CLM - P 301

UKAEA RESEARCH GROUP

Preprint

MHD STABILITY IN A TOKAMAK

F A HAAS
J D JUKES

CULHAM LABORATORY
Abingdon Berkshire

1972

CLM-P 301

MHD STABILITY IN A TOKAMAK

by

F.A. Haas and J.D. Jukes

(Submitted for publication in Nuclear Fusion)

ABSTRACT

The energy principle is used to derive a new necessary local criterion for MHD stability in axisymmetric confined plasmas. Applying this criterion to simple equilibrium models of a Tokamak leads to $q > 2$ as a necessary condition for stability, q being the inverse rotational transform in the vicinity of the magnetic axis. The theoretically predicted limits to both q and β are found to be consistent with experimental observations on T3 and ST Tokamaks.

UKAEA Research Group,
Culham Laboratory,
Abingdon,
Berks.

March, 1972

1. INTRODUCTION

The encouraging experimental results obtained in Tokamak T3-A[1] have led to a world-wide interest in machines of this type. In recent theoretical work[2 - 4] based on the MHD equations, the upper limit to the equilibrium-β at the magnetic axis has been evaluated for plasmas with both circular and non-circular cross-sections. Most stability studies of Tokamak including toroidal effects have been concerned with the localised interchange mode, and these have also been carried through for shaped plasma cross-sections[2 - 5].

In one such calculation, Jukes and Haas[2], working directly from the energy principle, have derived a new necessary criterion for stability. Applying their condition to a circular cross-section plasma with either a quasi-uniform or parabolic toroidal current distribution, they show that for stability in the vicinity of the magnetic axis it is necessary that $q > 2$, q being the inverse rotational transform. This result is, of course, more stringent than that usually quoted ($q > 1$), and the corresponding critical-β is considerably lower than that obtained by other authors.

The purpose of the present paper is to give a more detailed account of the derivation of these, and other results, and to relate them to earlier theoretical and experimental studies. In Section 2 we describe the coordinate systems associated with a general axisymmetric toroidal plasma. In Section 3, restricting attention to localised perturbations, we derive a general necessary criterion for stability. Section 4 discusses the application of this criterion to specific equilibrium models, the latter being described in an Appendix. Sections 5 and 6 contain a discussion and summary of results respectively.

2. EQUILIBRIUM AND COORDINATE SYSTEMS

Since we shall be concerned with axisymmetrix toroidal plasmas, it is convenient to choose a cylindrical coordinate system (R, ϕ, Z) in which all equilibrium quantities are independent of ϕ. It is assumed that the magnetic surfaces form nested toroids centred on a magnetic axis located $R = R_o$, $Z = 0$, encircling the Z-axis (see Figure 1). We then define ψ as the magnetic flux threaded the short way around this magnetic axis per radian in ϕ, within a flux surface labelled by ψ. Following Mercier[6] we choose a locally orthogonal set of coordinates (ψ, ϕ, χ) on the flux surfaces (see Figure 1), such that the line elements are defined by

$$dx_1 = \frac{d\psi}{RB_\chi} , \quad dx_2 = JB_\chi d\chi , \quad dx_3 = Rd\phi , \tag{2.1}$$

the corresponding unit vectors satisfying $\underline{e}_\psi \times \underline{e}_\phi = \underline{e}_\chi$, and where B_χ is given by

$$\underline{B}_\chi = \nabla \times (\frac{\psi}{R} \underline{e}_\phi) . \tag{2.2}$$

The volume element is

$$dx_1 dx_2 dx_3 = J \, d\chi d\psi d\phi , \tag{2.3}$$

J being the Jacobian of the transformation.

The MHD equilibrium equations are

$$j \times \underline{B} = \nabla p \,, \qquad (2.4)$$

$$j = \nabla \times \underline{B} \,, \qquad (2.5)$$

$$\text{and} \quad \nabla \cdot \underline{B} = 0 \,, \qquad (2.6)$$

where the symbols have their usual meanings. Now for equilibria having axial symmetry it is well-known that

$$p = p(\psi) \text{ and } RB_{\phi} \equiv I = I(\psi) \,, \qquad (2.7)$$

where I is the current stream function. It follows that Equations (2.4) and (2.5) can be written as

$$p' = \frac{j_{\phi}}{R} - \frac{II'}{R^2} \,, \qquad (2.8)$$

$$\text{and} \quad j_{\phi} = -\frac{R}{J}\frac{\partial}{\partial\psi}(JB^2) \,, \qquad (2.9)$$

respectively, the prime denoting $d/d\psi$. Working in (R, ϕ, Z) coordinates, Equations (2.4) and (2.5) can be combined to give[7]

$$R\frac{\partial}{\partial r}\left(\frac{1}{R}\;\frac{\partial\psi}{\partial R}\right) + \frac{\partial^2\psi}{\partial Z^2} = -Rj_{\phi} = -R^2p' - II' \,, \qquad (2.10)$$

which for prescribed $p(\psi)$ and $I(\psi)$ determines the equilibrium ψ-surfaces. In order to solve Equation (2.10), and to carry through the subsequent stability analysis, it has been found convenient to transform the local polar coordinates (r, θ, ϕ) based on the magnetic axis (see Figure 2), that is, we take

$$R = R_o + r\cos\theta \quad \text{and} \quad Z = r\sin\theta \,. \qquad (2.11)$$

Assuming the inverse aspect ratio of the torus, κ, to be small, the solution of Equation (2.10) can be developed as an expansion for any suitably prescribed $p(\psi)$ and $I(\psi)$.

