as large as possible. With large values of G and H, this leads to a profile which follows the model profile closely until b approaches the required mirror ratio R_V, and then turns over rapidly. The profiles are rather insensitive to the value of δ_o. In Figure 4 we show the profiles of $n(z)$, $p_\perp(z)$ and $B_V(z)$ for R=3, 5 and 10, with the value of δ_o taken as β_{max}, the maximum value of β permitted by eqn.(22): however very similar profiles are obtained for any value of δ_o in the range 0-1. In these calculations we have for simplicity neglected the effect of the electrostatic potential - i.e. we have set $R_e = R$.

6. CONCLUSIONS

The best available estimates of the collisional distribution function in a mirror machine indicate that the distributions of plasma density and pressure along the magnetic field lines are significantly non-uniform. This has two consequences for the design of a mirror reactor. First, there exists a maximum value of the plasma pressure at the centre of the machine, above which it is impossible to ensure both that the plasma is in magneto-hydrodynamic equilibrium in the radial direction and that the vacuum magnetic field possesses an absolute minimum at the centre, a condition which we show to be necessary for stability. This leads to a β limitation for mirror reactors, given by (21). Fortunately this limitation is comparatively unrestrictive, particularly in comparison with the β limitations which arise in toroidal systems.

Secondly, the thermonuclear power density falls off rather rapidly away from the centre of the machine. This does not directly affect the Q value of the reactor (the ratio of the thermonuclear to injection power) since it is already implicitly taken into account in such calculations. However it does adversely affect the capital cost of the reactor, and it is important to design the magnetic field profile in such a way that this adverse effect is minimised. It might appear at first sight that the appropriate profile would have a long flat central section with constant magnetic field (and hence maximum thermonuclear power output). However it emerges from a detailed cost analysis that such a flat section necessarily entails a disproportionate expenditure upon the mirror sections if the minimum B property is to be preserved. In the absence of a flat section, one can only attempt to make the increase in the magnetic field strength as slow as possible. However the constraint imposed by the minimum B requirement, which is embodied in the differential equations which determine the profile [(23), (24)], gives little freedom for manoeuvre in this direction. The optimum vacuum field profile is insensitive to both the plasma pressure and the mirror ratio, being well approximated by the 'cosh' profile (27) except in the immediate neighbourhood of the throat of the mirror. The ratio F of the thermonuclear power produced in such a mirror to the power produced in a machine with the same dimensions but with a square-well magnetic field profile is approximately 1/4.

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Appendix 1

To determine the self-consistant electrostatic potential from (12) and (13), we represent the peaked distribution U_+ by

$$U_+ \sim \delta(\varepsilon - T_i)$$

(as discussed in Section 2), obtaining

$$n_i = n_o(1 - e\varphi/T_i)^{\frac{1}{2}} \left[\ell n \left(\frac{1+\alpha}{1-\alpha} \right) - 2\alpha \right] / \gamma_o \tag{A1}$$

where $\alpha^2(R,\varphi) \equiv 1 - \dfrac{T_i - e\varphi_m}{R(T_i - e\varphi)}$ and $\gamma_o \equiv \ell n \left(\dfrac{1 + \alpha(R_o,0)}{1 - \alpha(R_o,0)} \right) - 2\alpha(R_o,0)\cdot$

Thus if we expand (13) and (A1) about the mid-plane value in powers of φ and $R - R_o$:

$$n_i = n_o\left(1 - \frac{1}{2}\frac{e\varphi}{T_i} - \frac{\alpha_o}{\gamma_o}\left(\frac{e\varphi}{T_i} - \frac{R - R_o}{R_o}\right) \ldots\ldots \right) \tag{A2}$$

$$n_e = n_o\left(1 + \frac{e\varphi}{T_e} \quad \ldots\ldots\right) \tag{A3}$$

and apply the quasineutrality condition, we obtain

$$e\varphi\left(\frac{1}{T_e} + \frac{1}{2T_i} - \frac{\alpha}{\gamma_o T_i}\right) = \frac{R - R_o}{R_o}\frac{\alpha_o}{\gamma_o} \tag{A4}$$

and hence

$$n_i = n_e = n_o\left(1 + \frac{\alpha_o}{\gamma_o}\left(\frac{R - R_o}{R_o}\right) / \left(1 + \frac{T_e}{2T_i} - \frac{\alpha_o T_e}{\gamma_o Ti}\right)\right) \cdot \tag{A5}$$

Since for reasonable reactor parameters $\alpha_o/\gamma_o \sim 1$ and $T_e \ll T_i$, it is seen that the correction to the density profile near the centre of the machine due to the electrostatic field variation is of order T_e/T_i, so that the main density fall-off is due to the magnetic field variation, and is affected by the electrostatic field solely through the dependence of the distribution function on R_e (and hence on φ_m).

