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HIGH FREQUENCY HEATING OF ELECTRONS IN A MIRROR MACHINE

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HIGH FREQUENCY HEATING OF ELECTRONS IN A MIRROR MACHINE

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ABSTRACT

The motion of electrons contained in a mirror field with superimposed R.F. field is analysed using a perturbation treatment.

The unperturbed motion (with no R.F. field present) is expressed as a multiperiodic system whose Hamiltonian is calculated as a function of the three adiabatic invariants. The R.F. field is considered as a perturbation. It is shown that the R.F. field (with electric field perpendicular to the mirror field) affects only the perpendicular invariant (which is proportional to the magnetic moment). Appreciable changes of the perpendicular invariant occur only to electrons having a certain amplitude of longitudinal oscillation determined by the applied frequency. When the electrons have such a small amplitude of longitudinal oscillation that they stay in a practically uniform field, the R.F. field causes a rapid increase of the magnetic moment. In all the other cases, however, the effect of the R.F. field is only a small periodic fluctuation of the magnetic moment.

The mechanism of R.F. heating in a mirror field is discussed. It is shown that the heating is a slow statistical process which is made possible by electron collisions or some equivalent randomising process.

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Introduction

It is well known that electrons in a uniform magnetic field can rapidly gain energy from a high-frequency electric field having a frequency equal to the electron cyclotron frequency and a direction perpendicular to the magnetic field lines. Apparently this principle has been successfully used in the Oak Ridge experiment[1], to heat electrons contained in a mirror field. However, the heating in a mirror field is obviously different from that in a uniform field because no simple cyclotron resonance exists. As the electron moves along a flux tube of the mirror field its cyclotron frequency changes with the position of its guiding centre. The electron cyclotron frequency is approximately equal to the applied frequency only when the electron happens to be in a particular region of the mirror field. On the other hand, any change in the perpendicular energy of the electron, caused by the high-frequency field, changes the electron motion along the field lines.

The purpose of this paper is to investigate the mechanism of high-frequency heating of electrons in a mirror field. The difficult problem of finding a self consistent field solution in the plasma is entirely omitted and a given high-frequency field is assumed. Another basic (although not necessary) simplification is a non-relativistic treatment.

The perturbation theory of classical mechanics will be used for studying the electron motion in a mirror field with superimposed R.F. field. The motion in a static mirror field is considered as the unperturbed motion and the R.F. field as a perturbation. To make full use of the perturbation theory the unperturbed motion must be represented as a multiperiodic system. This will be done in the first section which is an extension of the work done by J. Lacina[2]. In the second section the perturbed motion will be studied. In the last section the heating mechanism is discussed and it is shown that the heating results from the combined action of the R.F. field and electron collisions.

1. Multiperiodic Representation of Particle Motion in a Mirror Field

If the static and rotationally symmetrical magnetic field satisfies the adiabatic condition, three single-valued approximate integrals of motion exist and the contained particle performs approximately a multiperiodic motion with three degrees of freedom (Larmor rotation, drift rotation around the mirror axis and oscillation along the field lines). In principle, it is always possible to express this fact explicitly by introducing angle variables w_i and action variables P_i $(i = 1,2,3)$ corresponding to the three degrees of freedom. Such co-ordinates will now be introduced assuming that the field is not very different from a uniform field.

The magnetic field B will be defined along the axis of the system by the expression

$$\frac{B}{B_o} = 1 + a\,\eta^2 \; ; \quad 0 < a < 1 \qquad \ldots (1.1)$$

where B_o is the field in the centre of the mirror and η is the distance along the axis in units of an arbitrary length L. Particles which stay at all times in the interval $|\eta| \leqslant 1$ will be considered contained. The distance $2L$ is thus the distance between the 'mirror points' and $a = R - 1$ where $R = B_{max}/B_o$ is the mirror ratio.

Following reference (2), the position of a particle is defined in a curvilinear orthogonal co-ordinate system ξ,θ,η, Fig. 1. The surfaces ξ = constant are formed by the field lines, the surfaces θ = constant are planes containing the axis and the surfaces of constant (scalar) magnetic potential are η = constant. All lengths are expressed in units of L. The canonical momenta corresponding to these co-ordinates are denoted by p_ξ, p_θ, p_η.

The non-relativistic Hamiltonian function of a particle of charge e, mass m in this field is:

$$H = \frac{1}{2}\,\omega_o \left[\frac{p_\xi^2}{h_\xi^2} + \left(\frac{p_\theta - \frac{1}{2}\xi^2}{r'} \right)^2 + \frac{p_\eta^2}{h_\eta^2} \right] \qquad \ldots (1.2)$$

where $\omega_0 = eB_0/m$,

$$r' = \frac{r}{L} = \xi\left[1 - \frac{a}{2}\eta^2 + \frac{3}{8}a^2\eta^4 - \frac{1}{4}a^2\eta^2\xi^2\right]$$

$$\frac{1}{h_\xi^2} \approx 1 + a\eta^2(1 + 2a\xi^2) \qquad \dots (1.3)$$

$$\frac{1}{h_\eta^2} \approx 1 + 3a^2\eta^2\xi^2 .$$

In the last two expressions terms containing higher powers of a than the second have been neglected.

It is shown in the Appendix that the following canonical transformation replaces approximately the old co-ordinates and momenta by angle and action variables w_i, P_i :

$$\xi = \sqrt{2P_2} + \sqrt{2P_1}\cos Q_1 , \qquad p_\xi = -\sqrt{2P_1}\sin Q_1$$

$$\theta = w_2 - \sqrt{\frac{P_1}{P_2}}\sin Q_1 \qquad p_\theta = P_2 \qquad \dots (1.4)$$

$$\eta = \eta_m \sin w_3 \qquad p_\eta = \sqrt{2aP_1}\,\eta_m \cos w_3$$

where

$$Q_1 = w_1 - \frac{1}{4}\frac{P_3}{P_1}\sin 2w_3$$

$$\eta_m^2 = \sqrt{\frac{2}{a}}\,\frac{P_3}{\sqrt{P_1}}$$

This transformation is valid if the Larmor radius is much smaller than the guiding centre radius.

The transformed Hamiltonian can be written in the form

$$H = H_0 + H_1 \qquad \dots (1.5)$$

where

$$H_0 = \omega_0 P_1\left[\Upsilon_0(P_2) + \Upsilon_1(P_2)\frac{P_3}{\sqrt{P_1}} + \Upsilon_2(P_2)\left(\frac{P_3}{\sqrt{P_1}}\right)^2 + \dots\right] \qquad \dots (1.6)$$

with

$$\Upsilon_0(P_2) \approx 1 - aP_2 - \frac{1}{2}a^2P_2^2$$

$$\Upsilon_1(P_2) \approx \sqrt{2a}\,(1 + aP_2 - 3a^2P_2^2 + \dots) \qquad \dots (1.7)$$

$$\Upsilon_2(P_2) \approx 10a^2P_2(1 - 4aP_2 + \frac{9}{2}a^2P_2^2 + \dots)$$

The term H_1 in (1.5) does not depend on w_2, but is a periodic function of both w_1 and w_3 and goes to zero as $a \to 0$ and $\sqrt{P_1/P_2} \to 0$. Under the assumptions made, this term may be regarded as a small perturbation of the main part H_o. According to the perturbation theory such a perturbation can be neglected if the main part is not degenerate, i.e. if the angular frequencies $\nu_i = \partial H_o / \partial P_i$ are linearly independent.

With this simplification the equations of motion derived from (1.5) have the simple form

$$P_i = \text{constant}, \quad w_i = \nu_i t + \text{constant} \quad (i = 1,2,3) \qquad \ldots (1.8)$$

where $\nu_i = \dfrac{\partial H_o}{\partial P_i}$ are the angular frequencies corresponding to the three degrees of freedom. They are functions of the three adiabatic invariants P_i :

$$\nu_1 \approx \omega_o \left[1 - aP_2 + \frac{1}{2} a(1 + aP_2)\eta_m{}^2\right] \quad \text{(cyclotron rotation)}$$

$$\nu_2 \approx - \omega_o aP_1\left[1 - a\eta_m^2 - 5a^3\eta_m^4\right] \quad \text{(azimuthal drift)} \qquad \ldots (1.9)$$

$$\nu_3 \approx \omega_o \sqrt{2aP_1}\left[1 + aP_2 + 10a^2P_2\eta_m^2\right] \quad \text{(longitudinal oscillation)}$$

The physical meaning of the invariants P_i can be seen from equations (1.4) and Fig.2. $\sqrt{2P_1}$ is equal to the particle Larmor radius in the ξ,η co-ordinate system. As in the median plane $\xi = \frac{r}{L}$, where r is the distance from the axis, we have

$$P_1 = \frac{1}{2}\left(\frac{r_L}{L}\right)_o^2 = \left(\frac{W_\perp}{m\omega^2L^2}\right)_o = \frac{\mu}{\omega_o eL^2} \qquad \ldots (1.10)$$

the subscript o denoting that the value of the Larmor radius r_L and of the transverse energy $W_\perp$ are to be taken on the median plane ($\eta = 0$). Thus, P_1 is proportional to the magnetic moment $\mu = W_\perp/B$.

