CULHAM LABORATORY
LIBRARY
17 APR 1972

CULHAM LIBRARY
REFERENCE ONLY

CLM - P 299

This document is intended for publication in a journal, and is made available on the understanding that extracts or references will not be published prior to publication of the original, without the consent of the authors.

UKAEA RESEARCH GROUP

Preprint

THE ALFVÉN CYCLOTRON RESONANCE INSTABILITY IN MIRROR PLASMAS

J G CORDEY
R J HASTIE

CULHAM LABORATORY
Abingdon Berkshire

1972

Enquiries about copyright and reproduction should be addressed to the
Librarian, UKAEA, Culham Laboratory, Abingdon, Berkshire, England

CLM-P 299

THE ALFVÉN CYCLOTRON RESONANCE INSTABILITY IN MIRROR PLASMAS

by

J.G. Cordey and R.J. Hastie

(Submitted for publication in Physics of Fluids)

ABSTRACT

The stability of the collisional distribution of a mirror machine against the Alfvén cyclotron resonance mode is examined. This instability which is shown to have a growth rate which scales as $\gamma \sim \omega_{ci} \exp(-R^{3/4}/\beta^{1/4})$ for $R\beta < 1$ may be dangerous for low beta mirror plasmas as well as high beta plasmas.

UKAEA Research Group,
Culham Laboratory,
Abingdon,
Berks.

January, 1972

1. INTRODUCTION

The theory of electrostatic microinstabilities in mirror machines is now virtually complete[1] and a start has now been made on the investigation of electromagnetic modes. In this note we examine an electromagnetic instability - the Alfvén cyclotron resonance instability[2], [3] and show that this instability can be quite serious for mirror confinement even at very low beta.

The Alfvén Cyclotron resonance instability was first examined by Rosenbluth[2] and independently by Sagdeev and Shafranov[3] for a two temperature Maxwellian and it was found that the growth rate scaled as $\frac{\gamma}{\omega_{ci}} \sim \exp - \left(\frac{1}{\beta (T_\perp / T_\parallel - 1)^2} \right)$, and so for plasmas which were close to isotropy the growth rate was quite small. Since then there have been several papers[4 - 6] concerned with instabilities caused by the non-Maxwellian distribution of electrons, but as far as the authors are aware only Scharer and Trivelpiece[5] have looked at instabilities due to the non-Maxwellian nature of the ion distribution. These authors found the threshold for instability of a two temperature Maxwellian plasma.

In this paper we examine the stability of the ion collisional distribution[7] of a mirror plasma and the growth rate of the Alfvén cyclotron resonance instability is determined as a function of the plasma β and the mirror ratio R.

In section 2 an analytic theory is presented in the limit of small $k_\parallel \rho_i$ (ρ_i = ion larmor radius). This shows that for large mirror ratio R , the growth rate has the asymptotic behaviour

$$\frac{\gamma}{\omega_{ci}} \sim \exp\left\{-\frac{(\log \beta R)^2}{\beta}\right\}$$

while for smaller values of R such that $\beta R < 1$ the growth rate has the form

$$\frac{\gamma}{\omega_{ci}} \sim \exp\left\{-\frac{R^{3/4}}{\beta^{1/4}}\right\} .$$

Growth rates are calculated for various β and R, using the small $k\rho$ approximation to calculate the real part of the dispersion equation. Finally in section 3 an exact treatment of the dispersion equation for marginal stability ($\gamma = 0$) is given. This provides a check for the results of section 2, which are found to be accurate over most of the parameter range considered.

2. ANALYTIC TREATMENT OF THE DISPERSION RELATION

The relevant dispersion relation for electromagnetic modes propagating along a uniform magnetic field is[8]

$$\omega^2 = k^2c^2 + \sum_j \omega_{pj}^2 \int \left[\frac{(\omega + kv_{\parallel})f}{(\omega \pm \omega_{cj} + kv_{\parallel})} + \frac{\frac{1}{2}k^2 v_{\perp}^2 f}{(\omega \pm \omega_{cj} + kv_{\parallel})^2} \right] d^3v \tag{1}$$

where $\omega_{pj}^2 = \frac{4\pi n e_j^2}{m_j}$, $k \equiv k_{\parallel}$, and f has been normalised so that

$$\int f \, d^3v = 1 . \tag{2}$$

The $\pm$ signs refer to the polarization of the mode. In the following we take a form of the collisional distribution function[7] [9] appropriate in a mirror machine of mirror ratio R.