Appendix 2

Here we show that the stability condition against the "Mirror instability" is equivalent to having a vacuum mirror field which is monotonically increasing from the centre.

The first of the two necessary conditions for stability obtained in Ref.(1) (eqn.c3) is

$$1 + \frac{2p_\perp + c}{B^2} > 0 . \qquad \text{(A6)}$$

Now as is shown by Hastie and Taylor[9] (eqn. 3.8), for a mirror confined plasma

$$\frac{\partial p_\perp}{\partial s} = (c + 2\, p_\perp) \frac{1}{B} \frac{\partial B}{\partial s} .$$

The above expression may be rewritten in the form

$$\frac{\partial}{\partial s} (p_\perp + \tfrac{1}{2} B^2) \frac{\partial B}{\partial s} = (1 + \frac{2p_\perp + c}{B^2}) B \left(\frac{\partial B}{\partial s} \right)^2 . \qquad \text{(A7)}$$

From (A6) the r.h.s. of (A7) must be greater than zero for stability. Since for a mirror equilibrium $\partial B/\partial s > 0$ it follows that

$$\frac{\partial}{\partial s} (p_\perp + \tfrac{1}{2} B) > 0 .$$

Using equation (19) this reduces to

$$\frac{\partial}{\partial s} (B_v^2) > 0.$$

i.e. the vacuum mirror field must increase monotonically away from the centre.

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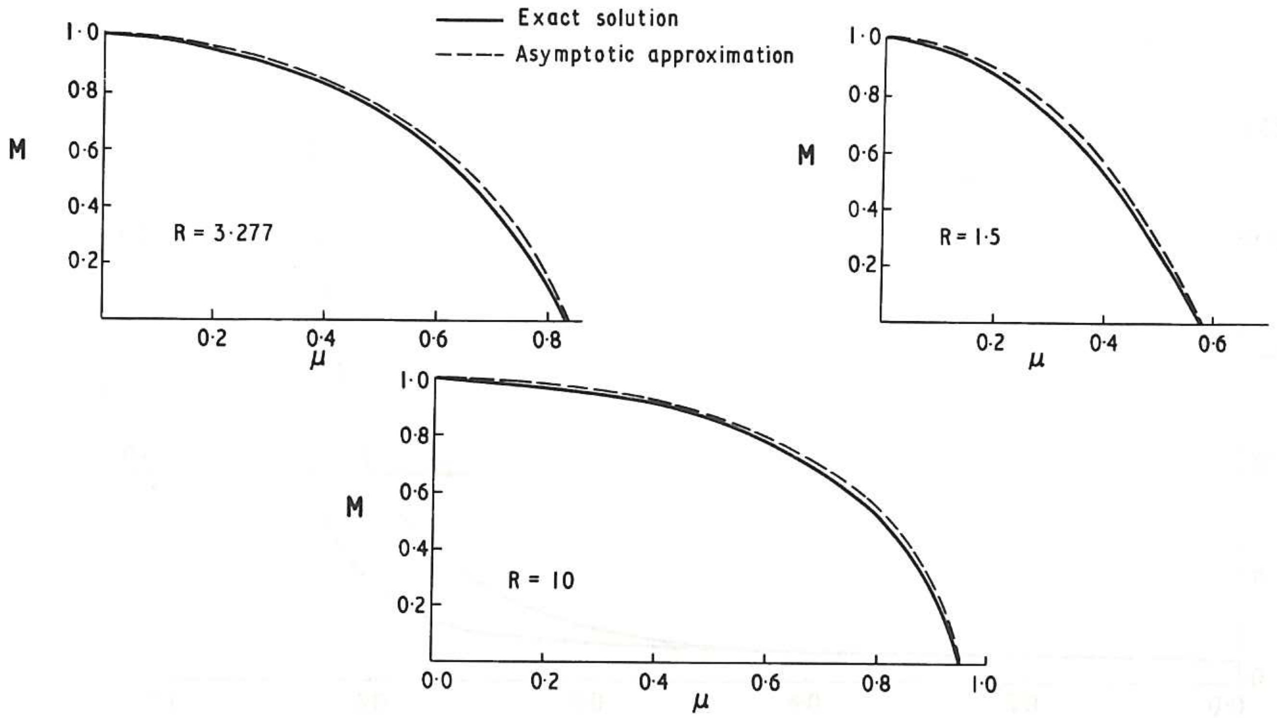

Fig. 1 M against μ for the exact solution and the asymptotic approximation for three mirror ratios R = 1.5, 3.277, 10.