Some special models for Tokamak are described in Appendix B. It should be noted that since the sense of ϕ in the sets of coordinates (ψ, ϕ, χ) and (r, θ, ϕ) is different, we shall change the sign of j_ϕ whenever we go into local polar coordinates.

3. LOCALISED INTERCHANGE MODES

Taking perturbations to vary in the ϕ coordinate as $\exp(in\phi)$, where n is an integer, the potential energy of a perturbed axisymmetric toroidal plasma can be written as[6]

$$\delta W_n = \frac{\pi}{2} \int J\, d\chi d\psi \left\{ \frac{1}{J^2R^2B^2} \left| \frac{\partial X}{\partial \chi} + i\, n\, \mu X \right|^2 + \frac{R^2}{J^2} \left| - \frac{1}{n}\frac{\partial U}{\partial \chi} + i\, I\, \frac{\partial}{\partial \psi}\left(\frac{JX}{R^2}\right) \right|^2 \right.$$

$$+ B^2 \left| U + \frac{\partial X}{\partial \psi} + \frac{jX}{RB^2} \right|^2 + \frac{\gamma p}{J^2} \left| \frac{\partial}{\partial \psi}(JX) + JU + \frac{\partial Z}{\partial \chi} + i\, n\, \mu\, Z \right|^2$$

$$\left. - 2K|X|^2 \right\}, \qquad (3.1)$$

where X, U, Z are three independent variables expressible in terms of the plasma displacement $(\xi_\psi, \xi_\phi, \xi_\chi)$ through

$$X = RB\xi_\psi\,, \quad U = \frac{\xi_\phi}{R} - \frac{I^2}{R^2} \cdot \frac{\xi_\chi}{B}\,, \quad Z = \frac{\xi_\chi}{B}\,. \qquad (3.2)$$

Note we have dropped the subscripts from j_ϕ and B_χ i.e. $B \equiv B_\chi$ and $j \equiv j_\phi$. The local pitch $\mu\ (\psi, \chi)$ and the quantity K are defined to be

$$\mu \equiv \frac{JI}{R^2} \qquad (3.3)$$

$$\text{and} \quad K = \frac{\partial}{\partial \psi}(\ell n\, R)\, \frac{II'}{R^2} - \frac{j}{R}\, \frac{\partial}{\partial \psi}(\ell n\, (JB))\,. \qquad (3.4)$$

Minimisation with respect to Z is simply accomplished by setting the appropriate term to zero. It is now convenient to introduce a new variable V which is defined by

$$V \equiv U + \frac{\partial X}{\partial \psi} . \qquad (3.5)$$

Since X and V are complex they can be expressed as

$$\left.\begin{aligned} X &= \hat{X}(\psi, \chi) \exp[i \int m(\psi,\chi) d\chi] \\ \text{and} \quad V &= \hat{V}(\psi, \chi) \exp[i \int m(\psi,\chi) d\chi] , \end{aligned}\right\} \qquad (3.6)$$

where $\hat{X}$ and m are real functions and $\hat{V}$ is complex. In order to obtain a necessary condition for stability, we shall assume m to depend on χ alone. The potential energy becomes

$$\delta W_n = \frac{\pi}{2} \int J d\chi d\psi \left\{ \frac{1}{J^2R^2B^2} \left(\left(\frac{\partial \hat{X}}{\partial \chi}\right)^2 + (m + n\mu)^2 \hat{X}^2 \right) - 2K\hat{X}^2 \right.$$
$$+ B^2 \hat{V}_I^2 + B^2 \left(\hat{V}_R + \frac{j\hat{X}}{RB^2}\right)^2 + \frac{R^2}{n^2J^2} \left(\frac{\partial \hat{V}_R}{\partial \chi} - m\hat{V}_I - \frac{\partial^2 \hat{X}}{\partial \psi \partial \chi}\right)^2$$
$$\left. + \frac{R^2}{J^2} \left(-\frac{1}{n} \left(\frac{\partial \hat{V}_I}{\partial \chi} + m\hat{V}_R - m\frac{\partial \hat{X}}{\partial \psi}\right) + I\frac{\partial}{\partial \psi}\left(\frac{J\hat{X}}{R^2}\right)\right)^2 \right\} , \qquad (3.7)$$