The invariant P_2 is related to the guiding centre radius r_g in the midplane by

$$P_2 = \frac{1}{2}\left(\frac{r_g}{L}\right)_o^2 \qquad \ldots (1.11)$$

The amplitude η_m (in units of L) of the oscillations along the field lines is determined by the two invariants P_1, P_3:

$$\eta_m^2 \approx \sqrt{\frac{2}{a}}\;\frac{P_3}{\sqrt{P_1}} \qquad \ldots (1.12)$$

By definition, the contained particles satisfy the relation $|\eta_m| \leqslant 1$. This means that the actions P_1, [illegible] corresponding to the contained particles must lie in the shaded area on Fig.3. To ensure the convergence of the expression (1.7) P_2 must be limited by

$$aP_2 \ll 1 \qquad \text{...(1.13)}$$

The total energy of the particle is

$$W = p_o L\, H_o = (W_\perp)_o + (W_\parallel)_o \qquad \text{...(1.14)}$$

where the perpendicular energy in the midplane is

$$(W_\perp)_o = m\, \omega_o^2 L^2 \Upsilon_o(P_2) \cdot P_1 \qquad \text{...(1.15)}$$

and the parallel energy in the midplane is

$$(W_\parallel)_o \approx m\, \omega_o^2 L^2 (\Upsilon_1(P_2) P_3 \sqrt{P_1} + \Upsilon_2(P_2) P_3^2) \qquad \text{...(1.16)}$$

It may be noted that the angular velocity of the Larmor rotation is

$$\frac{dQ_1}{dt} \approx \nu_1 - \frac{\nu_3}{2} \frac{P_3}{P_1} \cos 2w_3 \approx \omega_o\left[1 + \frac{1}{2} a\eta_m^2 (1 - \cos 2\nu_3 t)\right] \qquad \text{...(1.17)}$$

where the last expression is valid near the axis ($P_2 \to 0$).

$\frac{dQ_1}{dt}$ fluctuates between the minimum value $\dot{Q}_1 = \omega_o$ and the maximum value $\omega_o(1 + a\eta_m^2) = \omega_o \frac{B_m}{B_o}$ where B_m is the field at $\eta = \eta_m$ on the axis. The mean value of dQ_1/dt is equal to ν_1.

2. The Effect of the R.F. Field on Particle Motion

Let us assume that the R.F. field is a TE_o mode. The electric field has only an E_θ component which can be written in the form

$$E_\theta = + E_o(r) \cos kz \cos \omega t \qquad \text{...(2.1)}$$

assuming a cylindrical co-ordinate system r, θ, z. This field can be derived from a vector potential

$$A_\theta = - \frac{E_o(r)}{\omega} \cos kz \sin \omega t \qquad \text{... (2.2)}$$

If the mirror ratio is small the field lines of the mirror field are nearly parallel to the axis and the z-coordinate can be replaced by ηL. Moreover, the dependence of $E_o(r)$ on radius along a field line can be neglected.

The presence of the R.F. field changes the second term in the Hamiltonian (1.2) into

$$\frac{1}{2}\omega_o \left(\frac{P_\theta - \frac{1}{2}\xi^2 - \frac{r' A_\theta}{B_o L}}{r'} \right)^2$$

so that the Hamiltonian is now

$$H' = H + \Delta H \qquad \text{... (2.3)}$$

with H given by (1.2) and

$$\Delta H = -\omega_o \frac{p_\theta - \frac{1}{2}\xi^2}{r'} \frac{A_\theta}{B_o L} + \frac{1}{2}\omega_o \left(\frac{A_\theta}{B_o L}\right)^2 \quad \ldots (2.4)$$

In terms of angle and action variables (1.4), assuming again $\sqrt{P_1/P_2} \ll 1$ and using (1.2), this can be written as

$$H' = \omega_o \left[\frac{H_o}{\omega_o} - \varepsilon H_2\right] \quad \ldots (2.5)$$

where H_o is the unperturbed Hamiltonian (1.6),

$$\varepsilon = \sqrt{2}\,\frac{E_o}{\omega B_o L} \quad \ldots (2.6)$$

$$H_2 \approx \sqrt{P_1}\,\cos(w_1 - \frac{1}{4}\frac{P_3}{P_1}\sin 2w_3)\cos(kL\eta_m \sin 2w_3)\sin \omega t \quad \ldots (2.7)$$

If the amplitude of the R.F. field satisfies the condition

$$\varepsilon \ll \sqrt{P_1} \quad \ldots (2.8)$$

for all P_1 of interest, εH_2 is a small perturbation of the main term H_o/ω_o in (2.5). With this assumption the particle motion will now be approximately calculated using the perturbation theory of mechanics. The case of 'cylotron resonance' will be considered, i.e. it will be assumed that the angular frequency ω of the R.F. field is near to the mean value $\nu_1(P_i)$ of the angular velocity of Larmor rotation. As ν_1 is a function of the P_i's, the relation $\omega = \nu_1$ can be exactly satisfied only with definite values of the P_i's. It will be approximately satisfied, however, in a certain range of the P_i's. In this range the function $w_1' = w_1 - \omega t$ will be a slowly varying function of time.

Let us introduce new canonical co-ordinates w_i', P_i' by a transformation derived from the generating function

$$S = (w_1 - \omega t)P_1' + w_2 P_2' + w_3 P_3' \; .$$

The transformation is

$$w_1' = w_1 - \omega t, \quad w_2' = w_2, \quad w_3' = w_3, \quad P_i' = P_i \quad \ldots (2.9)$$

Omitting the dashes from the unchanged variables the new Hamiltonian is

$$H'' = H' - \frac{\partial S}{\partial t} = \omega_o \left[\frac{H_o}{\omega_o} - \frac{\omega}{\omega_o} P_1 - \varepsilon H_2\right] \quad \ldots (2.10)$$

The perturbation H_2 can be expressed as a Fourier series in w_3 :

$$H_2 = \sqrt{P_1}\left[e^{jw_1'}\sum_{\ell=-\infty}^{+\infty} h_{1\ell}\, e^{j\ell w_3} - e^{-jw_1'}\sum_{\ell=-\infty}^{+\infty} h_{2\ell}\, e^{j\ell w_3}\right. \qquad \ldots(2.11)$$

$$\left. + e^{j(w_1' + 2\omega t)}\sum_{\ell=-\infty}^{+\infty} g_{1\ell}\, e^{j\ell w_3} + e^{j(w_1' + 2\omega t)}\sum_{\ell=-\infty}^{+\infty} g_{2\ell}\, e^{j\ell w_3}\right]$$

According to the perturbation theory, the equations of motion are

$$\frac{\overline{dw_i'}}{dt} = \omega_0\left[\frac{1}{\omega_0}\frac{\partial H_0}{\partial P_i} - \varepsilon\frac{\partial \overline{H}_2}{\partial P_i} + O(\varepsilon^2)\right]$$

$$\frac{\overline{dP_i}}{dt} = \omega_0\left[\varepsilon\frac{\partial \overline{H}_2}{\partial w_i'} + O(\varepsilon^2)\right] \qquad \ldots(2.12)$$

where the velocities $\frac{\overline{dw_i'}}{dt}$, $\frac{\overline{dP_i}}{dt}$ are time averages over a time longer than the period T of the unperturbed longitudinal oscillation and $\overline{H}_2$ is averaged over the co-ordinate w_3 and time. It means that, to first order in ε, the perturbation H_2 can be replaced by its average:

$$\overline{H}_2 = -\frac{1}{2}\sqrt{P_1}\, f(P_1,P_3)\sin w_1' , \quad f(P_1,P_3) = \sum_{\ell=-\infty}^{+\infty} J_{2\ell}\left(\frac{1}{4}\frac{P_3}{P_1}\right)J_{4\ell}(kL\eta_m) \qquad \ldots(2.13)$$

J_k being Bessel functions of the first kind.

Thus, to first order in ε, the average values of the two invariants P_2, P_3 remain constant even in the presence of the R.F. field. However, P_1 is no more an invariant; its mean value $\overline{P}_1$ changes with time according to the equations

$$\frac{\overline{dP_1}}{dt} = -\omega\frac{\varepsilon}{2}\sqrt{\overline{P}_1}\, f(\overline{P}_1,P_3)\cos w_1'$$

$$\frac{\overline{dw_1'}}{dt} = \nu_1 - \omega + \omega_0\frac{\varepsilon}{2\sqrt{\overline{P}_1}}\left[\frac{1}{2}f + \overline{P}_1\frac{\partial f}{\partial \overline{P}_1}\right]\sin w_1' \qquad \ldots(2.14)$$

As the Hamiltonian (2.10) is time-independent in this approximation, the integral of the system (2.14) is simply H" = constant, or

$$\frac{H_0}{\omega_0} - \frac{\omega}{\omega_0}\overline{P}_1 - \varepsilon\,\overline{H}_2 = \text{constant} \qquad \ldots(2.15)$$

Expressing H_0 and H_2 explicitly, equation (2.15) can be written in the form

$$\sqrt{\overline{P}_1}\left[\frac{\sqrt{\overline{P}_1}}{\Upsilon_1 P_3}\left(\Upsilon_0 - \frac{\omega}{\omega_0}\right) + 1 + \frac{\varepsilon}{2\Upsilon_1 P_3} f(\overline{P}_1 P_3)\sin w_1'\right] = C \qquad \ldots(2.16)$$

This equation defines implicitly $\bar{P}_1$ as a function of w_1'. The phase angle w_1' is the angle between a vector rotating with angular velocity $\nu_1(P_1P_2P_3)$and a vector rotating with angular velocity ω.