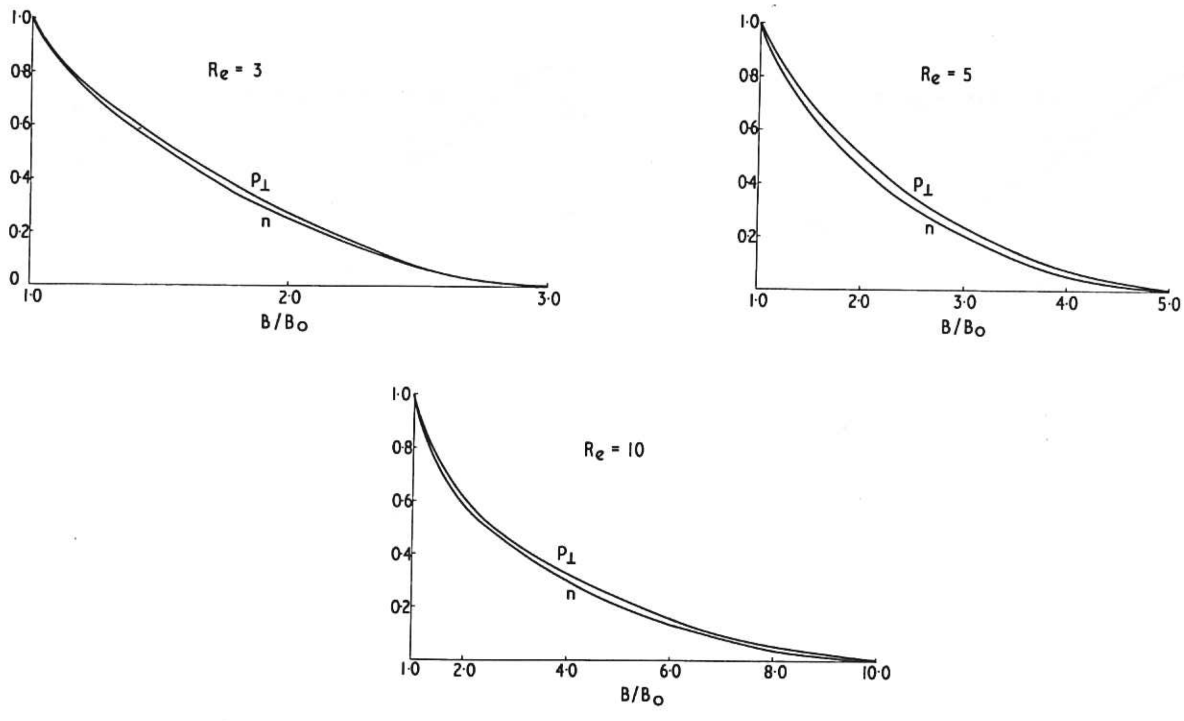

Fig. 2 The density n and perpendicular pressure $p_\perp$ against B/B_o for real mirror ratios 3, 5, 10.

