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VISCOSITY AND THERMAL CONDUCTIVITY EFFECTS ON THE CONTAINMENT OF A TOROIDALLY CONFINED RESISTIVE PLASMA WITH FLOW

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VISCOSITY AND THERMAL CONDUCTIVITY EFFECTS ON THE CONTAINMENT OF A TOROIDALLY CONFINED RESISTIVE PLASMA WITH FLOW

by

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A B S T R A C T

Plasma containment in toroidal systems is studied using a fluid model containing the effects of plasma inertia, resistivity (electron-ion momentum transfer), viscosity (ion-ion collisions), and thermal conductivity. Asymptotic expansion techniques are used to separate the problem into two distinct parts - determination of steady state configurations where time variations occur only on a diffusion time scale, and investigation of the stability of these equilibria. The previously known steady-state solutions are shown to be stable. However, terms in the stress tensor associated with the curvature of the magnetic field lines, which introduce parallel viscosity into the components of the momentum equation perpendicular to the field, are shown to restrict the possible solutions.

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I. INTRODUCTION

The effect of plasma inertia on the confinement of collisional plasma in axisymmetric toroidal configurations has received much attention. The work reported here was initiated as part of a program to extend the approach of GREENE et al. (1971). We generalize their model to include temperature variations (and thus allow for thermal conductivity) and collisional viscosity. Although it is still highly idealized - finite gyration radius terms are ignored in Ohm's Law and in the stress tensor, and ion-electron temperature relaxation is treated in a highly phenomenological manner - the present model contains most of the important features studied in earlier work and thus allows for a comparison of results.

One obvious way to attack the problem is by straightforward numerical solution as an initial value problem. The primary advantage of such an approach is that it correctly follows time development into the non-linear regime. When this was done with a simple fluid model (WINSOR et al. 1970, GREENE et al., 1971) it was found that an initially static plasma would start to rotate, increasing its speed up to a value somewhat below fv_{th} where v_{th} is the ion thermal speed and f is the ratio of the poloidal to the toroidal magnetic fields. The original numerical program has been generalized (BOWERS and WINSOR, 1970) to include the effects of finite gyration radius terms in Ohm's Law and the stress tensor, as well as a phenomenological parallel viscous contribution (DAWSON et al., 1970). As part of this work we extended the numerical model to include thermal conductivity, recognizing that it should not be difficult to merge the improvements. HAINES (1970) also approached toroidal confinement as an initial value problem. Since his coordinate surfaces do not coincide with magnetic surfaces, however, it is difficult to interpret his results in terms of plasma diffusion through the field lines.

Some progress can be made analytically. ZEHRFELD and GREEN, (1970) showed that in the limit of small resistivity the equilibrium problem can be reduced to one of solving a Bernoulli type equation and demonstrated a technique for constructing solutions. Expansion techniques, using the inverse-aspect-ratio as a small parameter, greatly facilitate the analysis. The first approach in this manner was the work by STRINGER (1969) who obtained steady-state expressions for the density and velocity distributions of the ions and electrons separately in a large-aspect-ratio torus, such that to lowest order the distributions are functions of the radius alone. Then the ambipolar condition, that the charge densities of the two species be identical, could be satisfied only if a radial electric field evolved. This provided a mechanism for the buildup of the poloidal velocity, given by

$$\partial v_\theta / \partial t \propto v_\theta / D$$

with $D \equiv f^2 v_{th}^2 - v_\theta^2$. He noted that the static equilibrium $v_\theta = 0$ is unstable, and that with any initial value of v_θ the system quickly attempts to approach a rotation speed $v_\theta = \pm f v_{th}$ (where the expansion is not valid).

This work was quickly followed by several papers in which more physics was introduced into the model in order to eliminate the singular behaviour. ROSENBLUTH and TAYLOR (1969) incorporated a viscous term into the ion momentum equation, showed that the ambipolar condition could be satisfied not only for static systems but for two other specific values of v_θ, and argued that these two would be stable. GALEEV (1969) noted that when electron thermal conductivity is inserted into the equations, the dielectric constant D is modified to the form

$$(\lambda f^2/r)^2 \ (f^2 v_{th}^2 - v_\theta^2)^2 + \rho^2 v_\theta^2 \ (\gamma f^2 v_{th}^2 - v_\theta^2)^2 \ ,$$

with λ the thermal conductivity and γ the ratio of specific heats, and the solution is no longer singular. This also allows two ambipolar rotational equilibrium solutions with poloidal velocity less than the thermal velocity. POGUTSE (1970) has incorporated all these features into one model and arrived at similar conclusions.

HAZELTINE et al., (1971) showed that, in the absence of these effects, an ambipolar solution should exist in which the density possesses a weak discontinuity as a function of θ - a shock solution.

In all these studies the poloidal velocity was assumed constant for the determination of the equilibrium density distribution, although, in actual fact, it was found to vary on a time scale associated with the time it would take an acoustic mode to propagate around the torus. Since this is much faster than the resistive diffusion time, the treatment is not self-consistent and one should not try to infer "stability" from such a calculation. In an attempt to rectify this problem on the simplest possible model, GREENE et al. (1971) ordered (KRUSKAL,1963) all the physical parameters to affect the results in a significant manner. Self-consistent equilibria were found, with time variation only on the resistive time scale for small poloidal velocity v_θ or large toroidal velocity v_Z, of the same order of magnitude as the thermal velocity. They then determined the stability properties by following the normal mode behaviour of a system which is perturbed only slightly from the steady-state solution. With their ordering the equation for the time variation of the poloidal velocity couples with those for acoustic motion, so they had to study a fifth order set of equations to determine the stability properties. The basic conclusion was that for subsonic velocities, $v_\theta < f v_{th}$, the steady state system was always unstable.

GREENE and WINSOR extended this analysis to equilibria with discontinuous density variations on the magnetic surfaces, and WINSOR and BOWERS incorporated a viscous term and finite gyration radius effects into the momentum equation and Ohm's law.

In this work we take account of the exact collisional stress tensor and thermal conductivity (BRAGINSKII, 1965) in the limit of large $\Omega_i \tau_i$ (where the particles undergo many gyrations between collisions so that parallel viscosity dominates), and small $v_{th}\tau_i/R$ (where collisions are sufficiently frequent that trapped particle effects are not important). To keep the discussion simple we ignore the effects of finite gyration radius. This is justified if $f > v_{th}/\Omega_i r$, or the ratio of the ion gyration radius to the plasma radius be less than the inverse aspect ratio, a condition barely satisfied in tokamaks.

In the next section we describe our model. In Sec. III we determine equilibrium solutions with time dependence associated with diffusion, and in Sec. IV we perturb about these solutions and obtain a dispersion relation for stability. In Sec. V we investigate the dependence of equilibria on the various parameters and solve this dispersion relation. The results are discussed in Sec. VI.

II MODEL

We discuss the behaviour of a low-β collisional plasma in a toroidal device, allowing for the effects of plasma inertia, resistivity, viscosity and thermal conduction. We take the usual axisymmetric magnetic field,

$$\underline{B} = (B_0/N)\ (f(r)\ \underline{e}_\theta + \underline{e}_z)\ , \qquad (1)$$

$$N \equiv 1 - \varepsilon\ \cos\theta,\ \varepsilon \equiv r/R\ ,$$

i.e., a system in which the magnetic surfaces consist of a set of nested toroids of circular cross section in the (r, θ) plane, (see Fig. 1). Our plasma is described by the usual fluid equations (using standard notation and Gaussian units) :

$$\rho \frac{d\underline{v}}{dt} = \frac{1}{c} \underline{J} \times \underline{B} - \underline{\nabla} \cdot \underline{\underline{P}} \ , \tag{2}$$

$$\underline{\nabla} \phi = \frac{1}{c} \underline{v} \times \underline{B} - \eta \underline{J} \ , \tag{3}$$

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \rho \underline{v} = 0 \ , \tag{4}$$

$$\rho^{\gamma} \frac{d}{dt} (p\rho^{-\gamma}) = \underline{\nabla} \cdot \underline{\underline{\lambda}} \cdot \underline{\nabla} (P/\rho) \tag{5}$$

where γ is the ratio of specific heats,

$$\underline{\nabla} \cdot \underline{J} = 0 \ ; \tag{6}$$

we use an ideal gas law relating the total hydrostatic pressure to the mean temperature $T = (T_i + T_e)/2$ and density ρ,

$$p = (kT/M)\, \rho \ , \tag{7}$$

with k Boltzmann's constant and M the reduced mass. To simplify the notation we absorb k/M into T so that $T \equiv {v_{th}}^2$ has the dimension of velocity squared. We decompose vectors as

$$\underline{v} = v_r \, \underline{e}_r + v_s \, B_o \frac{\underline{B} \times \underline{e}_r}{B^2} + v_b \, B_o \frac{\underline{B}}{B^2} \tag{8}$$

etc.