We thus see that the only effect of the R.F. field is that the quantity P_1 (proportional to the magnetic moment μ or to the transverse energy in the midplane) is no more invariant. However, there exists another invariant defined by the equation (2.16)

To get an idea how $\bar{P}_1$ varies let us trace the curves (2.16) in the phase-plane $\bar{P}_1, w_1'$. Consider first particles which have a very small amplitude of longitudinal oscillations ($P_3/\sqrt{P_1} \to 0$). These particles stay in a practically uniform field and their cyclotron frequency is $\Upsilon_0\omega_0$. Cyclotron resonance occurs if $\omega = \omega_0\Upsilon_0$. This condition makes the first term in (2.16) equal to zero. As $|f(\bar{P}_1,P_3)| \leqslant 1$ there will be no appreciable variation in $\bar{P}_1$ if $\varepsilon/2\Upsilon_1P_3 < 1$; there will be however large variation in $\bar{P}_1$ if $\varepsilon/2\Upsilon_1P_3 \gg 1$. Thus, given a certain value of the longitudinal invariant P_3, the amplitude of the R.F. field must be larger than a critical value.

$$E_{oc} = 2\sqrt{2}\ \omega B_0 L\ \Upsilon_1 P_3 \qquad \ldots(2.17)$$

if appreciable variations in $\bar{P}_1$ should occur. It may be noted that if $P_3 = 0$ this critical amplitude is zero as might be expected because the particles stay in a strictly uniform field.

Fig.4 shows an example of the curves (2.16) in the phase plane corresponding to $\Upsilon_0\omega_0 - \omega = 0$; $\varepsilon = 10^{-5}$; $L/\lambda = 2$; $P_3 = 10^{-6}$. In this case $\varepsilon/2\Upsilon_1P_3 = 5$. The horizontal lines correspond to $\bar{P}_1$ values which make the function $f(\bar{P}_1,P_3)$ equal to zero. Essentially they are the zeros of $J_0(\frac{1}{4}\frac{P_3}{\bar{P}_1}) = 0$, i.e.

$$\bar{P}_1/P_3 = 0.104; \quad 4.5 \times 10^{-2}; \quad 2.9 \times 10^{-2}; \ \ldots$$

Particles having $\bar{P}_1$ values lying on these lines experience in the average a zero R.F. field due to phase-changes caused by longitudinal oscillations. Each horizontal line is crossed only by two trajectories in two singular points. Because the velocity $d\bar{P}_1/dt$ in the singular points is zero, these horizontal lines cannot be crossed by any phase point in a finite time. We see that an indefinite increase of $\bar{P}_1$ can occur only to particles which start above all the horizontal lines. These particles have such a small amplitude of longitudinal oscillations that they stay in an essentially uniform field.

The phase-trajectories plotted in Fig.5 correspond to the same parameters as in the previous figure except that $P_3 = 10^{-5}$ ($\varepsilon/2\gamma_1 P_3 = 0.5$). The amplitude of the R.F. field is now smaller than the critical value and only periodic fluctuations in $\bar{P}_1$ occur.

Let us now consider particles which have an appreciable amplitude of longitudinal oscillations. The perturbation $\varepsilon\ \bar{H}_2$ in (2.15) can cause a noticeable change in $\bar{P}_1$ only if the term $H_o/\omega_o - (\frac{\omega}{\omega_o})\bar{P}_1$ changes slowly with $\bar{P}_1$. This will occur in the vicinity of such $\bar{P}_1$ values which make the derivative $\frac{\partial}{\partial \bar{P}_1}(H_o - \omega\bar{P}_1)$ equal to zero. Remembering that $\frac{\partial H_o}{\partial \bar{P}_1} = \nu_1$, we come to the conclusion that the R.F. field can affect only those particles whose average cyclotron angular frequency ν_1 is near to the value

$$\nu_1 = \omega \qquad ,..(2.18)$$

Given a certain mirror field, the frequency ν_1 is, according to (1.9), completely determined by the amplitude η_m of the longitudinal oscillations. The resonance condition (2.18) thus implies that the R.F. field appreciably affects only those particles which have a certain amplitude of longitudinal oscillations. This resonant amplitude is determined by the applied angular frequency ω and goes to zero as ω approaches $\gamma_o\omega_o$ and is equal to 1 for $\omega = \omega_o(\gamma_o + \gamma_1\sqrt{a/8})$.

Figs.6 and 7 give an idea of the actual variations of $\bar{P}_1$. The phase trajectories (2.16) are plotted in the phase-plane $\bar{P}_1, w_1'$ assuming $\omega - \gamma_o\omega_o = 0.02$; $\varepsilon = 10^{-5}$; $L/\lambda = 2$. The resonant amplitude is $\eta_m = 0.28$ according to (2.18) and (1.9) and assuming $a = 0.5$. Fig.6 corresponds to $P_3 = 10^{-5}$ (resonance occurring in the neighbourhood of $\bar{P}_1 = 6 \times 10^{-8}$) and Fig.7 corresponds to $P_3 = 4 \times 10^{-4}$ (with resonance at $\bar{P}_1 = 10^{-4}$). Remembering that $\bar{P}_1$ is proportional to the perpendicular energy, we see that Fig.6 corresponds to low-energy electrons and Fig.7 to high-energy electrons. The main feature of these figures is that $\bar{P}_1$ is subject only to relatively small periodic fluctuations, especially for low $\bar{P}_1$ values. If the amplitude of longitudinal oscillations is far from the resonant amplitude the phase-trajectories are slightly waved curves. Near the resonance the phase-trajectories are closed loops confined between horizontal lines corresponding to $\bar{P}_1$ values which make the function $f(\bar{P}_1, P_3)$ equal to zero. As in the previous case, these lines cannot be crossed in a finite time. They correspond to particles which experience, on average, a zero R.F. field due to phase changes caused by longitudinal oscillations.

The maximum possible fluctuation of $\bar{P}_1$ for small $\bar{P}_1$ values is nearly equal to the distance $\Delta\bar{P}_1$ between two neighbouring horizontal lines. In general, this distance is a complicated function of $\bar{P}_1$ and P_3. If however $\frac{1}{4}\frac{P_3}{\bar{P}_1} \gg 1$ the approximate formula

$$\frac{\Delta\bar{P}_1}{\bar{P}_1} = 4\pi\,\frac{\bar{P}_1}{P_3} = 4\pi\sqrt{\frac{2}{a}}\,\frac{1}{\eta_m^2}\,\sqrt{\bar{P}_1} \qquad \ldots(2.19)$$

is valid. In the last expression P_3 was replaced by the amplitude η_m using the last of the equations (1.4).

Let us estimate the period of the variations of $\bar{P}_1$ for the case that the phase trajectories form closed loops contained between two horizontal lines as those in Fig.6. Such phase-flow can be approximately represented by Fig.8. Let us cut the phase-flow by the line L . The phase points crossing the line L in a time interval Δt will occupy in the phase-plane an area

$$\Delta S = \int_0^{\pi/2} \left|\frac{d\bar{P}_1}{dt}\right| \, dw_1' \cdot \Delta t \qquad \ldots(2.20)$$

The velocity

$$\left|\frac{d\bar{P}_1}{dt}\right| = \omega\,\frac{\varepsilon}{2}\sqrt{\bar{P}_1}\, f \cos w_1' \qquad \ldots(2.21)$$

is the velocity of the phase points on the line L. After a time $\Delta t = T$ equal to the mean period of oscillation the phase points will fill the whole accessible area $\Delta S = \pi\Delta\bar{P}_1$ on the phase plane. Thus, the mean period of oscillations is

$$T = \frac{\pi \Delta \overline{P}_1}{\int_0^{\pi/2} \left| \frac{dP_1}{dt} \right| dw_1'} \qquad \ldots(2.22)$$

using equation (2.21) this may be written in the form

$$\frac{T}{T_o} = \frac{\pi}{2} \frac{\sqrt{\overline{P}_1}}{\varepsilon} \frac{\Delta \overline{P}_1}{\overline{P}_1} \frac{1}{f} \qquad \ldots(2.23)$$

where T_o is the period of R.F. oscillations. At low energies $\Delta\overline{P}_1/\overline{P}_1$ is of the order of ten per cent, the value of f is about 0.1 and the theory is valid if $\sqrt{\overline{P}_1}/\varepsilon \gg 1$. Assuming $\sqrt{\overline{P}_1}/\varepsilon \approx 1000$ the period of oscillations is of the order of 10^3 R.F. periods.

In concluding this single-particle analysis it may be pointed out that the quantities $\overline{P}_1$ and $\overline{P}_3$ are averages of the actual values over a period of longitudinal oscillation (an averaging over the period of the cyclotron motion lies in the very concept of the 'invariants' P_1, P_3). The time-dependence of these averages has been calculated to first order in ε. The theory is valid within a time interval shorter than T_3/ε where T_3 is the period of longitudinal oscillations. The quantities P_1 and P_3 are subject to periodic variations with a period T_3 about the calculated averages $\overline{P}_1$ and $\overline{P}_3$. These periodic variations have not been investigated.