where $\hat{V}_R$ and $\hat{V}_I$ denote the imaginary parts of $\hat{V}$ respectively. In this paper we shall only be concerned with modes for which $n \to \infty$, although the ratio m/n will be regarded as finite. In this limit δW can now be further minimised by setting $\hat{V}_I = 0$. Minimisation with respect to $\hat{V}_R$ is achieved for

$$\hat{V}_R = \left(m^2 + \frac{J^2B^2n^2}{R^2}\right)^{-1} \left\{ nm\left[\left(\frac{m}{n} + \mu\right)\frac{\partial \hat{X}}{\partial \psi} + I\hat{X}\frac{\partial}{\partial \psi}\left(\frac{J}{R^2}\right)\right] - \frac{J^2 j n^2 \hat{X}}{R^3} \right\} . \qquad (3.8)$$

The potential energy now becomes

$$\delta W_{\infty} = \frac{\pi}{2}\int J d\chi d\psi \Bigg\{ \frac{1}{R^2B^2}\left[\left(\frac{\partial \hat{X}}{J\partial\chi}\right)^2 + (m+n\mu)^2\frac{\hat{X}^2}{J^2}\right] - 2K\hat{X}^2$$

$$+ \frac{B^2S}{m^2}(m+\mu n)^2\left(\frac{\partial\hat{X}}{\partial\psi}\right)^2 + \frac{SB^2n^2}{m^2}\left[\frac{jm}{RB^2n} + I\frac{\partial}{\partial\psi}\left(\frac{J}{R^2}\right)\right]^2\hat{X}^2$$

$$- \frac{\hat{X}^2}{J}\frac{\partial}{\partial\psi}\left[JSB^2\frac{n}{m^2}(m+n\mu)\left(\frac{jm}{RB^2n} + I\frac{\partial}{\partial\psi}\left(\frac{J}{R^2}\right)\right)\right]\Bigg\}, \quad (3.9)$$

where

$$S \equiv \left(1 + \frac{\mu^2n^2B^2R^2}{m^2I^2}\right)^{-1}, \quad (3.10)$$

and the terms linear in $\partial\hat{X}/\partial\psi$ have been removed by a ψ-integration by parts. Reverting to the variable $\hat{\xi}_{\psi}$ (dropping the subscript) and performing further partial integrations, Equation (3.9) can be expressed in the form

$$\delta W_{\infty} = \frac{\pi}{2}\int J d\chi d\psi\left\{ SB^4R^2\left(1+\frac{\mu n}{m}\right)^2\left(\frac{\partial\hat{\xi}}{\partial\psi}\right)^2 + S\Lambda\hat{\xi}^2 + \frac{1}{R^2B^2}\left(\frac{\partial\hat{X}}{J\partial\chi}\right)^2\right\}, \quad (3.11)$$

where

$$\Lambda \equiv \frac{m^2I^2}{\mu^2R^4}\left(1+\frac{\mu n}{m}\right)^2\left[S^{-1} - \frac{\mu^2B^4R^6}{m^2I^2}\left(\frac{\partial}{\partial\psi}\ell n\left(\frac{JB}{R}\right)\right)^2\right]$$

$$+ \left(1 - \frac{\mu^2n^2}{m^2}\right)\left[\frac{B^2R^2}{SJ}\frac{\partial}{\partial\psi}\left(JB^2S\frac{\partial}{\partial\psi}\ell n\left(\frac{JB}{R}\right)\right) + B^2R^2\frac{dp}{d\psi}\cdot\frac{\partial}{\partial\psi}\ell n R^2\right]$$

$$+ \frac{\mu^2n^2}{m^2}B^2R^2\frac{dp}{d\psi}\left[\frac{\partial}{\partial\psi}\ell n R^2 - \frac{jR}{I^2} + \frac{B^2R^2}{I^2}\frac{\partial}{\partial\psi}\ell n J\right]. \quad (3.12)$$

We now investigate the localised interchange modes[6,8,9], which satisfy the conditions $m, n \to \infty$ and $|m + n\mu| \ll m$. Thus any criterion so derived will be necessary, although possibly not sufficient for stability. On a particular surface $\psi = \psi_s$ (the 'singular' surface), $n\oint\mu\, d\chi + \bar{m} = 0$, where $\bar{m}$ is an integer. On a neighbouring surface $\psi = \psi_s + \Delta\psi$, $m + n\mu \simeq n\Delta\psi\frac{\partial\mu}{\partial\psi}$. With these substitutions