Most of the approximations and assumptions implied by the use of this model were described and justified by Greene et al. (1971) ; here we comment only on the additional effects introduced in this paper, viz., the correct treatment of viscous terms in the pressure tensor $\underline{\underline{P}}$, and the physical interpretation of Eq (5) the energy equation.

In the limit of large $\Omega_i \tau_i$ (Ω_i is the ion-cyclotron frequncy, τ_i the collision time), the stress tensor is (BRAGINSKII, 1965; SHKAROVSKY et al., 1966)

$$\underline{\underline{P}} = p\ \underline{\underline{\delta}} + \underline{\underline{\Pi}} \tag{9}$$

where

$$\underline{\underline{\Pi}} = -\ {}^3\!/_2\ \mu\rho T(\underline{b}\underline{b} - \tfrac{1}{3}\underline{\underline{\delta}})\ (\underline{b}\underline{b} - \tfrac{1}{3}\underline{\underline{\delta}})\ :\ \underline{\underline{W}} \tag{10}$$

represents collisional viscosity. Here μ is the ion-ion collision time, $\underline{\underline{\delta}}$ is the unit tensor, $\underline{b} = \underline{B}/|B|$ and $\underline{\underline{W}}$ is the zero-order traceless rate-of-strain tensor,

$$\underline{\underline{W}} = \left\{ \nabla\underline{v} + (\nabla\underline{v})^T - \tfrac{2}{3}\,\nabla.\underline{v}\ \ \underline{\underline{\delta}} \right\}. \tag{11}$$

Since this expression for $\underline{\underline{\Pi}}$ has been obtained independently by several authors using both the guiding-center model (CONNOR and STRINGER) and Grad's thirteen moment approach (BYDDER and LILEY, 1968)*, it may be adopted with some confidence.

In a set of coordinates locally aligned with the magnetic field, the stress tensor reduces to†

$$\underline{\underline{\Pi}} = -3\mu\rho T\ (\underline{b}\underline{b} - \tfrac{1}{3}\underline{\underline{\delta}})\ [\underline{b}.\nabla\ (\underline{v}.\underline{b}) - (\underline{b}.\nabla\underline{b}).\underline{v} - \tfrac{1}{3}\nabla.\underline{v}]. \tag{12}$$

*We are indebted to Dr R.J. Hosking for demonstrating this equivalence to us.

†This differs from the expression of RUTHERFORD et al.(1970) by the inclusion of the field line curvature term $(\underline{b}.\nabla\underline{b}).\underline{v}$.

The anisotropy of the plasma, which in more sophisticated models (CONNOR and STRINGER) is described by energy equations for both parallel and perpendicular pressure, may enhance momentum transfer from perpendicular to parallel directions (magnetic pumping, DAWSON and UMAN, 1965). Thus it is reasonable to write Eq. (2) as

$$\rho \frac{d\underline{v}}{dt} = \frac{1}{c} \underline{J} \times \underline{B} - \underline{\nabla} p + {}^3/_2 \mu \underline{\nabla} \cdot \left[\rho T \left(\underline{b}\underline{b} - \tfrac{1}{3}\underline{\underline{\delta}} \right) \left[\underline{b} \cdot \underline{\nabla} \left(\underline{v} \cdot \underline{b} \right) - \left(\underline{b} \cdot \underline{\nabla} \underline{b} \right) \cdot \underline{v} - \kappa/3 \, \underline{\nabla} \cdot \underline{v} \right] \right] , \tag{13}$$

where if $\kappa > 1$ additional bulk viscosity is present. Because of the curvature of the magnetic field lines, these parallel viscosity terms modify the perpendicular components of the momentum equation and affect the buildup of rotation in the system.

To describe the evolution of energy in the system one should adopt equations for both the ion and electron temperatures, as was done by POGUTSE (1970). On the other hand, some simplification is possible because of the enormous differences between ion and electron thermal conductivities (BRAGINSKII, 1965). In the limit of large electron-ion collision frequency, for example, relaxation between the species is sufficiently large that the electron and ion temperatures are very nearly equal. Then Eq. (5) is appropriate with $\underline{\underline{\lambda}} \approx \underline{\underline{\lambda}}_e$. In the other extreme (small coupling between the electron and ion fluids), Eq. (5) applies to the ion temperature $(\underline{\underline{\lambda}} \approx \underline{\underline{\lambda}}_i)$, while the appropriate electron energy equation becomes $\underline{\nabla} \cdot \underline{\underline{\lambda}}_e \cdot \underline{\nabla} T_e \approx 0$, satisfied with T_e constant on a magnetic surface. Thus it is reasonable to simplify the model by including only one temperature equation, which in both extremes properly describes the total temperature in the plasma. It is difficult to justify the use of a single temperature for situations lying between these; clearly one should expect significant entropy transport when $\omega\tau_{ei} \sim 1$. Since it is difficult to justify the use of fluid models under such resonance conditions (CONNOR and STRINGER), it is reasonable to adopt a simple model.

Just as with the treatment of viscosity, the assumption that $\Omega_i\tau_i$ is large makes it possible to retain only the $\underline{\underline{\lambda}}:\underline{B}\underline{B}/B^2$ term in the thermal conductivity tensor. Thus, using Eqns. (4) and (7) we write Eq. (5) as

$$\rho \frac{\partial T}{\partial t} + \rho \underline{v}.\underline{\nabla} T = \lambda \, \underline{B}.\underline{\nabla}\left(\frac{\underline{B}.\nabla \underline{T}}{B^2} \right) - (\gamma - 1)\, \rho T \underline{\nabla}.\underline{v} . \qquad (14)$$

To the order we consider, collisional heating is negligibly small.

In order to study equilibrium and stability, we use the expansion techniques of GREENE et al. (1971). We employ the same ordering as in that work; in terms of the inverse aspect ratio, we order

$$\varepsilon \sim f \sim \eta/f^2 \sim \lambda f^2 \sim \mu f^2 \sim v_s/v_{th} \sim f v_b/v_{th} \ll 1 .$$

The ordering of the viscosity and thermal conductivity is chosen so that they affect both the equilibrium and stability results. Their magnitude can be adjusted inside the ordering to approximate physical situations.

We allow parallel flow to be large (zero order) to facilitate the comparison of the various approaches and effects.

III EQUILIBRIUM

As emphasized earlier, a realistic discussion of plasma diffusion in toroidal systems must utilize the physical time scales inherent in the problem. Properties characterizing the equilibrium configuration change on a time scale associated with resistive diffusion [$0(\varepsilon^3)$ in this calculation], while the stability of such an equilibrium must be investigated by considering small linear perturbations which vary on the more rapid time scale associated with wave motion (whose frequencies are of first order in the aspect ratio). The details of the work parallel GREENE et al. (1971); for completeness we outline the calculation, concentrating on those aspects not included in the earlier discussion.

The calculation proceeds up to the second order in ε in the following straightforward manner: we solve the perpendicular components of the equation for momentum conservation Eq.(13) and Ohm's Law Eq.(3) for $\underline{J}_\perp$ and $\underline{v}_\perp$ respectively. Now charge continuity Eq.(6) and the parallel component of Ohm's Law yield two magnetic differential equations (KRUSKAL and KULSRUD, 1958) which can be integrated to determine the variation of J_b and ϕ on a magnetic surface. The solvability conditions for these equations determine the net current along the field and the average potential on the surface. The remaining equations, the parallel component of the equation of motion, the continuity of mass and the energy equation, yield a set of three coupled partial differential equations which can be solved to find ρ, v_b and T.