$$f = N e^{-\frac{v^2}{\alpha^2}} \left[\frac{R v_\perp^2}{v^2} - 1 + (2R - 3) \log \frac{v_\perp^2 R}{v^2} \right] \qquad (3)$$

where the normalization factor is given by

$$(\pi^{3/2} \alpha^3 N) = \left\{ (2R - 3) \left[\log R + 2 \log \left(1 + \sqrt{\frac{R-1}{R}} \right) \right] + \frac{2}{3} \sqrt{\frac{R-1}{R}} \, (8 - 5R) \right\}^{-1} . \qquad (4)$$

In equation (1) we shall neglect ω^2 compared to $k^2 c^2$, take $\omega + k v_\parallel \ll \omega_{ce}$ in the electron term, and $\omega \pm \omega_{ci} \gg k v_\parallel$ in the calculation of the real part of the ion term. The resulting (real) dispersion equation is

$$0 = \omega_p^2 \left\{ \frac{3 k^2 \rho^2}{2\beta} \mp \frac{\left(\frac{\omega}{\omega_c}\right)^2}{\left(\frac{\omega}{\omega_c} \pm 1\right)} + k^2 \rho^2 G(R) \left[\frac{1}{\left(\frac{\omega}{\omega_c} \pm 1\right)^2} + \frac{T_e}{T_i} \right] \right\} \qquad (5)$$

where $\omega_p \equiv \omega_{pi}$; $\omega_c \equiv \omega_{ci}$; $\rho = \frac{\alpha}{\omega_{ci}}$, the ion larmor radius; $\beta = 2(p_\perp + \frac{1}{2} p_\parallel)/B^2$, is the ion β; $\frac{T_e}{T_i}$ is the temperature ratio which we will take to be zero in the following; and

$$G(R) = \int (v_\perp^2 - 2 v_\parallel^2) \frac{f}{2\alpha^2} d^3 v$$

$$= \frac{6}{5} (N \pi^{3/2} \alpha^3) R \left(1 - \frac{1}{R}\right)^{5/2} \qquad (6)$$

is the anisotropy term which gives rise to the firehose instability when sufficiently negative. It follows from (6) that G(R) is always positive for distributions of the form given by (3).

Assuming moderately small values of β (and the upper polarization sign) equation (5) has the solutions :

$$\frac{\omega}{\omega_c} = \pm \sqrt{\frac{3}{2\beta}}\, k\rho \quad ; \quad k^2\rho^2 < \frac{2}{3}\beta \tag{7}$$

which are the two branches of the shear Alfven wave, separating into

$$\frac{\omega}{\omega_c} = + \frac{3k^2\rho^2}{2\beta} \quad ; \quad k^2\rho^2 > \frac{2}{3}\beta \tag{8}$$

the whistler mode, and

$$\frac{\omega}{\omega_c} = - 1 + \frac{2\beta}{3k^2\rho^2} \quad ; \quad \frac{2}{3}\beta < k^2\rho^2 < \left(\frac{2}{3}\beta \right)^{2/3} \tag{9}$$

a cyclotron mode.

For large values of $\beta \gtrsim 1$, the Alfvén mode continues up to $k^2\rho^2 \lesssim 1$ as

$$\frac{\omega}{\omega_c} = \pm\, k\rho \sqrt{\frac{3}{2\beta} + G(R)\left(1 + \frac{T_e}{T_i}\right)}\ . \tag{10}$$

These modes are shown in Figure 1.

Of these modes the whistler is right hand circularly polarised and interacts strongly with the electrons at the electron cyclotron frequency(4) (5) (6). The ion cyclotron mode is left hand circularly polarised and interacts with the ions.