Equation (3.11) gives

$$\delta W_\infty = \frac{\pi}{2} \int J\, d\chi\, d\psi \left\{ L(\psi, \chi)\ (\Delta\psi)^2 \left(\frac{\partial \hat{\xi}}{\partial \psi}\right)^2 + M(\psi,\chi)\hat{\xi}^2 + N(\psi,\chi) \left(\frac{\partial \hat{X}}{J\partial \chi}\right)^2 \right\}, \tag{3.13}$$

where

$$L \equiv \frac{B^4R^2}{\bar{D}} \left(\frac{1}{\mu}\ \frac{\partial \mu}{\partial \psi}\right)^2, \qquad N \equiv \frac{1}{R^2B^2} \tag{3.14}$$

and

$$M \equiv \frac{B^2R^2}{\bar{D}}\ \frac{dp}{d\psi} \left[\frac{\partial}{\partial \psi}\ (\ell n R^2) - \frac{jR}{I^2} + \frac{B^2R^2}{I^2} \frac{\partial}{\partial \psi} \ell n\ J \right], \tag{3.15}$$

with $\bar{D}$ given by

$$\bar{D} \equiv 1 + \frac{B^2R^2}{I^2} . \tag{3.16}$$

The components of the expression for δW_∞ have a straightforward physical interpretation. The first component represents shear stabilization. The second component, after averaging over a surface through the χ-integration, represents in part the "average magnetic well". However, the first term in M (proportional to $B^2R^2 \frac{\partial}{\partial \psi}(\ell n\ R^2)$) alternates in sign over the flux surface, and when modulated by the amplitude of $\hat{\xi}^2$, which also varies over χ, the average of the product can give a negative contribution proportional to $(p')^2$. This is the so-called 'ballooning' term(10, 11) and is in addition to the average, unmodulated part. The final term is the magnetic (positive) energy of the modulation. The modulation in χ of both equilibrium and perturbed quantities is therefore an essential feature of the present calculation. An equivalent expression to Equation (3.13) has been obtained by Mercier(6), but this leads to a less stringent stability condition than that derived in the present paper. The difference between the two calculations will be discussed later.

To further minimise δW_∞ it is convenient to transform to the local polar coordinates (r, θ, ϕ). For a line element ds along $B_\chi \equiv B$ we have (see Appendix A),

$$\frac{ds}{B} = J\,d\chi = \frac{rR}{\left.\frac{\partial\psi}{\partial r}\right|_\theta} \cdot \frac{d\theta}{2\pi} = \alpha\,\frac{d\theta}{2\pi} \quad . \tag{3.17}$$

In the vicinity of the magnetic axis the flux surfaces are displaced circles, and since we are particularly interested in this region we shall assume

$$J d\chi M(\psi, \chi) = \frac{d\theta}{2\pi}\left(\bar{G}(\psi) + 2\tilde{G}(\psi)\cos\theta\right) , \tag{3.18}$$

where $\bar{G}(\psi) \sim \kappa^2$ and $\tilde{G}(\psi) \sim \kappa$, κ being the inverse aspect ratio which we shall treat as small. To obtain a necessary criterion for stability we adopt the trial-function

$$\hat{\xi}(\psi, \chi) = \bar{\xi}(\psi) + \tilde{\xi}(\psi)\cos\theta , \tag{3.19}$$

where $\bar{\xi}(\psi) \sim 1$ and $\tilde{\xi}(\psi) \sim \kappa$. Similarly we write,

$$h \equiv RB = \bar{h}(\psi) + \tilde{h}(\psi)\cos\theta , \tag{3.20}$$

where $\bar{h}(\psi) \sim 1$ and $\tilde{h}(\psi) \sim \kappa$. By Equations (3.19) and (3.20) we have

$$\hat{X} = RB\hat{\xi}_\psi = \bar{X}(\psi) + \tilde{X}(\psi)\cos\theta , \tag{3.21}$$

where $\bar{X}(\psi) \sim 1$ and $\tilde{X}(\psi) \sim \kappa$. Lastly we define $Q(\psi, \theta)$ through

$$L(\psi, \chi) J d\chi = Q(\psi, \theta)\,\frac{d\theta}{2\pi} . \tag{3.22}$$

Using the above forms δW_∞ can now be written as

$$\delta W_\infty = \frac{\pi}{2}\int d\psi\Big\{\bar{Q}(\Delta\psi)^2\left(\frac{d\bar{\xi}}{d\psi}\right)^2 + \bar{G}(\psi)\ \bar{\xi}^2(\psi) + 2\bar{\xi}(\psi)\tilde{G}(\psi)\tilde{\xi}(\psi)$$

$$+ \frac{1}{2}\cdot\frac{1}{\bar{\alpha}\ \bar{h}^2}\left(\bar{h}(\psi)\tilde{\xi}(\psi) + 2\bar{\xi}(\psi)\tilde{h}(\psi)\right)^2\Big\}, \qquad (3.23)$$

where

$$\frac{1}{\bar{\alpha}} = \int_0^{2\pi}\frac{d\theta}{2\pi\alpha} = \oint\frac{J d\chi}{\alpha^2}\ . \qquad (3.24)$$