In lowest order the equations are satisfied when $\rho^{(o)}$, $\phi^{(1)}$, $v_b^{(o)}$ and $T^{(o)}$ are arbitrary functions of r only. In the next order we calculate $\underline{J}_\perp^{(1)}$ and hence obtain

$$J_b^{(o)} = (cr\cos\theta/fB_oR)\,[\rho^{(o)}(v_b^{(o)2} + 2T^{(o)})]'.$$

Knowing $J_b^{(o)}$, we evaluate $\phi^{(2)}(r,\theta)$ from the parallel component of Ohm's law and then $\underline{v}_\perp^{(2)}$ from the perpendicular components. Inserting these into the other equations, we obtain the coupled system :

$$\rho^{(o)}\frac{v_\theta^{(1)}}{r}\frac{\partial v_b^{(1)}}{\partial\theta} + \frac{fT^{(o)}}{r}\frac{\partial\rho^{(1)}}{\partial\theta} + \frac{f\rho^{(o)}}{r}\frac{\partial T^{(1)}}{\partial\theta} - \frac{\mu^* f^2}{r}\rho^{(o)}T^{(o)}\frac{\partial^2 v_b^{(1)}}{\partial\theta^2} = \alpha_1\sin\theta + \beta_1\cos\theta \qquad (15)$$

from the parallel equation of motion,

$$\frac{f\rho^{(o)}}{r}\frac{\partial v_b^{(1)}}{\partial\theta} + \frac{v_\theta^{(1)}}{r}\frac{\partial\rho^{(1)}}{\partial\theta} = \alpha_2\sin\theta + \beta_2\cos\theta \qquad (16)$$

from mass continuity, and

$$f(\gamma-1)\frac{\rho^{(o)}T^{(o)}}{r}\frac{\partial v_b^{(1)}}{\partial\theta} + \frac{\rho^{(o)}v_\theta^{(1)}}{r}\frac{\partial T^{(1)}}{\partial\theta} - \frac{\lambda f^2}{r^2}\frac{\partial^2 T^{(1)}}{\partial\theta^2} = \alpha_3\sin\theta + \beta_3\cos\theta \qquad (17)$$

from the energy equation. Here

$$\alpha_1 \equiv -\rho^{(o)}(2v_\theta^{(1)} - fv_b^{(o)})\,v_b^{(o)}/R\ ,$$

$$\alpha_2 \equiv -2\rho^{(o)} v_\theta^{(1)}/R \ , \quad \alpha_3 \equiv -2(\gamma-1)\ \rho^{(o)} T^{(o)} v_\theta^{(1)}/R,$$

$$\beta_1 \equiv -\upsilon\rho^{(o)} v_b^{(o)\prime}/r + \mu^+ f\rho^{(o)} T^{(o)} v_\theta^{(1)}/R \ ,$$

$$\beta_2 \equiv -\upsilon\rho^{(o)\prime}/r \ , \qquad \beta_3 \equiv -\upsilon\rho^{(o)} T^{(o)2}/r \ ,$$

$$\upsilon \equiv (\eta_c^2\ r^2/f^2 Bo^2 R)\ [\rho^{(o)}\ (v_b^{(o)2} + 2T^{(o)})]^\prime ,$$

$$\mu^* \equiv 2\mu\ (1 - \kappa/3)\ /r \ , \quad \mu^+ \equiv 2\mu\ (1 - 2\kappa/3)\ /r \ ,$$

$$v_\theta^{(1)} \equiv v_s^{(1)} + f v_b^{(o)} \ ;$$

primes denote differentiation with respect to r.

It is easily seen that these equations admit a solution of the form

$$\rho^{(1)}(r,\ \theta) = \rho_s^{(1)}(r)\ \sin\theta + \rho_c^{(1)}(r)\ \cos\theta$$

(with similar expressions for $v_b^{(1)}$ and $T^{(1)}$), where the coefficients are obtained by solving the set of six simultaneous algebraic equations in $\rho_s^{(1)}$, $\rho_c^{(1)}$, $v_{bs}^{(1)}$, $v_{bc}^{(1)}$, $T_s^{(1)}$ and $T_c^{(1)}$. The determinant of the matrix of coefficients of this system of equations (which corresponds to STRINGER's (1969, 1970) dielectric constant is $\rho^{(o)4}\Gamma/r^6$, where

$$\Gamma \equiv [\ (\bar\lambda D - \bar\mu\ v_\theta^{(1)2})^2 + (D\gamma + \bar\lambda\bar\mu)^2 v_\theta^{(1)2}] \ , \qquad (18)$$

$$\bar\lambda \equiv f^2\lambda/\rho^{(o)} r \ , \quad \bar\mu \equiv \mu^* f^2 T^{(o)} \ ,$$

$$D \equiv f^2 T^{(o)} - v_\theta^{(1)2} \ , \quad D\gamma \equiv \gamma f^2 T^{(o)} - v_\theta^{(1)2} \ .$$

The solution of these equations is given in the Appendix. Since Γ is positive-definite collisional effects remove the singularity. Of course this still leaves open the possibility of a shock solution (weak discontinuity) for small values of Γ .

Finally the solvability condition for the conservation of charge

$$\oint NJ_r d_\theta = 0 \; ,$$

provides in second order an equilibrium condition relating the zero-order density, temperature and parallel velocity profiles to the lowest order rotational velocity $v_\theta^{(1)}$

$$2\rho^{(o)}\langle T^{(1)}\sin\theta\rangle + (v_b^{(o)2} + 2T^{(o)})\langle\rho^{(1)}\sin\theta\rangle$$
$$+ 2\rho^{(o)}v_b^{(o)}\langle v_b^{(1)}\sin\theta\rangle + \tfrac{1}{2}\mu^* f\rho^{(o)}T^{(o)}\langle v_b^{(1)}\cos\theta\rangle = \tfrac{1}{2}\varepsilon\mu^+\rho^{(o)}T^{(o)}v_\theta^{(1)} . \qquad (19)$$

Here $\langle\;\rangle$ refers to an average over the magnetic surface. It is interesting to observe that although the viscosity terms which are introduced into the perpendicular components of the momentum equation provide a divergence-free contribution to $\underline{J}_\perp^{(1)}$ (so that they do not affect the determination of the first-order functions $\rho^{(1)}$, $v_b^{(1)}$ and $T^{(1)}$), they modify $J_r^{(2)}$ so as to provide the last two terms in this equilibrium equation. This drastically limits the values of μ for which equilibria can be found with $v_b^{(o)'} = 0$.

At this point our equilibrium calculation is completed. The equilibrium is not a completely stationary one; the zeroth-order parameters change on a diffusion time scale (order ε^3) . Since $v_r^{(3)}$ is a sinusoidal function of θ and thus contributes no net flow in this order, we can evaluate the appropriate diffusion rate with no additional work. Thus

$$\frac{\partial \rho^{(o)}}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left\{ r\rho^{(o)}\left[\eta c^2 (1 + 2r^2/f^2R^2)\ (\rho^{(o)}T^{(o)})'/B_o^2 + \eta c^2 r^2\ (\rho^{(o)} v_b^{(o)2})'/f^2 B_o^2 R^2 \right]\right\} - \frac{1}{r}\frac{\partial}{\partial r}\{ \nu\rho_c^{(1)} \}. \quad (20)$$

The first term on the right-hand side is easily recognised as representing the Pfirsch-Schlüter (1962) toroidal correction to the diffusion coefficient for a straight cylinder. The last term provides an additional contribution arising from the shift of the constant density contours away from the magnetic surfaces (r = constant). Zero-order parallel flows also contribute to the diffusion. Analogous expressions relating to transport of toroidal angular momentum and heat flow are given in the Appendix.

It should be emphasized that a diffusion calculation is meaningless if all the above equations are not satisfied - in particular the charge conservation condition Eq.(19) - or if the equilibrium configuration is unstable. Then the concept of diffusion is not applicable. In the next section we investigate the stability criterion.

IV STABILITY

In this section we consider the behaviour of small amplitude linear perturbations about an equilibrium state. We assume that

$$\rho = \rho_{eq} + \tilde{\rho},$$

etc., where the equilibrium quantities (like ρ_{eq}) are given in the Appendix. To carry out a normal mode analysis we assume an exponential variation with time $\tilde{\rho}(r,\theta,t) \sim \tilde{\rho}(r,\theta)\exp(-i\omega t)$. We are led to the ordering $\omega \sim \varepsilon$; the modes grow or damp on the time scale associated with propagation of acoustic waves the long way around the system.