3. The Mechanism of R.F. Heating

Let us first summarise the results of the one-particle analysis using a numerical example. We shall refer to a mirror field defined by equation (1.1) with $B_o = 3.6$ kG; $a = 0.5$; $L = 6$ cm. The electron cyclotron frequency in the centre is $\omega_o/2\pi = 10^{10}$ c/s.

The multiperiodic description of particle motion given in section 1 is valid for non-relativistic energies, small spatial variations of the field and if the Larmor radius r_L is much smaller than the guiding centre radius r_g. The first condition may be satisfied by arbitrarily limiting the energy to a maximum value of 10 keV. (Even at this small energy the electron-mass is two per cent higher than the rest mass. This shows that a relativistic treatment of the problem is desirable.) The second condition is satisfied by the choice of a and L. As the Larmor radius of a 10 keV electron in the midplane is $r_L \approx 1$ mm, the relative change in the field intensity at a distance of a Larmor radius is $2ar_L/L = 1/60$. The third condition implies that only particles far enough from the axis can be considered. A ratio $r_L/r_g \leqslant 0.05$ may be sufficient. This corresponds to a field-line 2 cm away from the axis at the midplane.

The chosen field line is characterised by the invariant $P_2 = 0.055$. The parameters (1.7) are $\Upsilon_o \approx \Upsilon_1 \approx 1$; $\Upsilon_2 \approx 0.12$. The two invariants P_1 and P_3 are related to the total energy W and perpendicular energy in the midplane $(W_\perp)_o$ by the relations (1.14) to (1.16) which give

$$(W_\perp)_o = 7.4 \times 10^7 \times P_1 \quad (\text{eV})$$
$$W = 7.4 \times 10^7 \times \frac{H_o}{\omega_o} \quad (\text{eV}) \qquad \ldots(3.1)$$

Replacing the invariant P_3 in the Hamiltonian (1.6) by the amplitude of longitudinal oscillations η_m we have

$$\frac{H_o}{\omega_o} = P_1 \left[\Upsilon_o + \Upsilon_1 \sqrt{\frac{a}{2}}\, \eta_m^2 + \Upsilon_2 \frac{a}{2} \eta_m^4 + \ldots\right] \qquad \ldots (3.2)$$

The three invariants P_1 to P_3 represent a simple means for describing the particle motion in a static mirror field. The invariant P_2 determines the rotational surface formed by the field lines along which the guiding centre of the particle is moving. The other two invariants P_1 and P_3 determine the energy of the particle, the amplitude of longitudinal oscillations and the three characteristic frequencies (1.9). Fig.9 illustrates the dependence of the total energy W, of the perpendicular energy $(W_\perp)_o$ and of the oscillations amplitude η_m on P_1 and P_3. Particles with an amplitude $\eta_m > 1$ are considered as lost. In the absence of collisions and with no R.F. field present any fixed point lying below the line $\eta_m = 1$ represents a contained electron.

The perturbation analysis of the effect of the R.F. field on electron motion is valid if the condition (2.8) is satisfied. The amplitude of the R.F. field inside the plasma can hardly exceed the value $E_0 = 100$ v/cm. The parameter ε then becomes $\varepsilon = 10^{-5}$ and the theory is valid if $P_1 \gg 10^{-10}$ or $(W_\perp)_0 \gg 0.01$ eV. Having not investigated the convergence of the perturbation treatment we are not in a position to specify the validity condition in more detail. Assuming, however, that $\varepsilon/\sqrt{P_1} \leqslant 0.01$ is sufficient we can tolerate an energy $(W_\perp)_0 \geqslant 70$ eV $(P_1 \geqslant 10^{-6})$ if $E_0 = 100$ V/cm. For an R.F. field with $E_0 = 10$ V/cm the allowed energy becomes $(W_\perp)_0 \geqslant 0.7$ eV $(P_1 \geqslant 10^{-8})$. Low energy electrons are excluded because the perturbation treatment assumes that the relative change in electron energy during one longitudinal oscillation is small.

The effect of the R.F. on the electron motion may be summarised as follows. The 'invariants' P_1 and P_3 are no longer constant. They may be split into two terms: $P_i = \overline{P}_i + \delta P_i$. The term $\overline{P}_i$ is an average of P_i over a period of longitudinal oscillation. The term δP_i is a double periodic function of time. The average value $\overline{P}_3$ of the longitudinal invariant remains constant even in the presence of the perpendicular R.F. field. The average value $\overline{P}_1$ of the perpendicular invariant changes with time. Appreciable changes in $\overline{P}_1$ occur, however, only for electrons whose average cyclotron frequency $\nu_1/2\pi$ is approximately equal to the applied frequency $\omega/2\pi$. The representative points of these resonant electrons in Fig.9 lie on the line η_m = constant, the resonant amplitude η_m satisfying the equation

$$\frac{\omega}{\omega_0} = \Upsilon_0 + \frac{1}{2}\sqrt{\frac{a}{2}}\ \Upsilon_1\,\eta_m^2 \approx 1 + \frac{1}{4}\,\eta_m^2 \qquad \ldots (3.3)$$

The last expression refers to our numerical example.

If $\omega = \omega_0\Upsilon_0$ the resonant amplitude is equal to zero. The averaged perpendicular invariant $\overline{P}_1$ of all electrons having sufficiently small amplitude will increase indefinitely with time (sometime after a temporary decrease). This case, shown in Fig.4, corresponds to a uniform magnetic field because the electrons stay practically in the midplane. If the amplitude is not small enough (i.e. if $\varepsilon/2\Upsilon_1 P_3 < 1$) $\overline{P}_1$ undergoes only periodic changes as shown in Fig.5.

Increasing the applied frequency we can induce variations in $\overline{P}_1$ to electrons performing large longitudinal oscillations. These variations are, however, only relatively small periodic fluctuations, as shown in Fig.6 and Fig.7.

What are the implications of the above results to the R.F. heating? It is clear that, under the assumptions made, electrons contained in a mirror field cannot absorb appreciable energy from an R.F. field apart from the trivial case when the electrons stay in a uniform magnetic field. In all other cases the heating must be caused by some effect not considered up to now. An obvious mechanism to look for is a randomising mechanism which can lead to stochastic acceleration. As all the electrons are created by ionization of neutral gas which is continually fed into the system, electron collisions are an apparent randomising mechanism. In principle, stochastic acceleration can also be caused by noise-modulation of the R.F. source.

Let us first consider the heating caused by electron collisions. The situation is very similar to heating in an R.F. discharge in the absence of a magnetic field. We shall discuss the heating mechanism in a mirror field using the $\overline{P}_1, P_3$ plane of Fig.9. Fig.10 shows a small part of the $\overline{P}_1, P_3$ plane. Heating occurs in the neighbourhood of the resonant amplitude represented by the line η_m = constant. Imagine that the point E_1 is suddenly occupied by an electron as a result of a scattering or ionizing process. Due to the R.F. field this point will oscillate along a line $\overline{P}_3$ = constant between the points A and B and about the centre C_1. The period of oscillation T is, according to equation (2.23), of the order of 10^3 R.F. oscillations, i.e. $T \approx 0.1$ μsec in our numerical example. The average collision time τ of an electron is much longer than the period T in typical cases. Taking only collisions with neutrals into account, the product of the ionization collision cross-section and electron velocity may be estimated as $\sigma v \approx 10^{-8}$ cm^3/s for hydrogen and electrons in the keV range[3]. The collision time $\tau = (N\sigma v)^{-1}$ is thus of the order τ = 100 μsec for a typical neutral density of $N = 10^{12}$ cm^{-3} corresponding to a pressure 3.10^{-5} mmHg. An electron performs on the average about a thousand oscillations in $\overline{P}_1$ between two successive collisions causing small angle scattering. The large angle scattering cross section is in the keV energy range, an order of magnitude smaller.

Let us assume that a collision occurred at an instant when the state of the electron was represented by the point E_2. If the energy of the electron is high enough the collision does not cause an appreciable change in the electron energy so that the representative point E_2 is displaced along a curve of constant total energy H_0 = constant to some other point E_3. This point oscillates again about some average value C_2 until a further collision replaces the representative point E_4 to E_5.

The result of the process described is small changes in electron energy between two successive collisions. If the electron energy at a collision is higher than the energy at the preceding collision a net increase in the electron energy results. The most probable energy of an oscillating electron is just the average value represented by the centres C. If more electrons are created having an energy below the average value than above it, the electrons will in the average gain energy from the R.F. field between the collisions. This will happen when the number of electrons decreases with increasing energy, but such an energy distribution is to be expected because new electrons, being the result of ionization, are fed into the system at a low energy (< 10 eV). As time proceeds their energy slowly increases but their density decreases due to scattering out of the mirror field.

Let us estimate the minimum time necessary for heating up these cold electrons. The maximum possible increase in $\overline{P}_1$ between two successive collisions is the difference $\Delta\overline{P}_1$ between two neighbouring horizontal lines in Fig.6 which is approximately given by equation (2.19). Thus, $\overline{P}_1$ can increase at most with a time rate

$$\frac{d\overline{P}_1}{dt} = \frac{\Delta\overline{P}_1}{\tau} = \frac{C}{\tau}\,\overline{P}_1^{\frac{3}{2}} \qquad \ldots (3.4)$$

where τ is the collision time and

$$C = 4\pi\sqrt{\frac{2}{a}}\;\frac{1}{\eta_m^2} \qquad \ldots (3.5)$$

Assuming that τ is a constant the integral of equation (3.4) is

$$\overline{P}_1 = \frac{4}{\left(\frac{2}{\sqrt{P_o}} - C\,\frac{t}{\tau}\right)^2} \qquad \ldots (3.6)$$

P_o being the initial value of $\overline{P}_1$ at $t = 0$.