By completing the square Equation (3.23) can now be minimised with respect to $\tilde{\xi}(\psi)$. Thus we obtain

$$\delta W_\infty = \frac{\pi}{2}\int d\psi\left\{\bar{Q}(\Delta\psi)^2\left(\frac{d\bar{\xi}}{d\psi}\right)^2 + A\ \bar{\xi}^2\right\}, \qquad (3.25)$$

where

$$A = \bar{G}(\psi) - 2\bar{\alpha}\ \tilde{G}^2(\psi) - \frac{4}{\bar{h}}\ \tilde{h}(\psi)\tilde{G}(\psi)\ . \qquad (3.26)$$

The condition for $\min(\delta W_\infty) \geqslant 0$ is therefore

$$4A + \bar{Q} > 0\ , \qquad (3.27)$$

which can be written in the form

$$\oint\frac{ds}{B}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial\psi}\right)^2 B^4R^2 + 4\frac{dp}{d\psi}\left\{\oint\frac{ds}{B}E - 2\frac{dp}{d\psi}\frac{\left(\oint\frac{ds}{B}E\cos\theta\right)^2}{\oint\frac{ds}{B}\cdot\frac{1}{\alpha^2}}\right.$$

$$\left. - \frac{8\oint\frac{1}{r}\left.\frac{\partial\psi}{\partial r}\right|_\theta\cos\theta\ ds}{\oint\frac{1}{r}\left.\frac{\partial\psi}{\partial r}\right|_\theta ds}\oint\frac{ds}{B}E\cos\theta\right\} > 0\ , \qquad (3.28)$$

where

$$E = B^2R^2\left[\frac{\partial}{\partial\psi}(\ell n\ R^2) - \frac{2jR}{I^2} - \frac{R^2}{I^2}\frac{\partial B^2}{\partial\psi}\right], \qquad (3.29)$$

and where since $\bar{D} = 1 + O(\kappa^2)$ for a Tokamak, we have replaced $\bar{D}$ by unity. Equation (3.28) is a necessary local condition for stability and should be applied to every flux-surface within the plasma. It is a generalization of Suydam's criterion[8] and reduces to it in the limit.

4. STABILITY OF SPECIAL EQUILIBRIA

In this section we shall apply our necessary criterion to some special models for equilibria in Tokamak. It is shown in Appendix B that for an equilibrium in which the toroidal current density is quasi-uniform or parabolic the flux surfaces in the neighbourhood of the magnetic axis are given by

$$y = \rho^2 + \rho^3 C_1 \cos\theta , \qquad (4.1)$$

where y and ρ are the dimensionless flux and radial coordinates respectively, i.e. $y = \psi/\bar{\psi}$ and $\rho = r/\kappa R_o$. The quantity $\bar{\psi}$ is the total flux due to the poloidal field. The constant C_1 can be expressed as

$$C_1 = \frac{\kappa}{4} (1 + \tau\beta_I) , \qquad (4.2)$$

where β_I is the 'poloidal-β' and is defined by[20]

$$\beta_I \equiv \frac{8\pi(\kappa R_o)^2}{I^2} \int_0^1\int_0^{2\pi} \rho\, p(\rho, \theta) d\rho d\theta ,$$

where I is the toroidal current. The parameter τ takes the values 4 or 3 according as the toroidal current distribution is quasi-uniform or parabolic. We can also write the pressure gradient for our models as

$$\frac{dp}{dy} = - \frac{\tau \bar{\psi}^2}{\kappa^2 R_o{}^4} \beta_I . \qquad (4.3)$$

We now evaluate the coefficients $\bar{h}$, $\tilde{h}$, $\bar{G}$, $\tilde{G}$ etc., which arise in the stability condition (3.27). By (A.1) and (A.2) of Appendix A, the quantity h is given by

$$h^2 = R^2B^2 = \left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right)^2 + \left(\frac{\partial\psi}{\partial r}\right)^2 , \qquad (4.4)$$

which, using the above dimensionless forms can be written as

$$h^2 = R^2B^2 = \frac{\bar{\psi}^2}{\kappa^2 R_o^{\,2}}\left(\left(\frac{\partial y}{\partial\rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial y}{\partial\theta}\right)^2\right) . \qquad (4.5)$$

Using Equation (4.1) this becomes

$$h^2 = \frac{4\,\bar{\psi}^2\,\rho^2}{\kappa^2 R_o^{\,2}}\,(1 + 3\rho C_1 \cos\theta) , \qquad (4.6)$$

where higher-order terms have been neglected. Since all θ-integrations must be performed around a constant y surface, it is necessary to write any ρ occurring in an integrand as a function of y and θ. In order to do this we invert Equation (4.1), obtaining

$$\rho = y^{1/2}(1 - \frac{C_1}{2}\cos\theta\, y^{1/2}) . \qquad (4.7)$$

Equation (4.6) now becomes

$$h^2 = \frac{4\,\bar{\psi}^2\,y}{\kappa^2 R_o^{\,2}}\,(1 + 2\,C_1 \cos\theta\, y^{1/2}) . \qquad (4.8)$$