Again the calculation proceeds in an analogous fashion to GREENE et al. (1971); conservation of mass, momentum and energy in the system leads to a set of three coupled partial differential equations for the zero-order perturbations $\tilde{v}_s^{(o)}$, $\tilde{\rho}^{(o)}$, $\tilde{v}_b^{(o)}$ and $\tilde{T}^{(o)}$:

$$- i\omega\tilde{\rho}^{(o)} + \frac{v_\theta^{(1)}}{r} \frac{\partial\tilde{\rho}^{(o)}}{\partial\theta} = - 2\rho^{(o)} \tilde{v}_s^{(o)} \sin\theta/R - \frac{f\rho^{(o)}}{r} \frac{\partial\tilde{v}_b^{(o)}}{\partial\theta} - \frac{\tilde{v}_s^{(o)}}{r} \frac{\partial\rho^{(1)}}{\partial\theta} \quad (21)$$

from Eq. (4),

$$- i\omega \tilde{v}_b^{(o)} + \frac{v_\theta^{(1)}}{r} \frac{\partial\tilde{v}_b^{(o)}}{\partial\theta} - \frac{\bar{\mu}}{r} \frac{\partial^2\tilde{v}_b^{(o)}}{\partial\theta^2}$$

$$= - \frac{f}{r} \frac{\partial\tilde{T}^{(o)}}{\partial\theta} - \frac{f\,T^{(o)}}{\rho^{(o)} r} \frac{\partial\tilde{\rho}^{(o)}}{\partial\theta} -$$

$$- \left[2v_b^{(o)} \sin\theta/R + \frac{1}{r} \frac{\partial v_b^{(1)}}{\partial\theta} - \mu^+ fT^{(o)}\cos\theta/R\right] \tilde{v}_s^{(o)} \quad (22)$$

from Eq. (13) and

$$- i\omega \tilde{T}^{(o)} + \frac{v_\theta^{(1)}}{r} \frac{\partial\tilde{T}^{(o)}}{\partial\theta} - \frac{\bar{\lambda}}{r} \frac{\partial^2\tilde{T}^{(o)}}{\partial\theta^2}$$

$$= - (\gamma-1) \frac{fT^{(o)}}{r} \frac{\partial\tilde{v}_b^{(o)}}{\partial\theta} - \left[2(\gamma-1)T^{(o)}\sin\theta/R + \frac{1}{r} \frac{\partial T^{(1)}}{\partial\theta}\right] \tilde{v}_s^{(o)} \quad (23)$$

from Eq. (14). Charge conservation requires that

$$\left[-i\,\omega + \mu^{\tau}\varepsilon T^{(o)}/R\right]\tilde{v}_s^{(o)} = 2\langle \tilde{T}^{(o)}\sin\theta\rangle/R + (2T^{(o)} + v_b^{(o)2})\,\langle\tilde{\rho}^{(o)}\sin\theta\rangle/\rho^{(o)}R$$

$$+ 2v_b^{(o)}\langle\tilde{v}_b^{(o)}\sin\theta\rangle/R + \mu^{*}f\,T^{(o)}\langle\tilde{v}_b^{(o)}\cos\theta\rangle/R \,. \qquad (24)$$

As in the equilibrium calculation, the viscosity coefficient enters in terms associated with both $\tilde{v}_s$ and $\tilde{v}_b$.

We proceed by introducing the Fourier decomposition

$$\tilde{\rho}^{(o)}(r,\theta) = \sum_m \left[\tilde{\rho}_{ms}^{(o)}(r)\,\sin m\theta + \tilde{\rho}_{mc}^{(o)}(r)\,\cos m\theta\right].$$

It is clear from Eq. (24) that only the $m = 0$ component of $\tilde{v}_s^{(o)}$ is non-vanishing, and that it couples only to the $m = 1$ components of the other variables. As before, (GREENE et al., 1971), the modes for $m \geq 2$ decouple, giving a series of damped, doppler-shifted acoustic oscillations. The terms of these equations associated with $m = 1$ form a seventh-order set; making the determinant vanish provides a dispersion relation (see Appendix). The equilibrium is stable if and only if there are no roots of this dispersion equation in the upper half plane. This can be determined most easily by numerical means (T.J.MARTIN).

V RESULTS

It is clear that the determination of equilibrium configurations is the major part of any diffusion discussion. As shown in Sec. III, the most essential feature is the satisfaction of Eq. (19), the condition of charge neutrality. Once such a solution is found one must check to see that it is stable.

It follows from Eq. (19) and the Appendix that the static solution where $v_\theta^{(1)} = v_b^{(o)} = v_b^{(o)\prime} = 0$ (net flow present only on a resistive time scale) is an equilibrium. In this case the modes can be identified analytically. The dispersion relation separates into

$$(\omega + i\,\bar{\lambda}/r)\,[\omega(\omega + i r\, a^2\mu^*) - a^2] - (\gamma - 1)\omega\, a^2 = 0 \;, \quad (25)$$

and

$$(\omega + i\,\bar{\lambda}/r)\,[\omega^2(\omega + i r\, a^2\,\tilde{\mu}) - (\omega - i q)\, g^2] - (\gamma - 1)\,\omega^2 g^2 = 0 \,. \quad (26)$$

Here

$$\tilde{\mu} \equiv \mu^* + \mu^+ r^2/f^2R^2 \;,$$

$$a^2 \equiv f^2 T^{(o)}/r^2 \;, \qquad g^2 \equiv a^2\,(1 + 2r^2/f^2R^2) \;,$$

$$q \equiv 2\eta c^2 r^2 T^{(o)} \rho^{(o)\prime 2} / f^4 R^2 B_o^2 \rho^{(o)} (1 + 2r^2/f^2R^2) .$$

Clearly the third-order equation determines two acoustic waves and an entropy mode associated with thermal conductivity. It is independent of resistivity. The other equation contains a second entropy mode associated with thermal conductivity and one associated with plasma rotation, and two geodesic waves (acoustic waves where the electric fields associated with the toroidal variation of the magnetic field modifies the frequency).

To proceed further it is useful to make a second asymptotic expansion where the resistivity is made small inside the inverse-aspect-ratio ordering. Then we must order the thermal conductivity and the viscosity in terms of this resistivity parameter. To investigate the situation where these new collisional effects provide a small departure from the previously studied model (GREENE et. al., 1971),

we assume that $\lambda^{-1} \sim \mu \sim q \to 0$. Then
$\omega_{1,2} \sim [\pm a - i\, r\, a^2(\mu^* + (\gamma-1)/\bar{\lambda})/2]$, $\omega_3 \sim \omega_4 \sim -i\,\bar{\lambda}/r$,
$\omega_5 \sim iq$ and $\omega_{6,7} \sim \pm g - i[r\, a^2\tilde{\mu} + (\gamma-1)\, r\, g^2/\bar{\lambda} + q]/2$.
The acoustic and the geodesic waves are both damped. The thermal entropy modes are strongly damped. The one unstable mode, associated with rotation, is not affected by viscosity or thermal conductivity in this ordering.

As the viscosity is increased the frequencies of both the acoustic waves and the geodesic waves decrease, so that, when $\mu^* = 2/ra$ the acoustic modes coalesce at $\omega_1 = \omega_2 \sim -i\, r\, a^2\, \mu^*/2$, and when $\tilde{\mu} = 2g/r\,a^2$ the geodesic modes become $\omega_6 = \omega_7 \sim -i\, r\, a^2\, \tilde{\mu}/2$.
For even larger values of μ these modes separate along the imaginary axis. In the limit where $\mu^{-1} \sim \lambda^{-1} \sim q \to 0$ the modes approach $\omega_1 \sim -i\, r\, a^2\, \mu^*$, $\omega_2 \sim -i/r\,\mu^*$, $\omega_3 \sim \omega_4 \sim -i\,\bar{\lambda}/r$, $\omega_5 \sim -i\, r\, a^2\, \tilde{\mu}$ and $\omega_{6,7} \sim -(i\, g^2/2\, r\, a^2\, \tilde{\mu}\,[1 \pm (1 + 4\, r\, a^2\, \tilde{\mu}\, q/g^2)^{1/2}]$.
One mode is always unstable.

As the thermal conductivity is decreased (keeping the viscosity small) the frequencies of both the acoustic and the geodesic waves increase. When $\lambda \sim \mu \sim q \to 0$ we find
$\omega_{1,2} \sim \pm \gamma^{1/2} a - (i/2\gamma)[\gamma\, r\, a^2\, \mu^* + (\gamma - 1)\,\bar{\lambda}/r]$,
$\omega_3 \sim -i\,\bar{\lambda}/\gamma\, r$, $\omega_{4,5} \sim [i(q - \bar{\lambda}/r)/2\gamma]\{1 \pm [1 + 4\gamma\, q\bar{\lambda}/r(q - \bar{\lambda}/r)^2]^{1/2}\}$
$\omega_{6,7} \sim \pm \gamma^{1/2} g - (i/2\gamma)[q + \gamma\, r\, a^2\tilde{\mu} + (\gamma - 1)\bar{\lambda}/r]$. Again one and only one mode remains unstable.