The increase in $\overline{P}_1$ is very rapid if t/τ approaches the value

$$\frac{t_\infty}{\tau} = \frac{2}{C\sqrt{P_o}} = \frac{1}{2\pi}\sqrt{\frac{a}{2}}\,\frac{\eta_m^2}{\sqrt{P_o}} \qquad \ldots (3.7)$$

The time t_∞ can be considered as a rough estimate of the minimum time necessary to heat up the electrons from an initial perpendicular energy W_o corresponding to the perpendicular invariant $\overline{P}_1 = P_o$. The very approximate nature of the above equations is obviously caused by two reasons; equation (2.19) over-estimates the difference $\Delta\overline{P}_1$ at high energies and τ is a function of energy. However, equation (3.7) seems to be in agreement with experimental observations at least in the fact that the heating-up time is several orders of magnitude longer than the collision time. Following our numerical example and assuming $\eta_m \approx 0.5$, $W_o \approx 1$ eV ($P_o \approx 10^{-8}$) we obtain from equation (3.7) $t_\infty/\tau \approx 200$.

For the collision time τ can be taken the value $\tau \approx 10^{-4}$ sec corresponding to small angle scattering. This can be shown in the following way. It has been pointed out previously that the difference ΔP_1 between two neighbouring horizontal lines in Fig.6 is $\Delta P_1/P_1 \leqslant 0.1$. The corresponding change in the amplitude of longitudinal oscillations is $\Delta\eta_m/\eta_m \approx \Delta\overline{P}_1/4\overline{P}_1 \leqslant 0.025$. As the angle α between the direction of particle motion and the median plane is given by

$$\tan\alpha = \sqrt{\frac{(W_\parallel)_o}{(W_\perp)_o}} \approx \sqrt[4]{\frac{a}{2}}\,\eta_m$$

a small change $\Delta\eta_m$ in the amplitude corresponds to a change

$$\Delta\alpha \approx \frac{\sqrt[4]{a/2}}{1 + \sqrt{a/2}\,\eta_m^2}\,\Delta\eta_m$$

in the angle α. Thus, a transition between two neighbouring horizontal lines occurs if the electron is deflected by an angle $\Delta\alpha \leqslant 0.02\,\eta_m$ which is always smaller than 0.02 radians.

With a collision time $\tau \approx 10^{-4}$ sec, the shortest heating time is $t_\infty \approx 0.02$ sec.

The heating time actually observed in the Oak Ridge experiment is 0.1 sec[4].

The stochastic acceleration caused by a noise-modulated R.F. source has been investigated for several years as a possible (but inefficient) method of particle acceleration[5]. In order to estimate the R.F. spectrum bandwidth necessary for stochastic acceleration let us assume that the electric field has a time-dependence, $\cos(\omega t + \varphi(t))$. The phase angle $\varphi(t)$ is a random function of time. If φ is constant the phase points in the phase-plane $\overline{P}_1, w_1'$ perform a periodic motion. As $\overline{P}_1$ increases the mean angular frequency of the electron gyration changes as well and the electron is no more in a favourable phase for energy absorption from the R.F. field. If, however, the phase angle φ of the R.F. field changes by π radians in half a period of the $\overline{P}_1$ oscillation, the electron stays in a favourable phase and the increase of $\overline{P}_1$ continues. A random change of φ causes a random walk of phase points in the phase plane resulting in energy increase of some of the electrons. The required change in the frequency of the R.F. source is

$$\Delta\omega = \frac{d\varphi}{dt} \approx \frac{\pi}{T/2}$$

But according to equation (2.23) the average period T is of the order of 0.1 μsec so that $\Delta\omega \approx 60$ Mc/s. This is more than three orders of magnitude higher than the expected bandwidth of a D.C. magnetron under normal conditions. Thus, a natural noise modulation of the R.F. source seems to be excluded as a possible cause of electron heating. Electron collisions seem to be the only considerable randomising mechanism.

As the heating is a statistical process it can be expected that the electrons will reach a distribution corresponding to statistical equilibrium compatible with the containment condition.

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APPENDIX

Derivation of the Canonical Transformation (1.4)

Equations (4.4) to (5.4) of reference (2) give the Hamiltonian function of a particle in the natural co-ordinate system ξ', θ', η' of Fig. 1. We attached dashes to the co-ordinates of reference (2) to point out that they are slightly different from our co-ordinates ξ, θ, η which are normalised. The canonical transformation

$$\xi = \xi'/L, \quad \theta = \theta', \quad \eta = \eta'/L$$
$$p_\xi = p'_\xi/p_o, \quad p_\theta = p'_\theta/p_o L, \quad p_\eta = p'_\eta/p_o \qquad \ldots (A.1)$$

with $p_o = eB_oL$ replaces the (dashed) co-ordinates and momenta of reference (2) by (not dashed) co-ordinates and momenta used in this work. Applying this transformation to the Hamiltonian (4.4) of ref.(2), and taking into account our special field shape (1.1) we obtain the Hamiltonian (1.2) to (1.3).

The transformation from our co-ordinate system $\xi, \theta, \eta, p_\xi, p_\theta, p_\eta$ to a multi-periodic system w_i, P_i will be performed in two steps. In the first step angle variables for the transverse motion will be introduced leaving the longitudinal co-ordinate η and momentum p_η unchanged. In the second step the periodic character of the longitudinal motion will also be expressed explicitly by replacing η and p_η by an appropriate angle and action variable.

In order to carry out the first step we replace the transverse co-ordinates ξ, θ and momenta p_ξ, p_θ by new co-ordinates Q_1, Q_2 and momenta P_1, P_2 using the transformation

$$\xi = \sqrt{2P_2} + \sqrt{2P_1}\cos Q_1, \quad p_\xi = -\sqrt{2P_1}\sin Q_1$$
$$\theta = Q_2 - \sqrt{\frac{P_1}{P_2}}\sin Q_1, \qquad p_\theta = P_2 \qquad \ldots (A.2)$$

This is a canonical transformation derived from a generating function

$$S = \theta P_2 - \int\sqrt{2P_1 - (\xi - \sqrt{2P_2})^2}\, d\xi \qquad \ldots (A.3)$$

Let us express the Hamiltonian (1.2) in terms of the new co-ordinates. Assuming at first a uniform magnetic field ($a = 0$) we obtain

$$H_{a=0} = \omega_o P_1 \left[1 - \sqrt{\frac{P_1}{P_2}} \cos^3 Q_1 \; \frac{1 + \frac{3}{4}\sqrt{\frac{P_1}{P_2}} \cos Q_1}{\left(1 + \sqrt{\frac{P_1}{P_2}} \cos Q_1\right)^2} \right]$$

If $\sqrt{P_1/P_2} \ll 1$ the second term in the bracket may be neglected and the Hamiltonian has a simple form $H_{a=0} = \omega_o P_1$. This result can be obtained exactly using a more complicated transformation given in reference (2). The condition $\sqrt{P_1/P_2} \ll 1$ implies that the particle Larmor radius is much smaller than its guiding centre radius (see chapter 1). As this condition is satisfied in many mirror machines, apart from a small region near the axis, we prefer to use the much simpler transformation (A.2).

The complete Hamiltonian (1.2) in terms of the new co-ordinate will be a periodic function of Q_1 and can be written in the form of a Fourier series

$$H = \sum_{k=-\infty}^{+\infty} H_k \, e^{jkQ_1}$$

If $a < 1$ and $\sqrt{P_1/P_2} \ll 1$ all the periodic terms in this series are small perturbations of the average value $\bar{H} = H_o$. According to the perturbation theory, the motion, averaged with respect to the angular co-ordinate Q_1 can be derived from the averaged Hamiltonian H_o. A similar calculation as in reference (2) gives approximately for the latter

$$H_o = \omega_o P_1 \frac{B}{B_o} + \frac{1}{2} \omega_o \frac{p_\eta^2}{\bar{h}_\eta^2} \qquad \ldots (A.4)$$

where

$$\frac{B}{B_o} \simeq \Upsilon_o(P_2) + \alpha(P_2)\eta^2 \ , \ \Upsilon_o(P_2) \simeq 1 - aP_2 - \frac{1}{2} a^2 P_2^2 \ , \ \alpha(P_2) \simeq a(1 + 4aP_2) \qquad \ldots (A.5)$$

is the normalised magnetic induction and

$$\bar{h}_\eta^2 \approx 1 + 2aP_2 + 4a^2 P_2^2 - 10a^2 P_2 \eta^2 \qquad \ldots (A.6)$$

Both quantities are expressed as functions of P_2 and η.