To determine the growth rate of these modes for frequencies near the ion cyclotron frequency we require the imaginary contribution to the dispersion equation from the ion cyclotron resonance. The ion term can be simplified by first integrating the final term by parts with respect to $v_{\parallel}$, and using the relation

$$\frac{\partial f}{\partial v_{\parallel}} = 2v_{\parallel} \frac{\partial f}{\partial {v_{\perp}}^2} - 2v_{\parallel} N e^{-\frac{v^2}{\alpha^2}} \left[\frac{R}{v^2} + \frac{(2R - 3)}{{v_{\perp}}^2} \right] . \tag{11}$$

The resulting ion term reduces to the form

$$\omega_p^2 (\pi^{3/2}\alpha^3 N) \left\{ -\frac{8}{3} R(1-\frac{1}{R})^{3/2} + \frac{1}{\sqrt{\pi}\, k\rho} \int_{-\infty}^{+\infty} \frac{A(u^2) e^{-u^2} du}{(u + U_c)} \right\} \quad (12)$$

where $U_c \equiv \sqrt{\frac{R}{R-1}\left(\frac{\omega}{\omega_c} \pm 1\right)} \Big/ k\rho$ and

$$A(u^2) = \left(\frac{\omega}{\omega_c} \pm 1\right)(R-1)\left[3 - u^2 e^{u^2} E_1(u^2)\right]$$
$$+ \left(\frac{\omega}{\omega_c}\right)\left[(R-1)(1 - u^2 e^{u^2} E_1(u^2)) + (2R-3)\left\{e^{\frac{u^2}{R}} E_1\left(\frac{u^2}{R}\right) - e^{u^2} E_1(u^2)\right\}\right] \quad (13)$$

with

$$E_1(u^2) = \int_{u^2}^{\infty} \frac{e^{-t}}{t}\, dt .$$

This contributes an imaginary term D_i to equation (5)

$$D_i = \omega_{pi}^2 \, (\pi^{3/2}\alpha^3 N) \frac{\sqrt{\pi}}{k\rho} e^{-U_c^2} A(U_c^2) \quad (14)$$

and consequently a growth (or damping) given by

$$\frac{\gamma}{\omega_c} = (\pi^{3/2}\alpha^3 N) \frac{\sqrt{\pi}}{k\rho} e^{-U_c^2} A(U_c^2) \Big/ \left\{ \mp \frac{2\left(\frac{\omega}{\omega_c}\right)}{\left(\frac{\omega}{\omega_c} \pm 1\right)} \pm \frac{\left(\frac{\omega}{\omega_c}\right)^2}{\left(\frac{\omega}{\omega_c} \pm 1\right)^2} - \frac{2k^2\rho^2 G(R)}{\left(\frac{\omega}{\omega_c} \pm 1\right)^3} \right\} . \quad (15)$$

3. STABILITY

In the following we first look at the right-hand circularly polarized wave which is the whistler mode and then the left-hand wave which is the ion cyclotron mode.

1. Whistler Mode ; $\frac{\omega}{\omega_c} = + \frac{3}{2} \frac{k^2\rho^2}{\beta}$.

For this mode $A(U_c^2) > 0$ and $\gamma < 0$, i.e. this mode is always damped.

2. Ion cyclotron mode ; $\frac{\omega}{\omega_c} \sim -1 + \frac{2\beta}{3k^2\rho^2}$.

The denominator of the right hand side of equation (15) is positive, and for large enough R such that $\frac{U_c^2}{R} \lesssim 1$, while $U_c^2 \gg 1$, $A(U_c^2)$ is given approximately by

$$A(U_c^2) \propto \frac{2\beta}{3k^2\rho^2} - e^{\frac{U_c^2}{R}} E_1\left(\frac{U_c^2}{R}\right) < 0 \qquad (16)$$

so $\gamma < 0$ and the wave is damped.

However for moderate values of R such that $\frac{U_c^2}{R} \gg 1$, $A(U_c^2)$ is approximately given by

$$A(U_c^2) \propto 2\left(\frac{\omega}{\omega_c} + 1\right) + \frac{1}{U_c^2}\left[\left(\frac{\omega}{\omega_c} + 1\right) + 2(R-1)\frac{\omega}{\omega_c}\right] \qquad (17)$$

and γ becomes positive (growth) for

$$k^2\rho^2 < \left(\frac{2\beta}{3}\right)^{3/4} \frac{R^{1/4}}{(R-1)^{1/2}} \qquad (18)$$

provided that the real part of ω is still correctly given by equation (9) in this range, i.e. provided that

$$\frac{2\beta}{3}\frac{(R-1)^2}{R} < 1 . \qquad (19)$$

If this last condition is violated the marginal stability curve intersects the Alfvén section of the dispersion curve rather than the ion cyclotron section. The magnitude of the growth rate is dominated by the factor $e^{-U_c^2}$ which near marginal stability takes the value

$$\frac{\gamma}{\omega_c} \sim \exp\left\{-\left(\frac{3}{2\beta}\right)^{1/4} \frac{(R-1)^{3/2}}{R^{3/4}}\right\} . \qquad (20)$$