When $v_\theta^{(1)} \neq 0$ it is difficult to determine the equilibrium solutions and their stability properties analytically. We therefore employed numerical techniques (T.J. MARTIN). Solving the equilibrium relation, Eq. (19), for one parameter (which characterizes the equilibrium) in terms of the rest, we calculated numerical values for the equilibrium first-order plasma variables (see the Appendix) and hence determined the elements of the dispersion relation. Unfortunately, from a purely physical standpoint there is no good parameter to characterize the equilibrium. In the past there has been a tendency to equate the difference between the expression in Eq. (19) and zero to a growth term $\rho\, \partial v_\theta/\partial t$ and to plot this function in terms of v_θ. As stressed in this work, this can have meaning only in the vicinity of an equilibrium solution and one can have little confidence in the form of such a curve elsewhere. Therefore we chose to determine the $v_b^{(o)\prime}$ which is necessary to satisfy the equilibrium condition. Presentation in this manner has the attribute that it clearly illustrates how the various models are related.

In Figure 2 the value of $v_b^{(o)\prime}$ which is necessary to satisfy the equilibrium condition on a surface with prescribed zeroth-order density, density gradient, constant temperature, and no local net toroidal flow, is presented as a function of the rotation velocity $v_\theta^{(1)}$ for several values of the thermal conductivity and no viscosity. The values of $v_\theta^{(1)}$ for which $v_b^{(o)\prime} = 0$ are the ambipolar solutions of GALEEV (1969). In the infinite thermal conductivity limit, $\lambda \rightarrow \infty$, the minimum value of $v_b^{(o)\prime}$ occurs when $v_\theta^{(1)}$ approaches fv_{th} - a region where the expansion has broken down. As λ is decreased this knee occurs for smaller values of $v_\theta^{(1)}$. For λ below some critical number the value of $v_\theta^{(1)}$ where it occurs begins to increase again, so that as $\lambda \to 0$ the knee appears when $v_\theta^{(1)}$ approaches $(\gamma)^{1/2} fv_{th}$ as one should expect.

Numerical evaluation of the dispersion relation shows that the rotational mode is unstable for values of $v_\theta^{(1)}$ corresponding to the dotted parts of the curves. The growth rate becomes zero when $\partial v_b^{(o)\prime}/\partial v_\theta = 0$, and for larger $v_\theta^{(1)}$ it is damped. Thus the equilibria of GREENE et al. (1971) are unstable while that found by GALEEV (1969) is stable.

The effect of viscosity on equilibrium is shown in Figure 3 for several systems with infinite thermal conductivity. Again we have calculated the value of $v_b^{(o)\prime}$ which is necessary to achieve equilibrium for a given value of μ. For very small values of μ it is necesary to have $v_\theta^{(1)}$ very near fv_{th} (where the model breaks down) for solutions with $v_b^{(o)\prime} = 0$ to exist. Reasonable solutions with $v_b^{(o)\prime} = 0$ can be achieved with modest values of the viscosity, but not if μ is too large. This limitation on μ is caused by viscous terms in the radial component of the momentum equation associated with derivatives of the unit vectors. The curves are lower if $\kappa > 1$ (corresponding to the presence of an additional phenomenological viscous term associated with transfer of energy from perpendicular to parallel pressure), so that for a given value of μ the value of $v_\theta^{(1)}$ for which $v_b^{(o)\prime} = 0$ is smaller. Again the equilibrium is unstable with respect to rotation along the dotted parts of the curves and otherwise stable.

If both $^1/\lambda$ and μ are finite the equilibrium curves for $v_b^{(o)\prime}$ as a function of v_θ are what would be obtained by adding the two figures. Similar results are found when the other parameters are varied. It should not be surprising that solutions cannot be found with $v_b^{(o)\prime} = {}^1/\lambda = \mu = 0$ by adjusting $T^{(o)\prime}$, since T enters the problem only through the pressure.

The most striking thing about these results is that if $\mu > 0.6$ the equilibrium curve of $v_b^{(o)\prime}$ v.s $v_\theta^{(1)}$ increases monotonically - the only equilibrium solution with no large toroidal flow in the system ($v_b^{(o)} = 0$ on all magnetic surfaces) is the static $v_\theta^{(1)} = 0$ one that is unstable. The collision time for a typical laboratory plasma is larger than given by this value of μ. Thus the discussions of systems with weak discontinuities (HAZELTINE et al., 1971; BOWERS and WINSOR; GREENE and WINSOR) may be more pertinent than had been appreciated.

VI DISCUSSION

The major emphasis of this work has been to separate the discussion of plasma diffusion into two unique parts - equilibrium and stability. Obviously, calculation of diffusion has significance only if the system is in a stable equilibrium state. If one starts from an initial plasma distribution which is not a stable equilibrium, the condition that no net current flows from a magnetic surface requires that the poloidal flow must change on a time scale associated with the propagation of acoustic waves (~ ε in our model). The azimuthal distribution of density, parallel flow and temperature are then adjusted on the same time scale to satisfy the fluid equations. In turn, the non-linear character of these equations implies modifications to how v_θ must change.

In the model treated here considerable care was taken in choosing an expansion to describe the effects of ion-ion collisions. Braginskii's stress tensor (1965) with $\Omega_i \tau_i \gg 1$ provides a good representation provided $v_{th} \tau_i / R < 1$. In this limit the important viscous term arises from parallel viscosity. This term also contributes to the components of the momentum equation perpendicular to the field due to the curvature of the magnetic field lines. This feature modifies both the equilibrium condition and the stability criterion.

In order to keep things as simple as possible we have left many interesting and important physical effects out of the model. The major omission is the neglect of terms associated with finite gyration radius (the nondiagonal pressure tensor terms in the momentum equation and the Hall terms and the electron pressure gradient in Ohm's law). These terms provide significant modifications. The main effect of the finite gyration radius terms in the pressure tensor is to alter the inertial term in the component of the momentum equation along the field (STRINGER, 1969). The additional terms in Ohm's law can make the acoustic mode unstable (the familiar drift instability) under some conditions. Since these effects have been well explored (BOWERS and WINSOR), we chose to avoid including them in the calculation.

One should investigate the consequences of an anisotropic pressure, $T_\perp \neq T_\parallel$. In toroidal configurations where the mean free path for collisions is short enough that trapped particle effects can be neglected, this effect should not be very important. However, transfer from perpendicular to parallel energy could significantly enhance the bulk viscosity (DAWSON and UMAN, 1965). To incorporate this into the model we phenomenologically introduced a parameter $\kappa \geqslant 1$ into our viscosity term. Its primary effect is to make it somewhat easier to satisfy the equilibrium condition.

One should introduce two equations to treat the effect of thermal conductivity - one for the electrons and one for the ions. As described in our discussion of the model, a single equation suffices well in both regimes of large ion-electron relaxation, where both components maintain the same temperature and the large electron thermal conductivity dominates, and small relaxation, where the electron temperature is constant on a magnetic field line and only the ion thermal conductivity affects the motion.

The most significant feature of these calculations is the strong limitation on possible equilibrium configurations imposed by the charge conservation constraint, Eq. (19). In the earlier work (GREENE et al., 1971), which utilized the idealized model without viscosity or thermal conductivity, it was necessary to have large toroidal flows on some magnetic surfaces in the system ($v_b^{(o)\prime} \neq 0$) in order to construct an equilibrium. These flows provided a mechanism which made the rotational mode unstable and could also excite a Kelvin-Helmholz instability in the acoustic mode.

The presence of either thermal conductivity or viscosity modifies the equilibrium conditions as shown in Figures 2 and 3. In particular, they allow the possibility of equilibria with no net toroidal plasma flow - as had been found earlier. The demonstration that these particular equilibria are stable had not previously been given. The transition from instability to stability indicated in these figures occurs just where it would be predicted by a calculation of forces. Thus, if $v_b^{(o)\prime}$ is greater than this value we see from the charge conservation condition that $\partial v_\theta/\partial t$ exerts a restoring force if v_θ is shifted from its equilibrium value.