The equations of the transverse motion derived from the Hamiltonian (A.4) are

$$\frac{dQ_1}{dt} = \omega_o \frac{B}{B_o} \qquad P_1 = \text{constant}$$

$$\frac{dQ_2}{dt} = \omega_o P_1 \frac{\partial B/\partial P_2}{B_o} \qquad P_2 = \text{constant} \qquad \ldots (A.7)$$

$\frac{dQ_1}{dt}$ is the angular velocity of cyclotron rotation, $\frac{dQ_2}{dt}$ is the angular velocity of the azimuthal drift. This concludes our first step which introduced angle and action variables for the transverse motion.

Since the Hamiltonian (A.4) is now a function of only one co-ordinate η, it is a simple matter to introduce the third angular co-ordinate corresponding to the longitudinal oscillation. For this aim we replace the old co-ordinates Q_1, Q_2, η and momenta P_1, P_2, p_η by new co-ordinates w_1, w_2, w_3 and momenta P_1', P_2', P_3 using a canomical transformation derived from the generating function

$$S = P_1'Q_1 + P_2'Q_2 + \int p_\eta d\eta \qquad \ldots(A.8)$$

The old momentum

$$p_\eta = h_\eta \sqrt{2P_1} \sqrt{\frac{H_o}{\omega_o P_1} - \frac{B}{B_o}} \qquad \ldots(A.9)$$

must be expressed, of course as a function of the new momenta. As the obvious results of the transformation generated by (A.8) is $P_1' = P_1$, $P_2' = P_2$ it is only necessary to express H_o by means of the new momentum P_3 which is defined by

$$P_3 = \frac{1}{2\pi} \oint p_\eta d\eta = \frac{\sqrt{2P_1}}{2\pi} \oint \bar{h}_\eta \sqrt{\frac{H_o}{\omega_o P_1} - \frac{B}{B_o}} \; d\eta \qquad \ldots(A.10)$$

The integral is taken over one period of the longitudinal oscillation. The above equation implicitly defines the Hamiltonian H_o as a function of the action variables P_1, P_2, P_3. We shall now express the function $H_o(P_i)$ explicitly.

Denoting $P_3/\sqrt{P_1} = x$, $H_o/\omega_o P_1 = y$, equation (A.10) is

$$x = \frac{\sqrt{2}}{2\pi} \oint \bar{h}_\eta \sqrt{y - \frac{B(P_2,\eta)}{B_o}} \; d\eta \qquad \ldots(A.11)$$

It may be noted that this equation defines y as a function of only two variables x and P_2 and that $x \to 0$ as $y \to \left.\frac{B}{B_o}\right|_{\eta=0}$. Hence y can be expressed as a power series

$$y = \Upsilon_o(P_2) + \Upsilon_1(P_2)x + \Upsilon_2(P_2)x^2 + \ldots \qquad \ldots(A.12)$$

provided the series converges. The normalised field in the median plane has been denoted by $\Upsilon_o(P_2) = \left.\frac{B}{B_o}\right|_{\eta=0}$. The coefficients $\Upsilon_i(P_2)$ can be found by differentiating the definition equation (A.11) with respect to x and calculating the limit of the derivatives dy/dx, d^2y/dx^2, ... as $x \to 0$. The result of this calculation is in equations (1.6) and (1.7).

Having expressed H_0 by means of the action variables it remains but to find the transformation generated by the function (A.8). Using the equation (A.5) for the field the generating function can be expressed in the form

$$S = P_1Q_1 + P_2Q_2 + \sqrt{2P_1\alpha} \int \bar{h}_\eta \sqrt{\eta_m^2 - \eta^2}\, d\eta \qquad \text{...(A.13)}$$

where

$$\eta_m^2 = \frac{\Upsilon_1 x + \Upsilon_2 x^2 + \ldots}{\alpha} \qquad \text{...(A.14)}$$

It is obvious that η_m represents the maximum value of η in the course of the motion. Thus, η_m is the amplitude of the longitudinal oscillation. Respecting only the first term in the series and putting $\alpha \approx a$ we obtain approximately

$$\eta_m^2 = \sqrt{\frac{2}{a}}\, \frac{P_3}{\sqrt{P_1}} \qquad \text{...(A.15)}$$

In order to get the transformation in a simple form we shall assume $\bar{h}_\eta = 1$, $\alpha = a$ which is approximately valid if $aP_2 \ll 1$. With this simplification the transformation equations are

$$w_1 = \frac{\partial S}{\partial P_1} = Q_1 + \frac{1}{2}\sqrt{\frac{a}{2P_1}}\, \eta \sqrt{\eta_m^2 - \eta^2}$$

$$w_2 = \frac{\partial S}{\partial P_2} = Q_2$$

$$w_3 = \frac{\partial S}{\partial P_3} = \sin^{-1} \frac{\eta}{\eta_m}$$

$$p_\eta = \frac{\partial S}{\partial \eta} = \sqrt{2aP_1}\, \sqrt{\eta_m^2 - \eta^2}$$

Solving for the old co-ordinates we get finally

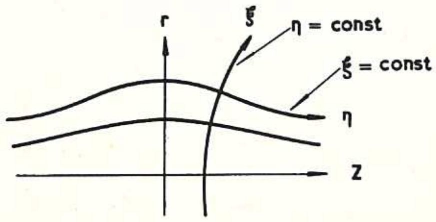
$$Q_1 = w_1 - \frac{1}{4}\frac{P_3}{P_1} \sin 2w_3$$

$$Q_2 = w_2 \qquad \text{...(A.16)}$$

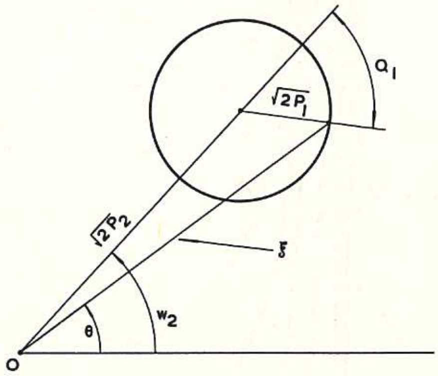
$$\eta = \eta_m \sin w_3$$

$$p_\eta = \sqrt{2aP_1}\, \eta_m \cos w_3$$

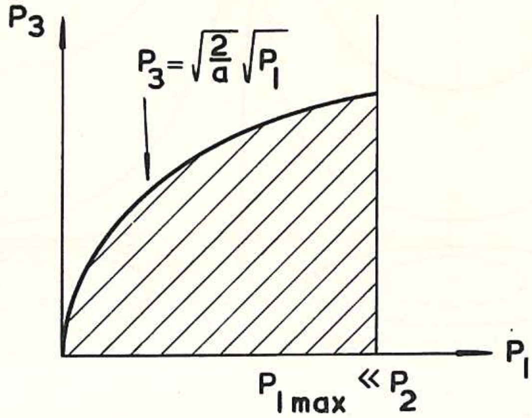
The canonical transformation (1.4) is obtained by the combination of our two transformations (A.2) and (A.16).



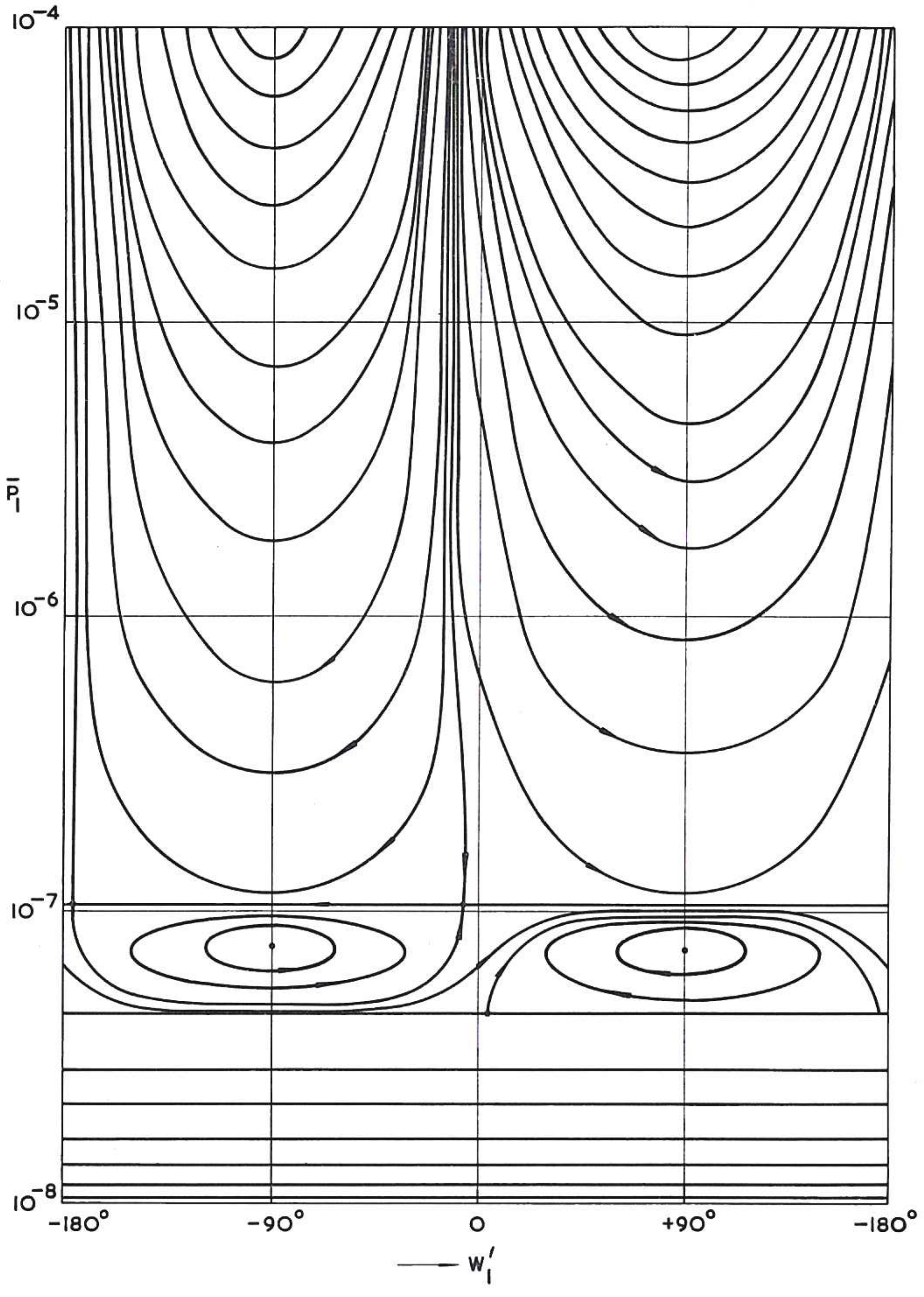
CLM-P 29 Fig. 1
The natural co-ordinate system ξ, η.