For existing mirror machines of moderate mirror ratio and very low β, this mode will be unstable unless the finite length criterion associated with the stability condition (18) is satisfied. This condition requires

$$\beta < \frac{3}{2} \frac{(R-1)^{2/3}}{R^{1/3}} \left(\frac{\pi \rho_i}{L}\right)^{8/3} \qquad (21)$$

for stability, where L is the length between mirrors.

3. Alfvén waves ; $\frac{\omega}{\omega_c} = \pm k\rho \sqrt{\frac{3}{2\beta}}$.

Of these modes the positive one which is the continuation of the Whistler mode discussed above is always damped, while the other mode (which continues into the ion cyclotron mode) is unstable if

$$A(U_c^2) \propto 1 - \frac{\omega}{\omega_c} e^{\frac{U_c^2}{R}} E_1\left(\frac{U_c^2}{R}\right) > 0 \ . \qquad (22)$$

For moderate values of R, such that $\frac{U_c^2}{R} \gtrsim 1$, this is always satisfied, and if $\frac{U_c^2}{R} < 1$ it is satisfied if

$$k\rho < \sqrt{\frac{2\beta}{3}} \Big/ \log\left(\frac{2\beta R}{3}\right) , \qquad (23)$$

In this case the amplitude of the growth rate near marginal stability is determined by

$$\frac{\gamma}{\omega_c} \sim \exp\left\{ - \frac{3}{2\beta}\left(\log\left(\frac{2\beta R}{3}\right)\right)^2 \right\} . \qquad (24)$$

The result of the foregoing analysis is that one branch of the Alfvén wave, and its continuation as an ion cyclotron wave, are unstable at long wavelengths with growth rates depending on R and β as outlined above.

Using equation (15) we have calculated $\frac{\gamma}{\omega_c}$ for various choices of R and β, and in Figure 1 is shown a plot of $\frac{\gamma}{\omega_c}$ against $\frac{\omega}{\omega_{ci}}$ for $R = 10$, $\beta = 1$. In general $\frac{\gamma}{\omega_c} \to 0$ exponentially as $k\rho \to 0$, and attains a maximum value near $k\rho = (k\rho)_{crit}$ the marginal stability value of $k\rho$. In Table 1 we give values of this maximum of $\frac{\gamma}{\omega_c}$, and of the corresponding values of $\frac{\omega}{\omega_c}$, $k\rho$ and U_c^2. Since the small parameter involved in the expansion of the ion term is U_c^{-2}, this last figure gives one indication of the reliability of the results. However in the next section we also give an exact treatment of the dispersion equation along the marginal stability curve, and compare the values of U_c^2 at marginal stability obtained by the two methods.

4. EXACT TREATMENT AT MARGINAL STABILITY

Returning to equations (12), (13) and (14) we see that the marginal stability curve is defined by the equation

$$A(U_c^2) = 0 ,$$

Now, taking $U_c \equiv \left(\frac{\omega}{\omega_c} \pm 1\right) \sqrt{\frac{R}{R-1}} \Big/ k\rho$, and $k\rho$ as the independent variables instead of $\frac{\omega}{\omega_c}$ and $k\rho$, the solution of this equation is simply

$$k\rho = \pm \sqrt{\frac{R}{R-1}} \, \frac{F(U_c^2)}{G(U_c^2)} \, \frac{1}{U_c}$$

where

$$F(U_c^2) = (R-1)\left(1 - U_c^2 e^{U_c^2} E_1(U_c^2)\right) + (2R-3)\left(e^{\frac{U_c^2}{R}} E_1\left(\frac{U_c^2}{R}\right) - e^{U_c^2} E_1(U_c^2)\right)$$

$$G(U_c^2) = (R-1)(3 - U_c^2 e^{U_c^2} E_1(U_c^2)) + F(U_c^2) \qquad (25)$$

Using this result explicitly in expression (12) for the ion contribution, this becomes

$$\omega_p^2 \left(\pi^{3/2} \alpha^3 N\right)\left\{-\frac{4}{3} R\left(1-\frac{1}{R}\right)^{3/2} + U_c \frac{\left(1-\frac{1}{R}\right)^{1/2}}{\pi^{1/2}} \int_{-\infty}^{\infty} e^{-u^2} \frac{\left[G(u^2) - \frac{G(U_c^2)}{F(U_c^2)} F(u^2)\right]}{(U + U_c)} du\right\} \tag{26}$$

where the singularity of the integrand in the second term has now been removed, and for a given value of U_c the integral may be computed without difficulty.