Plasma diffusion occurs on the resistive time scale (ε^3), an order higher than one must go to determine the equilibrium. Since the radial flow in this order, $v_r^{(3)}$, varies sinusoidally with θ, the equilibrium calculation provides an expression for plasma diffusion - the usual classical formula first derived by PFIRSCH and SCHLÜTER (1962), enhanced by the fact that the density is larger on the outside of the torus where plasma is moving away from the magnetic axis than on the inside where plasma is returning to the column. It is important to note that convective flow occurs with $\underline{v}_\perp^{(2)}$, an order larger than the net diffusion. This flow could be strongly affected by a change in boundary conditions - such as inserting a limiter or introducing a diverter so that the plasma is well separated from a conducting wall, or even changing the electrical potential on the limiter. Since the different magnetic surfaces are decoupled in our analysis it is difficult to study this effect analytically.

It is important that stability calculations, such as the ones reported here, be carried through before one believes an expression for plasma diffusion. Since in the vicinity of an equilibrium all the possible plasma motions couple together, it is difficult to be sure that a rough estimate is correct. The situation is significantly simplified by the demonstration by ROSENBLUTH and TAYLOR (1969) that non-axially symmetric perturbations or those with $m \geqslant 2$ do not couple with the toroidal effects and undergo only ordinary acoustic motion. It is also useful to note, as they demonstrated, that if the straight configuration analogous to the toroidal system of interest is unstable with respect to an axisymmetric mode with a $\cos\theta$ dependence (drift instabilities in their model), then a meaningful equilibrium cannot even be found. This situation is analogous to the one of finding toroidal stellarator equilibria (GREENE et al., 1966), where the homogeneous part of the equation which determined the equilibrium is the same as the Euler-Lagrange equation for determining stability of the straight system.

If the nonideal effects discussed here are very small ($\lambda \to \infty$ and $\mu \to 0$), then it is possible that the inverse-aspect-ratio expansion may break down where the equilibrium condition is satisfied ($\rho^{(1)}/\rho^{(o)} \gg \varepsilon$) . In this case one should reorder v_θ in the expansion and obtain a "shock" solution in the manner of GREENE and WINSOR. The new viscosity terms associated with field line curvature are too small to affect BOWERS and WINSOR'S calculation of equilibria with shocks.

The use of numerical simulation to supplement these calculations is valuable, primarily because it is impossible to see how behaviour on neighbouring surfaces couple and because the analytic formalism is applicable only for a discussion of equilibrium states and cannot be extended to treat the time development. We have modified a simulation code (WINSOR et al., 1970) to include the effect of thermal conductivity and used it to follow the behaviour of a system near the calculated equilibrium. As one should expect, we were able to observe some acoustic motion (present in the initial conditions) damp on the proper time scale and the plasma gradually diffuse. It should be easy to incorporate these changes into the more general code (BOWERS and WINSOR, 1970) that contains finite gyration radius effects.

We are presently modifying the numerical code (WINSOR et al., 1970) so that it can be used to follow plasma motion in a more realistic field geometry - a spherator or levitron.

Since the plasma flow is intimately connected with the curvature of the magnetic field lines, it should be useful to see the importance of these effects. It would be difficult to make much progress analytically because one cannot utilize a small inverse-aspect-ratio expansion.

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REFERENCES

BOWERS, E. and WINSOR, N.K. (to be published in the Proceedings of the 4th Conf. on Numerical Simulation of Plasmas, NRL, 2 - 3 Nov. 1970.)

BOWERS, E.C. and WINSOR, N.K. (submitted for publication to Phys. Fluids).

BRAGINSKII, S.I. Reviews of Plasma Physics (Consultants Bureau, New York, 1965), vol. 1, p.205.

BYDDER, E.L. and LILEY, B.S. Aust. J. Phys., 21, 609, (1968).

CONNOR, J.W. and STRINGER, T.E. (submitted for publication to Phys. Fluids).

DAWSON, J.M., WINSOR, N.K., BOWERS, E.C. and JOHNSON, J.L. IV European Conf. on Controlled Fusion and Plasma Physics, (C.N.E.N., Rome, 1970) p.15.

DAWSON, J.M. and UMAN, M.J. Nuclear Fusion 5, 242 (1965).

GALEEV, A.A. Zh.E.T.F. Pis. Red. 10, 353 (1969) [J.E.T.P. Letters 10, 225 (1970)].

GREENE, J.M., JOHNSON, J.L., WEIMER, K.E. and WINSOR, N.K. Phys. Fluids 14, 1258 (1971).

GREENE, J.M. and WINSOR, N.K. (to be published).

GREENE, J.M., JOHNSON, J.L. and WEIMER, K.E., Plasma Physics, (J. Nucl. Energy, Pt. C) 8, 145 (1966).

HAINES, M.G., Phys. Rev. Lett., 25, 1480 (1970).

HAZELTINE, R.D., LEE, E.P. and ROSENBLUTH, M.N. Phys. Fluids, 14, 361 (1971).

KRUSKAL, M.D., in Mathematical Models in Physical Sciences (S. Drobot, Ed.), Prentice-Hall, Englewood Cliffs, N.J., (1963), p.17.

KRUSKAL, M.D. and KULSRUD, R.M. Phys. Fluids, 1, 265 (1958).

MARTIN, T.J. (to be published).

PFIRSCH, D. and SCHLÜTER, A. Max-Planck Inst. Rep. No. MPI/PA/7/62, (1962).

POGUTSE, O.P. Nuclear Fusion, 10, 399 (1970).

ROSENBLUTH, M.N. and TAYLOR, J.B., Phys. Rev. Lett., 23, 367 (1969).

RUTHERFORD, P.H., KOVRIZHNIKH, L.M., ROSENBLUTH, M.N. and HINTON, F.L. Phys. Rev. Lett., 25, 1090 (1970).

SHKAROFSKY, I.P., JOHNSTON, T.W. and BACHYNSKI, M.P. The Particle Kinetics of Plasmas, Addison-Wesley (1966), Chapter 8.

STRINGER, T.E. Phys. Rev. Lett., 22, 770 (1969).

STRINGER, T.E. Phys. Fluids 13, 1586 (1970).

WINSOR, N.K., JOHNSON, J.L. and DAWSON, J.M. J. Comp. Phys. 6, 430 (1970).

ZEHRFELD, H.P. and GREEN, B.J., Nuclear Fusion 10, 251 (1970).

APPENDIX

Here we give explicit expressions for the first-order equilibrium variables, the equilibrium condition Eq. (19), the rates of change of $v_b^{(o)}$ and $T^{(o)}$, and the seventh-order dispersion relation for the coupled modes of Sec. IV. For completeness we retain $v_b^{(o)}$ and $v_b^{(o)\prime}$.

From Eqs. (15), (16) and (17) we find

$$
\begin{aligned}
\rho_s^{(1)} = -\frac{1}{\Gamma}\Big\{ \upsilon \Big[& v_\theta \left[(\gamma-1) f^2 T D_\gamma - D_\lambda \right] \rho' + f\rho D_\lambda v_b' \\
& - f^2 v_\theta D_\gamma \rho T' - \mu^* \bar{\lambda} f^4 T v_\theta \left[\rho T' - 2(\gamma-1) T\rho' \right] \\
& + \mu^{*2} f^4 T^2 v_\theta (v_\theta^2 + \bar{\lambda}^2) \rho' \Big] \\
& + \varepsilon f^2 \rho v_\theta T \left[\bar{\lambda}(\gamma-1) U_r + \mu^* v_\theta^2 U_\gamma + \mu^* \bar{\lambda}^2 U_T \right. \\
& \left. - \mu^+ D_\lambda \right] \Big\}, \qquad \text{(A.1)}
\end{aligned}
$$

$$
\begin{aligned}
\rho_c^{(1)} = \frac{\rho}{\Gamma}\Big\{ \upsilon \Big[& \bar{\lambda} f^2 T \left[DT'/T - (\gamma-1) d \right] - \mu^* f^2 T v_\theta^2 d_T \\
& - \mu^* \bar{\lambda}^2 f^2 T d \Big] - \varepsilon \Big[D_\lambda U_r - 2\mu^* \bar{\lambda} (\gamma-1) f^4 T^2 v_\theta^2 \\
& - 2\mu^{*2} f^4 T^2 v_\theta^2 (v_\theta^2 + \bar{\lambda}^2) \\
& + \mu^+ f^3 v_\theta^2 T^2 \left[\bar{\lambda}(\gamma-1) + \mu^* (v_\theta^2 + \bar{\lambda}^2) \right] \Big] \Big\}, \qquad \text{(A.2)}
\end{aligned}
$$