CLM-P 29 Fig. 2
Geometrical interpretation of the transformation (1.4).

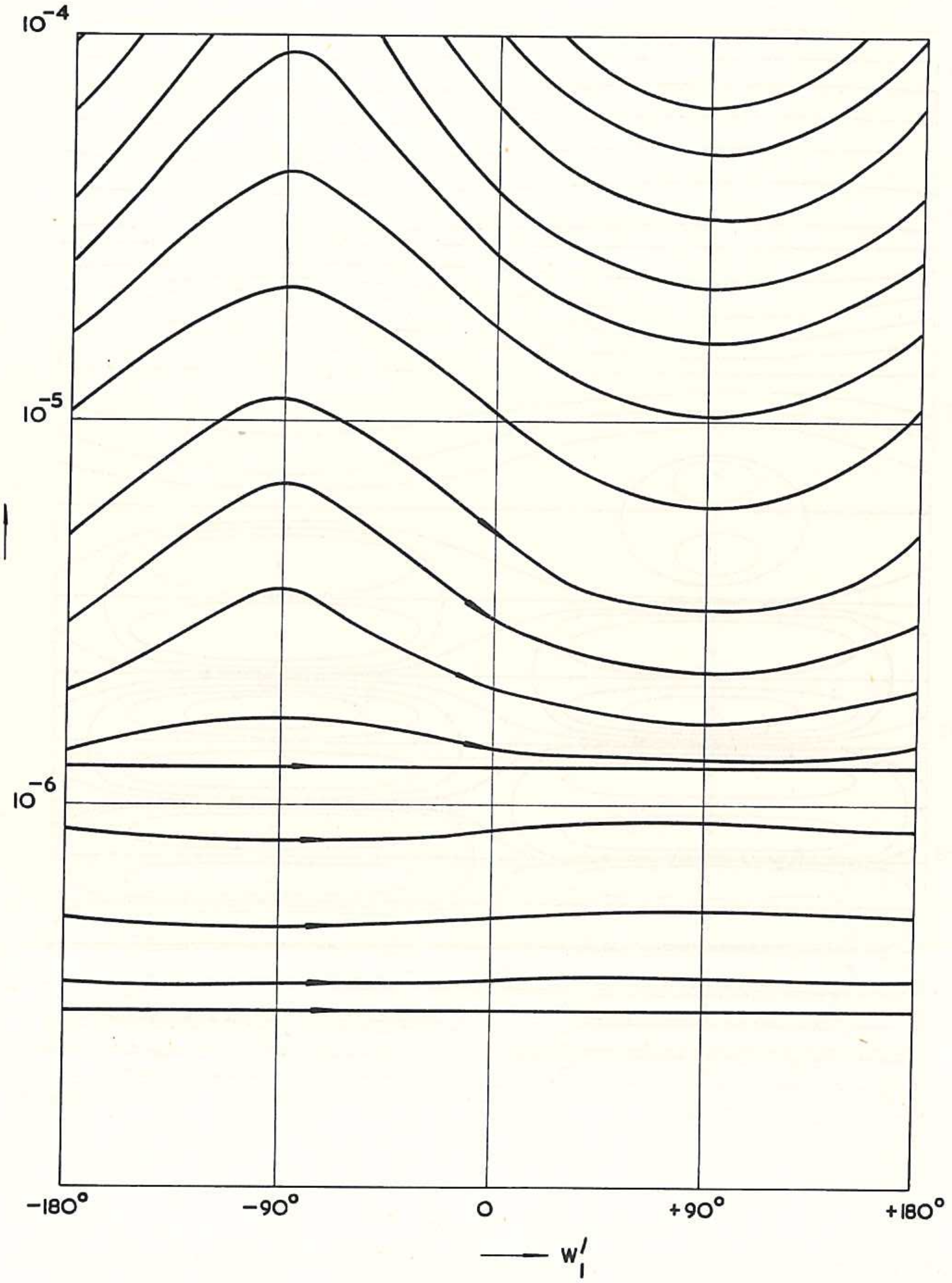


CLM-P 29 Fig. 3
Points representing particles contained in the mirror field must be in the shaded area of the P_1, P_3 plane.

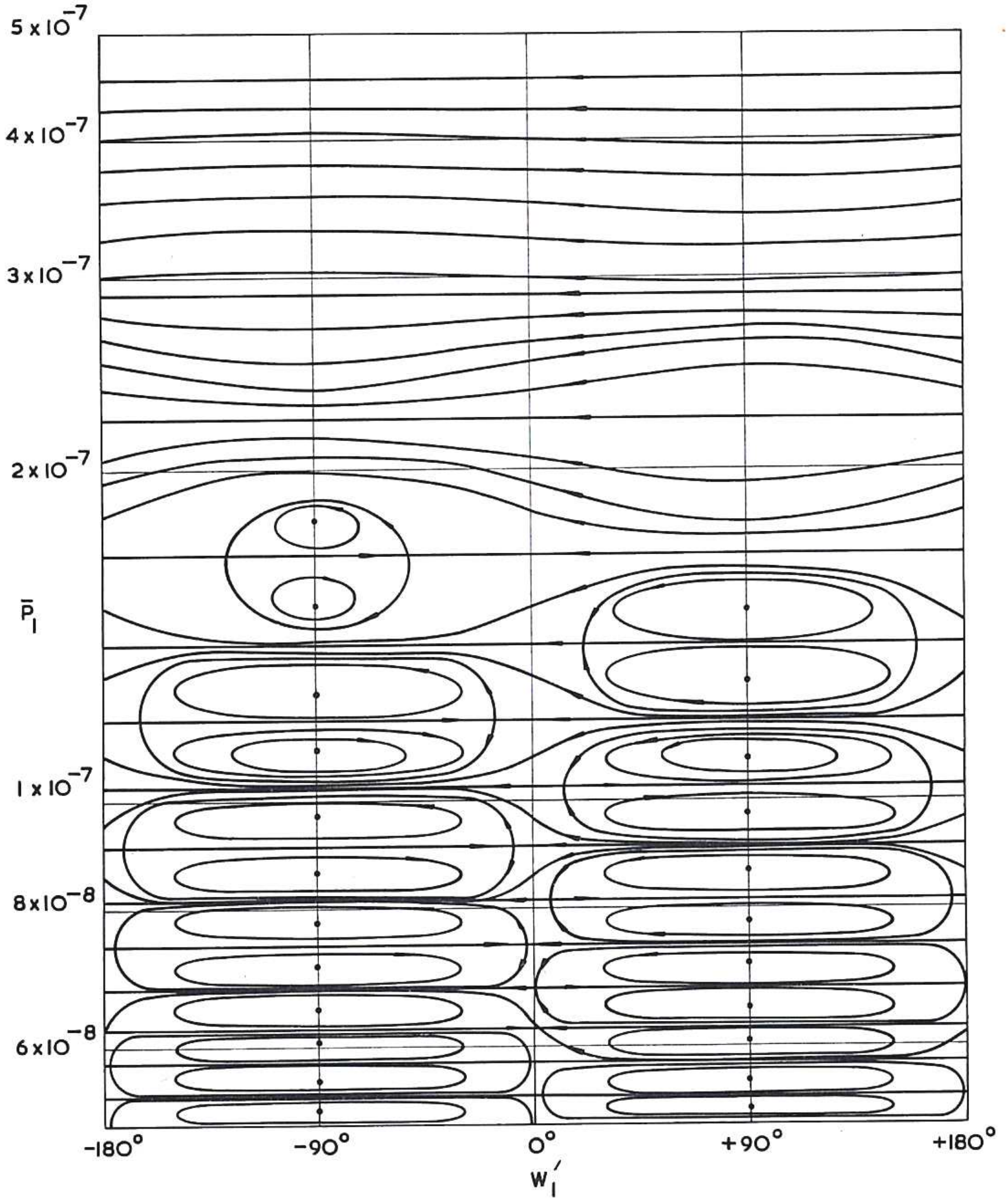


CLM - P 29 Fig. 4

Phase trajectories in the phase-plane $\overline{P}_1$, w'_1 corresponding to an applied frequency equal to the cyclotron frequency in the midplane and an R.F. field higher than the critical field ($\gamma_0\omega_0 = \omega$; $\epsilon = 10^{-5}$; $L/\lambda = 2$; $P_3 = 10^{-6}$).