From Equation (3.20), using Equation (4.8), we find that

$$\bar{h} = \frac{2\,\bar{\psi}\,y^{1/2}}{\kappa R_o} , \qquad (4.9)$$

to order 1, and that

$$\tilde{h} = \frac{\bar{\psi}\,C_1\,y}{\kappa R_o} , \qquad (4.10)$$

to $O(\kappa)$. Similarly, using the definition of α given in

Equation (3.17), Equation (3.24) gives

$$\frac{1}{\bar{\alpha}} = \frac{2\bar{\Psi}}{\kappa^2 R_o^3} \quad , \tag{4.11}$$

to $O(\kappa)$.

In order to evaluate the coefficients $\bar{G}(\psi)$ and $\tilde{G}(\psi)$ defined in Equation (3.18) and which derive from the quantity M given in Equation (3.15), we require to determine

$$j \equiv j_\phi = \frac{1}{r}\frac{\partial}{\partial r}(r\, B_\theta) - \frac{1}{r}\frac{\partial B_r}{\partial \theta} \ . \tag{4.12}$$

As mentioned earlier since the direction of ϕ in the (ψ, ϕ, χ) and (r, θ, ϕ) sets of coordinates is different, we shall reverse the sign of j_ϕ in M whenever we go into local polar coordinates. Writing Equation (4.12) in dimensionless form

$$j_\phi = \frac{\bar{\Psi}}{\kappa^2 R_o^3}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial y}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 y}{\partial\theta^2} + O(\kappa)\right) . \tag{4.13}$$

Using Equation (4.1) j_ϕ becomes

$$j_\phi = \frac{4\bar{\Psi}}{\kappa^2 R_o^3}\left(1 + O(\kappa)\right) . \tag{4.14}$$

To evaluate the $B^2R^2 \frac{\partial}{\partial\psi}(\ell n R^2)$ term we need to determine $B^2R^2 \frac{\partial R}{\partial\psi}$. From Appendix A we have

$$B^2R^2 \left.\frac{\partial R}{\partial\psi}\right|_\chi = \left.\frac{\partial\psi}{\partial r}\right|_\theta \cos\theta - \left.\frac{\partial\psi}{\partial\theta}\right|_r \frac{\sin\theta}{r} \ , \tag{4.15}$$

which, casting into dimensionless form and using Equations (4.1) and (4.7), can be written as

$$B^2R^2 \left.\frac{\partial R}{\partial\psi}\right|_\chi = \frac{\bar{\Psi} y^{1/2}}{\kappa R_o}\left\{2\cos\theta + 2\,C_1\cos^2\theta\, y^{1/2} + \sin^2\theta\, C_1 y^{1/2}\right\} , \tag{4.16}$$

where higher-order terms have been neglected. The last term in M involves $B^2R^2 \frac{1}{J}\frac{\partial J}{\partial \psi}$, which, using Equation (2.9), can be written as

$$B^2R^2 \frac{1}{J}\frac{\partial J}{\partial \psi} = - j_\phi R - R^2 \frac{\partial B^2}{\partial \psi} . \quad (4.17)$$

Using Equations (A.1), (A.2), (4.1) and (4.7) we find that

$$B^2R^2 \frac{1}{J}\frac{\partial J}{\partial \psi} \sim \kappa j_\phi R . \quad (4.18)$$

Noting that $\bar{\psi} \sim \kappa R_o^2 B_\theta$ and $I = B_\phi R$, then M can be written as

$$M = \frac{2}{\kappa R_o R} \cdot \frac{dp}{dy} \left\{ 2 \cos\theta \, y^{1/2} - C_1 \, y \cos 2\theta \right.$$
$$\left. + \frac{8 \bar{\psi}^2 \kappa y}{\kappa^4 R_o^4 B_{\phi o}^2} + 0(\kappa^2) \right\} , \quad (4.19)$$

where we have taken cognisance of the fact that $\bar{D} = 1 + 0(\kappa^2)$. Using Equations (3.17), (3.18), (4.7) and (4.19) we find that

$$\bar{G} = \frac{8 \kappa \bar{\psi} y}{\kappa^3 R_o^3 B_o^2} \frac{dp}{dy} , \quad (4.20)$$

where B_o is the leading-order part of B_ϕ ($B_o \equiv B_{\phi o}$), and

$$\tilde{G} = \frac{\kappa R_o y^{1/2}}{\bar{\psi}} \cdot \frac{dp}{dy} , \quad (4.21)$$

to leading-order.