The full dispersion relation now takes the form

$$D(U_c) = 0 \tag{27}$$

where

$$D = \left(\frac{3}{2\beta} + \frac{T_e}{T_i} G(R)\right) \frac{R}{R-1} \frac{F^2(U_c^2)}{G^2(U_c^2)} \frac{1}{U_c^2} \pm \left(\frac{G(U_c^2)}{F(U_c^2)} - 1\right)$$

$$+ \left(\pi^{3/2}\alpha^3 N\right)\left(1-\frac{1}{R}\right)^{\frac{1}{2}}\left\{-\frac{4}{3}(R-1) + \frac{U_c}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{\left[G(u^2) - G(U_c^2) F(u^2)/F(U_c^2)\right] du}{(u + U_c)}\right\} \tag{28}$$

and its solution gives U_c and hence $k\rho$ and $\frac{\omega}{\omega_c}$ at marginal stability. We have solved the equation for a variety of choices of R and β, and since the growth rate near marginal stability is dominated by the factor $\exp\{-U_c^2\}$, we give in Table 2 the values of U_c^2 at marginal stability. For comparison we have included in the last column the value of U_c^2 obtained (also for marginal stability) from the approximate treatment discussed in section 1.

5. CONCLUSION

The stability of the collisional distribution of a mirror machine has been examined for modes with $k_\perp = 0$.

The Alfvén cyclotron resonance instability is found to have a large growth rate even for small values of β. ($\gamma \sim \omega_{ci} \exp(-R^{3/4}/\beta^{1/4})$ for $R\beta < 1$). The analytic theory from which this result is derived is found to be in agreement with a numerical evaluation of the dispersion equation at marginal stability.

R	β	$\left(\frac{\gamma}{\omega_c}\right)_{max}$	$\left(\frac{\omega}{\omega_c}\right)$	$(k\rho)$	U_c^2
10	1	3.2×10^{-2}	0.43	0.37	2.6
10	0.5	1.6×10^{-2}	0.46	0.32	3.2
10	0.1	1.3×10^{-3}	0.55	0.20	5.5
5	1	1.0×10^{-1}	0.5	0.41	1.8
5	0.1	1.7×10^{-2}	0.61	0.23	3.6
5	0.01	2.7×10^{-4}	0.75	0.11	7.6
1.5	10^{-2}	3.1×10^{-1}	0.85	0.17	2.28
1.5	10^{-3}	3.8×10^{-2}	0.91	0.076	4.2
1.5	2×10^{-4}	3.3×10^{-3}	0.94	0.042	6.5
1.5	10^{-4}	8.1×10^{-4}	0.95	0.033	7.8
1.2	10^{-3}	5.1×10^{-1}	0.94	0.093	2.5
1.2	10^{-4}	5.2×10^{-2}	0.96	0.041	4.7
1.2	10^{-5}	8.1×10^{-4}	0.98	0.017	8.5

TABLE 1

Growth rate, frequency and wavelength of the Alfvén-cyclotron mode for various values of the mirror ratio (R) and plasma β, evaluated for the mode and maximum growth rate.

R	β	$\left(\frac{\omega}{\omega_c}\right)$ exact	$\left(\frac{\omega}{\omega_c}\right)$ approx.	exact U_c^2	approx. U_c^2
10	1	0.45	0.47	1.19	1.75
10	0.1	0.58	0.57	4.25	4.58
5	1	0.52	0.56	0.63	1.12
5	0.1	0.62	0.64	2.36	2.65
5	0.01	0.73	0.74	6.27	6.49
1.5	10^{-2}	0.87	0.87	1.25	1.45
1.5	10^{-3}	0.91	0.92	3.16	3.30
1.5	10^{-4}	0.95	0.95	6.81	6.90
1.2	10^{-3}	0.94	0.95	1.56	1.72
1.2	10^{-4}	0.97	0.97	3.66	3.78
1.2	10^{-5}	0.98	0.98	7.60	7.70
20	1	0.39	0.41	1.99	2.63
10^4	1	0.16	0.17	31.9	35.8

TABLE 2

Comparison of the approximate values of the frequency and the exponential growth factor (U_c^2) with exact results for these quantities at marginal stability ($\gamma = 0$).