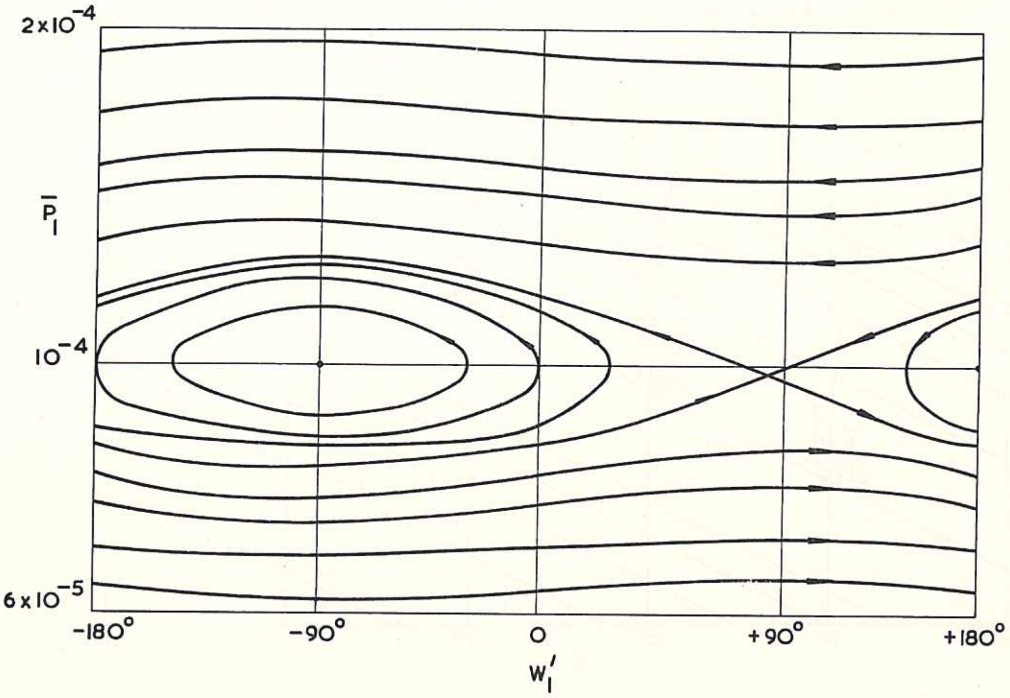


CLM - P 29 Fig. 5
Phase trajectories corresponding to the same parameters as in Fig. 4 except that $P_3 = 10^{-5}$
The increase in P_3 makes the critical field higher than the applied field.

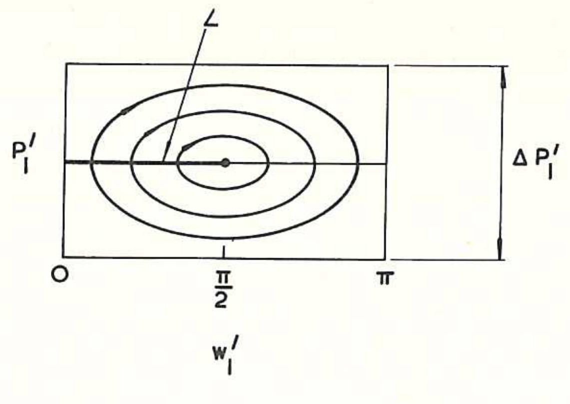


CLM-P 29 Fig. 6

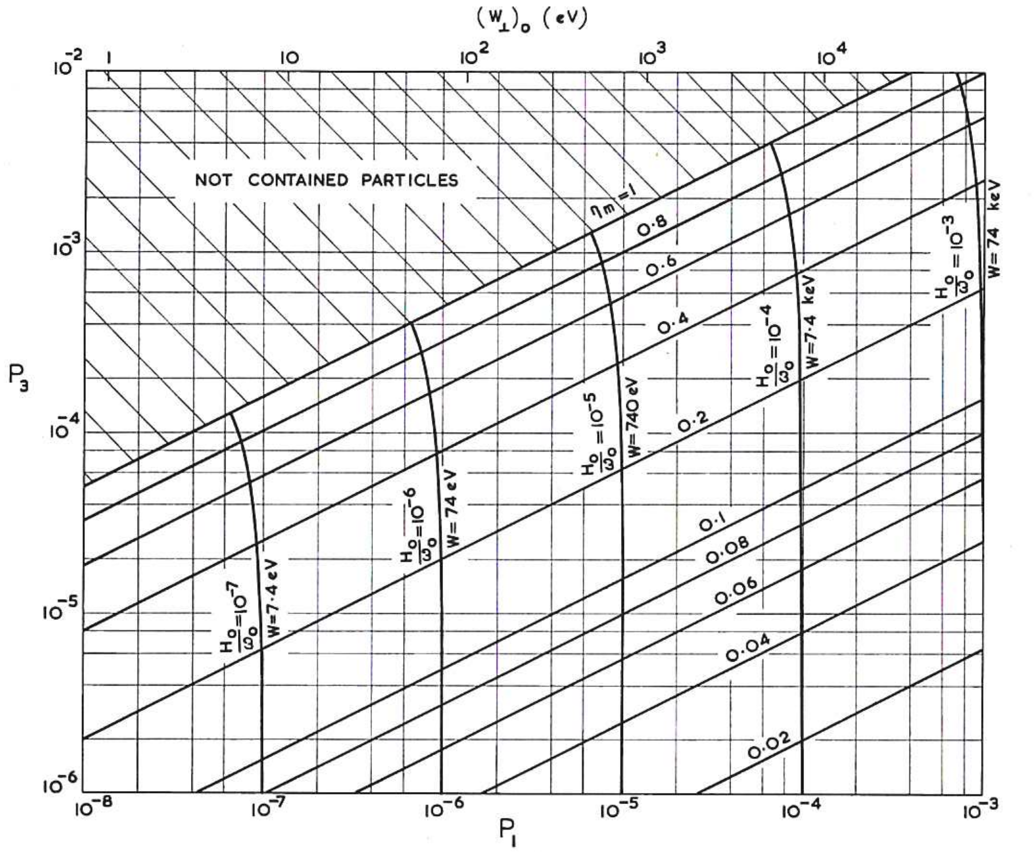
Phase trajectories in the phase-plane $\overline{P}_1$, w_1' corresponding to an applied frequency 2 per cent higher than the cyclotron frequency in the midplane and to low energy electrons ($\omega - \gamma_0\omega_0 = 0.02$; $\epsilon = 10^{-5}$; $L/\lambda = 2$; $P_3 = 10^{-5}$).



CLM-P29 Fig. 7
Phase trajectories corresponding to the same parameters as in Fig. 6 except that $P_3 = 4 \times 10^{-4}$. Resonance occurs for high energy electrons.

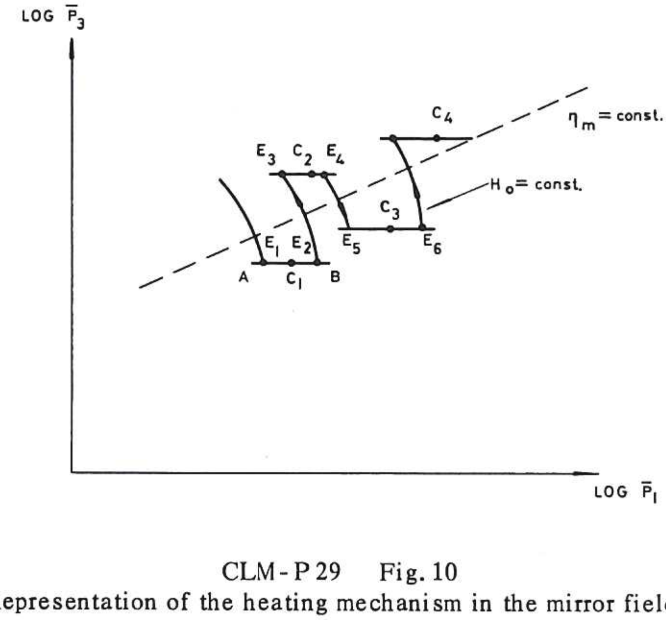


CLM-P29 Fig. 8
Approximate representation of the closed phase-trajectories in the phase-plane $\overline{P}_1, w_1'$.



CLM-P 29 Fig. 9

Curves of constant amplitude η_m of longitudinal oscillations and of constant Hamiltonian H_o plotted in the plane P_1, P_3 assuming $a = 0.5$. The transverse invariant P_1 is proportional to the transverse energy $(W_\perp)_o$ in the midplane and the Hamiltonian H_o is proportional to the total energy. The scales of both quantities in electron-volts are also indicated assuming electrons having a cyclotron frequency $\omega_o/2\pi = 10^4$Mc/s in the centre of the machine and a distance between the mirrors $2L = 12$ cm.



CLM-P 29 Fig. 10

Representation of the heating mechanism in the mirror field.