Using Equation (4.3) and the above forms for the various coefficients, the stability condition can be written as

$$\bar{Q} + \frac{\tau 8 \bar{\psi}}{q^2 R_o^3} y \beta_I \left[\frac{q^2}{4} - 1 - \frac{1}{4} \tau \beta_I q^2 \right] > 0 , \quad (4.22)$$

where q is the so-called 'safety factor' and is defined in Equation (B.10). Now

$$\bar{Q} = \oint J d\chi \left(\frac{1}{\mu}\frac{\partial\mu}{\partial\psi}\right)\frac{B^4R^2}{\bar{\bar{D}}}$$

$$\simeq \left(\frac{1}{q}\frac{dq}{d\psi}\right)^2 \oint \alpha\ B^4R^2\ \frac{d\theta}{2\pi}\ , \qquad (4.23)$$

which can be expressed as

$$\ddot{Q} = \frac{8\bar{\psi}}{\kappa^2 R_o^{\ 3}}\left(\frac{d\ \ell n\ q}{d\ \ell n\ y}\right)^2 . \qquad (4.24)$$

Thus finally Equation (4.22) can be written in the form

$$1 + \kappa^2\ \beta_I\ \frac{\tau y}{q^2}\ \left(\frac{d\ \ell n\ y}{d\ \ell n\ q}\right)^2\left(\frac{q^2}{4} - 1 - \frac{1}{4}\ \tau\ \beta_I\ q^2\right) > 0\ . \qquad (4.25)$$

The first term, which is always stabilising, represents the shear. For $q > 2$ the component containing $(q^2/4 - 1)$ is also stabilising - the so-called "average-magnetic well" effect. The last term, which is proportional to $\beta_I^{\ 2}$ and always destabilising, is referred to as the ballooning term. We now consider the three physical situations which arise when each of the components, shear, well, and ballooning, is neglected in turn.

Sufficiently close to the magnetic axis[12] $q(y) = q_o(1 + 0(y))$ and the well and ballooning terms, which are of order $1/y$, become dominant as $y \to 0$. Thus condition (4.25) can be simplified to give

$$\beta_I < (1 - 4q^{-2})\tau^{-1}\ , \qquad (4.26)$$

as a necessary condition for stability. It is clear that for (4.26) to be satisfied it is necessary that $q > 2$.

We also observe that as $q \to \infty$ the right-hand side of (4.26) approaches the asymptotic limit of τ^{-1}. Thus the upper limit for β_I is $\frac{1}{4}$ or $\frac{1}{3}$ depending on whether the toroidal current distribution is uniform or parabolic.

Using Equation (B.24) it is straightforward to show that the maximum value of β $(\equiv 2p/B_\phi^2)$ allowed by (4.26) is

$$\beta = \beta_{crit} = \frac{1}{5-\tau} \cdot \frac{\kappa^2}{32} , \qquad (4.27)$$

the optimum value of q being $2\sqrt{2}$. For $\kappa = 1/5$ and taking the current to be quasi-uniform, $\beta_{crit} \simeq 1.3 \times 10^{-3}$. The corresponding value for poloidal-β is $\beta_I \simeq 1.3 \times 10^{-1}$. Thus it is evident that in the vicinity of the magnetic axis the critical stable pressure is very low.

Away from the magnetic axis ($y \gtrsim \frac{1}{2}$) the ballooning component may be stabilised through the shear (for example $d(\ell n\, q)/d(\ell n\, y) \sim 1$ for a parabolic current), the average magnetic well effect being negligible. The critical value of β is then given by

$$\beta_{crit} \sim \frac{\kappa}{q^2} \left| \frac{d\, \ell n\, q}{d\, \ell n\, y} \right| , \qquad (4.28)$$

and which approaches the equilibrium limit obtained in Appendix B. When the shear is weaker, as with a quasi-uniform current for which $\frac{d\, \ell n\, q}{d\, \ell n\, y} \sim \kappa^2$ (see Equation (B.13)), we obtain the criterion

$$\left(\frac{d\, \ell n\, q}{d\, \ell n\, y}\right)^2 > 2\beta \left(1 - \frac{q^2}{4}\right) y \qquad (4.29)$$

providing β is sufficiently small to make the ballooning term negligible. Condition (4.29) is equivalent to Suydam's criterion[8], and which can be recovered in the appropriate limit. Thus for $\kappa \to 0$, and using Equations (4.1), (4.3) and (B.24), we obtain

$$\frac{\rho}{4}\left(\frac{q'}{q}\right)^2 + \left(1 - \frac{q^2}{4}\right)\frac{2}{B_o^2}\,\frac{dp}{d\rho} > 0 \; . \qquad (4.30)$$

5. DISCUSSION

In the previous section we obtained a necessary condition for stability and considered the three cases which arise when each of the three components, shear, well and ballooning, are neglected in turn. In particular it was shown that for a circular cross-section plasma with a quasi-uniform or parabolic toroidal current distribution, for stability in the neighbourhood of the magnetic axis, it is necessary that $q > 2$. This condition is, of course, more stringent than the generally accepted $q > 1$ result[12 - 16].