$$
\begin{aligned}
v_{bs}^{(1)} = -\frac{1}{\Gamma}\Big\{ & \upsilon \left[D_\lambda d/f + f v_\theta^2 D_\gamma T' + \mu^* \bar{\lambda} v_\theta^2 f^3 T T' \right] \\
& - \varepsilon f T v_\theta^2 \left[\bar{\lambda} (\gamma-1) U_r + \mu^* v_\theta^2 U_\gamma + \mu^* \bar{\lambda}^2 U_T \right. \\
& \left. + \mu^+ D_\lambda \right] \Big\}, \qquad \text{(A.3)}
\end{aligned}
$$

$$v_{bc}^{(1)} = -\frac{v_\theta}{\Gamma}\Big\{\upsilon\Big[\bar\lambda f T[DT'/T - (\gamma-1)d] - \mu^* f T v_\theta{}^2 d_T - \mu^*\bar\lambda^2 f T d\Big]$$

$$- \varepsilon\Big[v_\theta^2 D_\gamma U_\gamma/f + \bar\lambda^2 D\, U_T/f + 2\mu^*\bar\lambda(\gamma-1)f^3T^2v_\theta{}^2$$

$$+ \mu^+[\bar\lambda(\gamma-1) + \mu^*(v_\theta{}^2 + \bar\lambda^2)]f^3T^2v_\theta{}^2\Big]\Big\}, \quad (A.4)$$

$$T_s^{(1)} = \frac{v_\theta T}{\Gamma}\Big\{\upsilon\Big[D\gamma[(\gamma-1)d - DT'/T] + \mu^*\bar\lambda(\gamma-1)f^2Td$$

$$- \mu^{*2}f^4T v_\theta{}^2 T'\Big] + (\gamma-1)\varepsilon\Big[\bar\lambda D\, Ur - \mu^* f^2 T v_\theta{}^2 U_\gamma$$

$$- 2\mu^{*2}\bar\lambda f^4T^2 v_\theta{}^2 + \mu^+(\gamma-1)f^2T v_\theta{}^2(D_\gamma + \bar\lambda\mu^* f\ T)\Big]\Big\}$$

(A.5)

and

$$T_c^{(1)} = -\frac{T}{\Gamma}\Big\{\upsilon\Big[\bar\lambda D\,[DT'/T - (\gamma-1)d] + \mu^*(\gamma-1)f^2T v_\theta{}^2 d_T$$

$$+ \mu^{*2}\bar\lambda f^2T v_\theta{}^2 T'\Big]$$

$$+ \varepsilon(\gamma-1)v_\theta^2\Big[D_\gamma U_r + \mu^*\bar\lambda f^2T U_T - 2\mu^* f^4T^2 v_\theta^2$$

$$+ \mu^+(\gamma-1)f^2T v_\theta{}^2[\bar\lambda D + \mu^* f^2 T v_\theta{}^2]\Big]\Big\}. \quad (A.6)$$

Here Γ is given by Eq. (18) and

$\upsilon \equiv \eta c^2 r^2[\rho(v_b{}^2 + 2T)]'/f^2B_o{}^2R$,

$D \equiv f^2T - v_\theta{}^2$, $\quad D_\gamma \equiv \gamma f^2T - v_\theta{}^2$, $\quad D_\lambda \equiv \bar\lambda^2 D + v_\theta{}^2 D_\gamma$,

$Ur \equiv 2v_\theta{}^2 - 2f v_\theta v_b + f^2 v_b{}^2$,

$U_\gamma \equiv 2D_\gamma + Ur$, $\quad U_T \equiv 2D + Ur$,

$d_T \equiv f^2T\rho'/\rho - f v_\theta v_b' + f^2T'$,

$d \equiv d_T - f^2T'$, $\quad \mu^* \equiv 2\mu(1 - \kappa/3)/r$,

$\mu^+ \equiv 2\mu(1 - 2\kappa/3)/r$, $\quad \bar\lambda \equiv \lambda f^2/\rho r$.

To simplify notation we omitted the ordering label from the plasma parameters (all of which are of lowest order in ε).

The equilibrium condition represented by Eq. (19) is

$$
\begin{aligned}
&\frac{\upsilon}{\varepsilon}\Big\{\bar{\lambda}^2 D[(2Tv_\theta + v_\theta v_b^2 - 2fTv_b)\rho'/\rho - U_T v_b'/f] \\
&\quad + v_\theta D_\gamma[(UrT - D_\gamma v_b^2)\rho'/\rho - v_\theta U_\gamma v_b'/f + Ur\,T'] \\
&\quad - \mu^* \bar{\lambda} f^2 T v_\theta\,[2(\gamma-1)f^2 T(v_b^2 + T)\rho'/\rho + 2(\gamma-1)fTv_\theta v_b' \\
&\qquad\qquad - U_T T' + \tfrac{1}{2}(DT' - (\gamma-1)dT)] \\
&\quad - \mu^{*2} f^2 T^2 v_\theta^3\,[2f^2 T' + f^2(v_b^2 + 2T)\rho'/\rho - \tfrac{1}{2} d_T] \\
&\quad - \mu^{*2}\bar{\lambda}^2 f^2 T^2 v_\theta\,[f^2(v_b^2 + 2T)\rho'/\rho - \tfrac{1}{2} d]\Big\} \\
&- \Big\{\bar{\lambda}(\gamma-1)v_\theta T\,Ur^2 + \mu^* T v_\theta^3 U_\gamma (U_\gamma - \tfrac{1}{2}D_\gamma) \\
&\quad + \mu^*\bar{\lambda}^2 T v_\theta (U_T - \tfrac{1}{2}D)U_T + 3\mu^{*2}\bar{\lambda}(\gamma-1)f^4 T^3 v_\theta^3 \\
&\quad - \mu^+ v_\theta (v_\theta^2 T D_\gamma U_\gamma - \tfrac{1}{2} T\Gamma) + \mu^+\bar{\lambda}^2 v_\theta T D U_T \\
&\quad - 5/2\,\mu^+\mu^*\bar{\lambda}(\gamma-1)f^4 T^3 v_\theta^3 \\
&\quad - \tfrac{1}{2}\mu^{*2}\mu^+(v_\theta^2 + \bar{\lambda}^2) f^4 T^3 v_\theta^3\Big\} = 0\,. \qquad \text{(A.7)}
\end{aligned}
$$

Expressions for the rate of change of angular momentum and heat flow are quite complicated. They can be obtained from:

$$\frac{\partial v_b^{(o)}}{\partial t} = \frac{\eta c^2}{B_o^2} (\rho T)'' v_b' - \frac{\eta c^2 r}{f^2 B_o^2 R} [\rho (v_b^2 + 2T)]' v_b' \left(\frac{\rho_c^{(1)}}{\rho} - \frac{r}{2R} \right)$$

$$+ \frac{\eta c^2}{B_o^2 R^2} \left[\frac{r^2 [\rho (v_b^2 + 2T)]' v_b}{f^2} \right]'$$

$$- \frac{\eta c^2}{B_o^2 R r} \left[\frac{r^2 [\rho (v_b^2 + 2T)]' v_{bc}^{(1)}}{f^2} \right]$$

$$- \frac{v_\theta}{r} \left\langle \frac{\rho^{(1)}}{\rho} \frac{\partial v_b^{(1)}}{\partial \theta} \right\rangle - \frac{(2 v_s + f v_b)}{R} v_{bs}^{(1)} - \frac{v_\theta v_b}{R} \frac{\rho_s^{(1)}}{\rho}$$

$$+ \frac{\mu^* f^2}{r} \left\langle \frac{\partial v_b^{(1)}}{\partial \theta} \left(T^{(1)} + \frac{T}{\rho} \rho^{(1)} \right) \right\rangle$$

$$+ \frac{\mu^* f^2}{2R} T v_{bs}^{(1)} + \frac{\mu^+ f}{R} v_\theta \left(T_s^{(1)} + \frac{T}{\rho} \rho_s^{(1)} \right) \qquad (A.8)$$