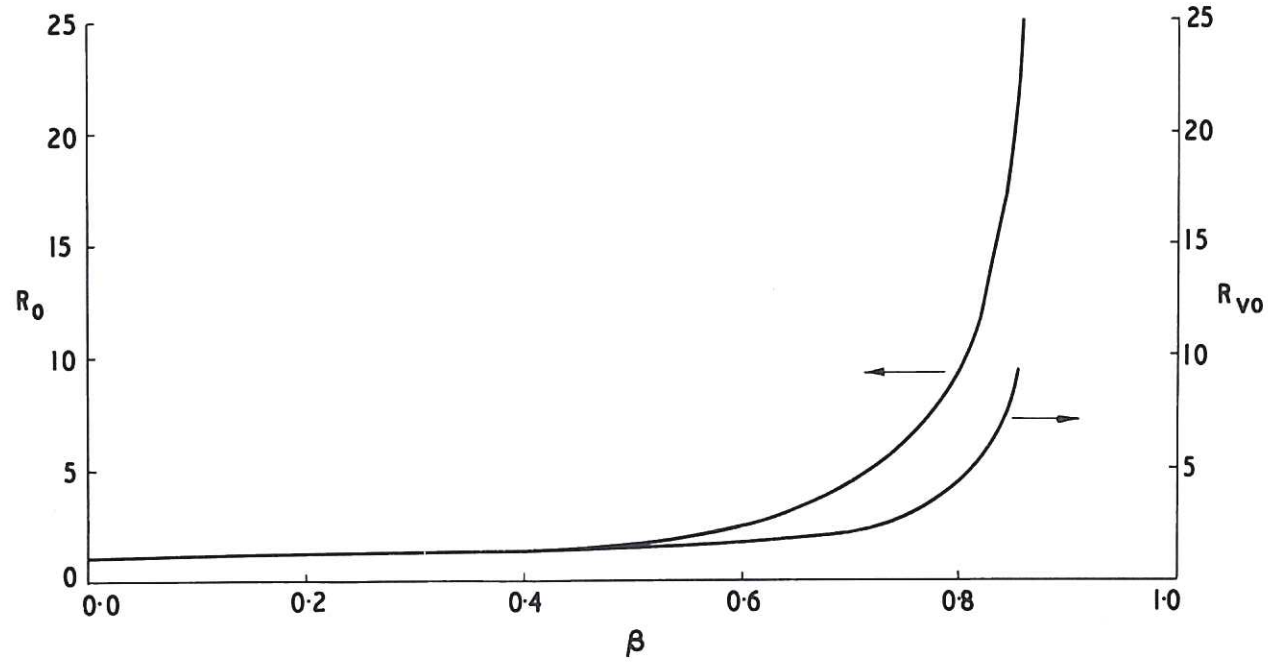

Fig. 3 Real effective mirror ratio R_e and vacuum mirror ratio R_{vo} against the critical β for stability. In relating R_{vo} to R_e we have assumed that the ambipolar potential is zero.

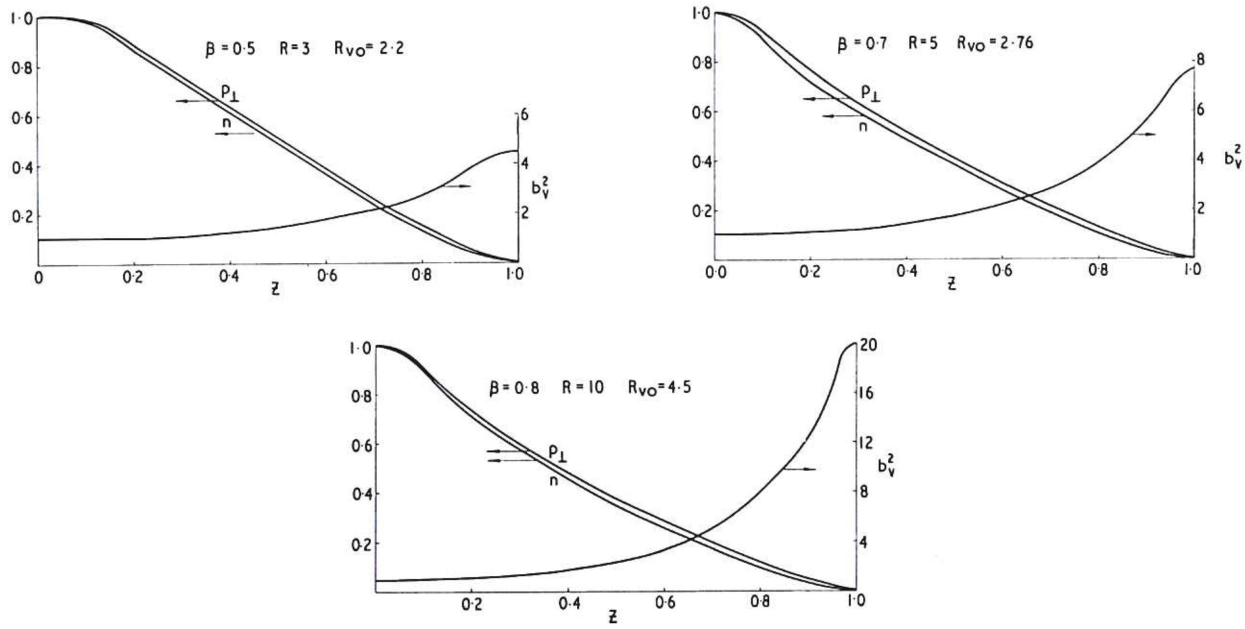

Fig. 4 The density n and perpendicular pressure $p_{\perp}$ against z the distance from the centre of the machine for the optmised minimum-B profiles. The profile of the vacuum magnetic field is also shown.