REFERENCES

1. D.E. Baldwin et al., Proc. 4th. Int. Conf. Plasma Physics and Controlled Nuclear Fusion Research, Madison, 17 - 23 June, 1971. (IAEA, Vienna, 1971). Vol. II, pp. 735 - 755.
2. M.N. Rosenbluth, Risö Report No. 18, p. 189, Lecture at International Summer Course in Plasma Physics, 1960.
3. R.Z. Sagdeev and V.D. Shafranov, Soviet Physics J.E.T.P., 12, 130 (1960).
4. R.V. Sudan, Phys. Fluids 6, 57 (1963).
5. J.E. Scharer and A.W. Trivelpiece, Phys. Fluids 10, 592 (1967).
6. J.E. Scharer, Phys. Fluids 10, 652 (1967).
7. J.P. Holdren, Lawrence Radiation Laboratory report UCRL - 72391 (1971).
8. E.G. Harris, J. Nucl. Energy, Part C: Plasma Physics 2, 138 (1961).
9 J.G. Cordey, Phys. Fluids 14, 1407 (1971).

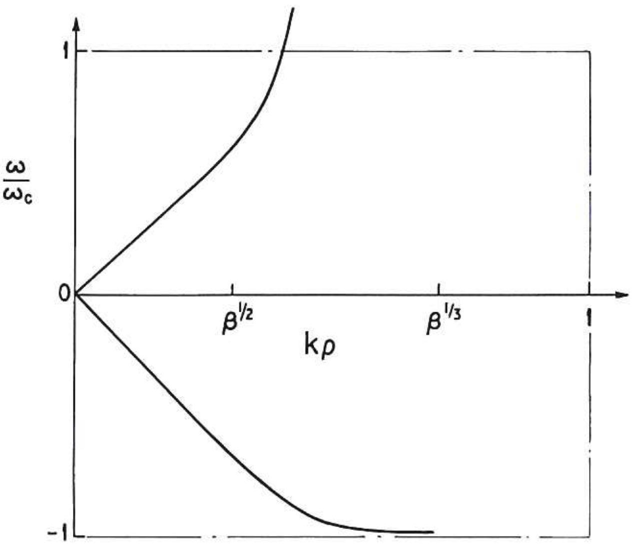

Fig.1. Dispersion diagram for Alfvén and Whistler modes when $\beta \ll 1$

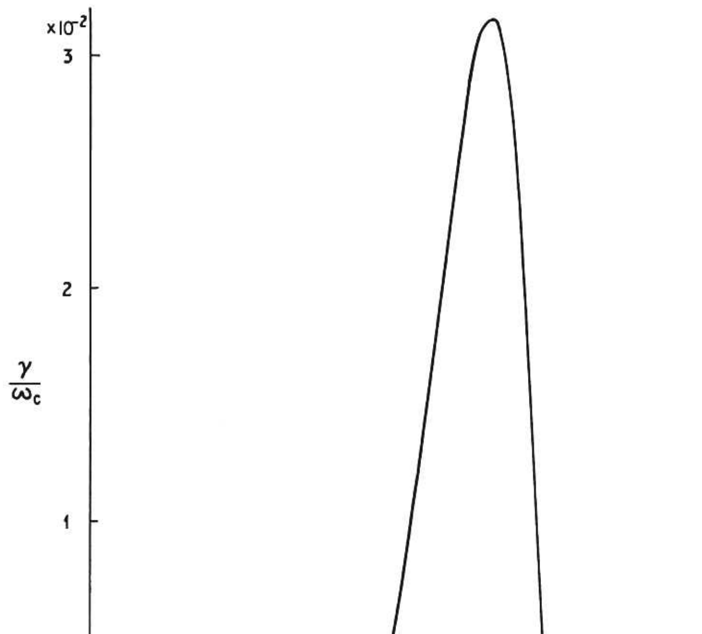

Fig.2. Growth rate as a function of frequency for $R = 10$ and $\beta = 1$