and

$$\frac{\partial T^{(o)}}{\partial t} = \frac{\eta c^2}{B_o^2} (\rho T)' T' - \frac{\eta c^2 r}{f^2 B_o^2 R} [\rho (v_b^2 + 2T)]' T' \left(\frac{\rho_c^{(1)}}{\rho} - \frac{3r}{2R} \right)$$

$$+ (\gamma - 1) \frac{\eta c^2}{B_o^2} \frac{T}{r} [r (\rho T)']'$$

$$+ \frac{3(\gamma - 1)}{2} \frac{\eta c^2}{B_o^2 R^2} \frac{T}{r} \left[\frac{r^3 [\rho (v_b^2 + 2T)]'}{f^2} \right]'$$

$$- \frac{\eta c^2}{B_o^2 R r} \left[\frac{r^2 [\rho (v_b^2 + 2T)]' T_c^{(1)}}{f^2} \right]'$$

$$- \frac{v_\theta}{r} \left\langle \frac{\rho^{(1)}}{\rho} \frac{\partial T^{(1)}}{\partial \theta} \right\rangle + (\gamma - 2) \frac{f}{r} \left\langle v_b^{(1)} \frac{\partial T^{(1)}}{\partial \theta} \right\rangle$$

$$- (\gamma - 1) \frac{Tf}{\rho r} \left\langle \rho^{(1)} \frac{\partial v_b^{(1)}}{\partial \theta} \right\rangle . \qquad (A.9)$$

Unless explicitly labelled, the plasma parameters referred to here, are those of lowest order in ε.

Finally, from the determinant of the coefficients of the stability equations, Eqs. (21) through (24), we obtain the dispersion relation,

$$\begin{vmatrix} -i\omega + \varepsilon\mu^{+}T/R & -(v_b^2+2T)/2\rho R & 0 & -v_b/R & -\mu^{*}fT/R & -\frac{1}{R} & 0 \\ \frac{2\rho}{R} - \frac{\rho_c{}^{(1)}}{r} & -i\omega & -v_\theta/r & 0 & -f\rho/r & 0 & 0 \\ \rho_s{}^{(1)}/r & v_\theta/r & -i\omega & f\rho/r & 0 & 0 & 0 \\ 2v_b/R - v_{bc}{}^{(1)}/r & 0 & -fT/\rho r & -i\omega + \bar{\mu}/r & -v_\theta/r & 0 & -f/r \\ v_{bs}{}^{(1)}/r - \mu^{+}fT/R & fT/\rho r & 0 & v_\theta/r & -i\omega + \bar{\mu}/r & f/r & 0 \\ 2(\gamma-1)T/R - T_c{}^{(1)}/r & 0 & 0 & 0 & -(\gamma-1)fT/r & -i\omega + \bar{\lambda}/r & -v_\theta/r \\ T_s{}^{(1)}/r & 0 & 0 & (\gamma-1)fT/r & 0 & v_\theta/r & -i\omega+\bar{\lambda}/r \end{vmatrix} = 0 . \qquad (A.10)$$

Since this determinant can be written in the form $|- i\omega\,\delta + A| = 0$, where δ is the unit matrix and A is a real matrix, $-\omega^{*}$ is a root if ω is. This result simplifies the problem to that of finding the roots only in the right half (including the imaginary axis) of the complex ω plane.

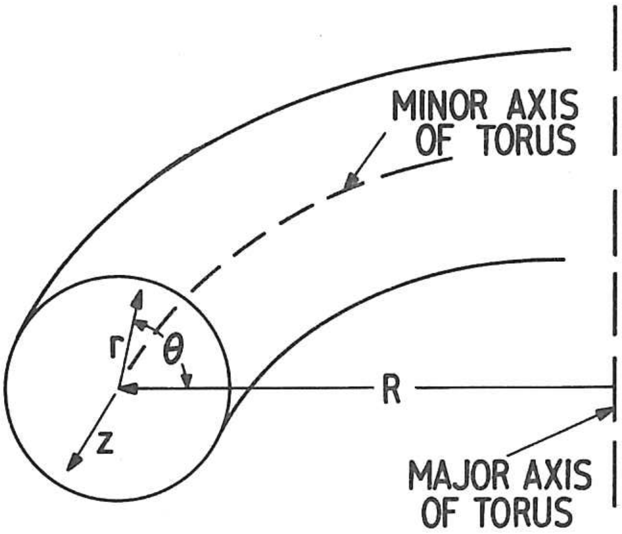

Fig.1. The toroidal coordinate system.

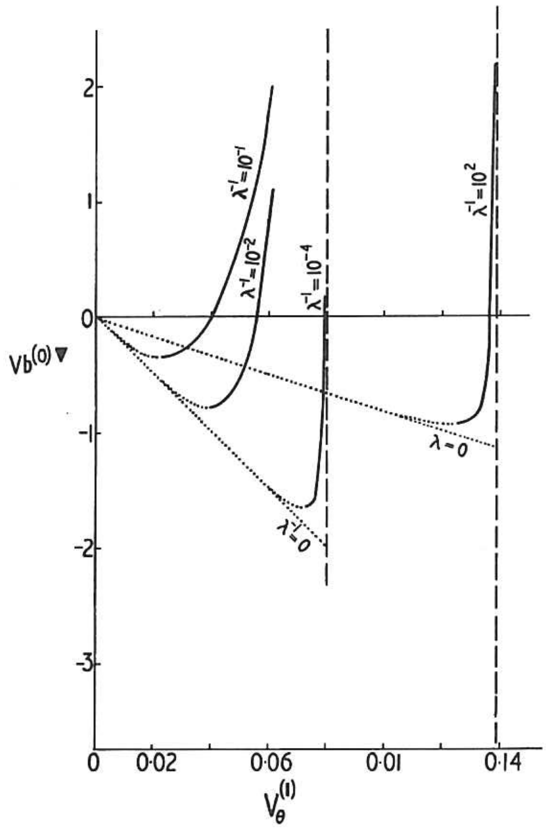

Fig.2. Values of $v_b^{(0)\prime}$ satisfying the equilibrium condition Eq.(19) as a function of $v_\theta^{(1)}$ for finite thermal conductivity. In the units of GREENE et al., (1971), $\rho^{(0)} = 1$, $\rho^{(0)\prime} = -2$, $T^{(0)} = 1$, $v_b^{(0)} = 0$, $r = 0.8$, $f = 0.08$, $R = 20.0$, $\eta = 1.35 \times 10^{-5}$, $\gamma = 3$, $\mu = 0$. The dotted parts of the curves are unstable.

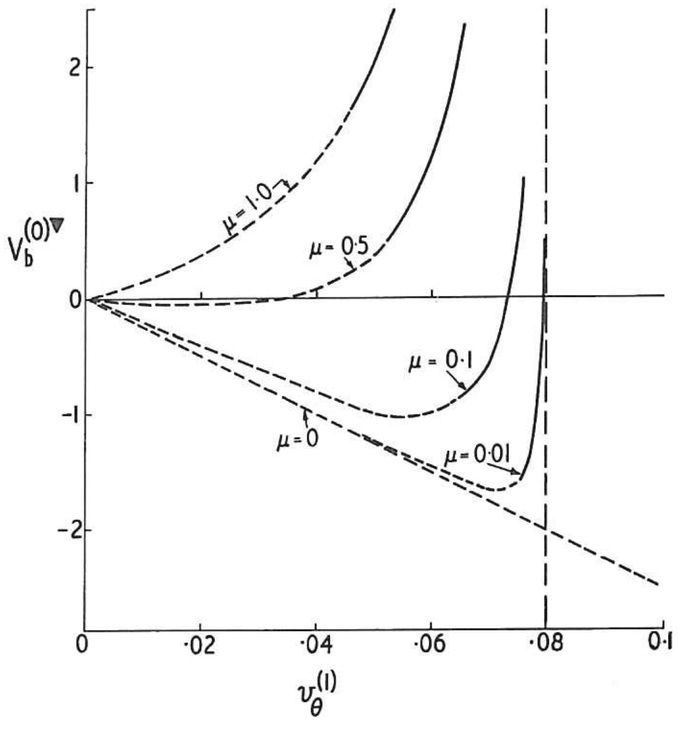

Fig.3. Values of $v_b^{(0)\prime}$ satisfying the equilibrium condition Eq.(19) as a function of $v_\theta^{(1)}$ for infinite thermal conductivity and finite viscosity. The remaining parameters have the values in Figure 2. The dotted parts of the curves are unstable.