The latter is usually obtained from Mercier's[6] general criterion for stability against localised modes in an axisymmetric system. In going from Equation (25) to Equation (26) in his paper, Mercier[6] takes the average of the shear over χ and pulls it through the χ-integrals. Application of the result obtained to the quasi-uniform or parabolic current distributions leads to the $q > 1$ result. If this approximation were not made, however, then the $q > 2$ result could be deduced from Mercier's formulae. Considering the limit of zero-shear Ware and Haas[13] also obtain the $q > 1$ condition for localised modes. However, in their work it was assumed that the azimuthal mode number $n \sim 1$, whereas in the present work we have taken $n \to \infty$. This suggests that the condition obtained by Ware and Haas is less stringent because it does not take account of the modes $n, m \to \infty$ ($\frac{n}{m}$ finite). Thus the earlier criteria, being only necessary for stability, are not inconsistent with the present one; they are simply less stringent. The more stringent $q > 2$ criterion can in fact be deduced from an earlier paper by Ware(17).

Applying the guiding centre equations to the quasi-uniform and parabolic current models, and taking the limit of zero-shear and small poloidal-β, Jukes(18) has again derived the $q > 2$ condition.

Finally we note that the critical-β values estimated on the basis of the present work, and which are considerably lower than those calculated by other authors [see for example Laval et al.(3)], are in quite good agreement with the T3-A(19) and ST(21) results. Thus the present limit on β_I found in experiments could be attributed to instability to localised modes near the magnetic axis, or elsewhere, if there is too little shear present. The theoretical limit on β_I as a function of q is given by Equation (4.26).

6. CONCLUSIONS

Using the energy principle for an axisymmetric plasma a new and more stringent necessary criterion for stability has been derived. Applying this criterion to simple models for Tokamak in which the toroidal current is taken quasi-uniform or parabolic, gives $q > 2$ as necessary for stability in the neighbourhood of the magnetic axis. The optimum critical pressure on axis corresponding to $q \sim 3$ (at the magnetic axis) is very low. Thus for the quasi-uniform model $\beta_{crit} \sim 0.03\kappa^2$ and for the parabolic model $\beta_{crit} \sim 0.01K^2$, whereas in the limit $q \to \infty$, $\beta_I \to \frac{1}{4}, \frac{1}{3}$ respectively. Both β and q limiting values agree quite well with the T3-A and ST results.

REFERENCES

1. PEACOCK, N.J., ROBINSON, D.C., FORREST, M.J., WILCOCK, P.D. and SANNIKOV, V.V., Nature 224, 488 (1969).

2. JUKES, J.D. and HAAS, F.A., Paper CN-28/F-11, Proc. 4th. Int. Conf. Plasma Physics and Controlled Nuclear Fusion Research, Madison (1971). (to be published by I.A.E.A.).

3. LAVAL, G., LUC, H., MASCHKE, E.K., MERCIER, C., and PELLAT, R., Paper CN-28/F-12, Proc. 4th. Int. Conf. Plasma Physics and Controlled Nuclear Fusion Research, Madison (1971).

4. SHAFRANOV, V.D. and YURCHENKO, E.I., Paper CN-28/F-13, Proc. 4th. Int. Conf. Plasma Physics and Controlled Nuclear Fusion Research, Madison (1971).

5. SOLOVEV, L.S., Soviet Physics JETP 26, 400 (1968).

6. MERCIER, C., Nuclear Fusion 1, 1, 47 (1960).

7. LAING, E.W., ROBERTS, S.J. and WHIPPLE, R.T.P., J. Nucl. Energy, Part C: Plasma Physics 1, 49 (1959).

8. SUYDAM, B.R., 2nd. Int. Conf. Peaceful Uses Atom. Energy (Proc. Conf. Geneva, 1958), 31, 157, U.N., New York (1958).

9. GREENE, J.M. and JOHNSON, J.L., Physics Fluids 5, 5, 510 (1962).

10. FURTH, H.P., KILLEEN, J., ROSENBLUTH, M.N. and COPPI, B., Proc. of Conf. on Plasma Physics and Controlled Nuclear

Fusion Research, I.A.E.A., Vienna, 1, 103 (1965).

11. KADOMTSEV, B.B. and POGUTSE, O.P., Soviet Physics - Doklady, 11, 10, 858 (1967).

12. MERCIER, C., Nuclear Fusion 3, 2, 89 (1963) and 4, 3, 213 (1964).

13. WARE, A.A. and HAAS, F.A., Physics Fluids 9, 5, 956 (1966).

14. SHAFRANOV, V.D. and YURCHENKO, E.I., Zh. Eksperim., Teor. Fiz., 53, 3, 1157 (1967) (Translated Soviet Physics, J.E.T.P. 26, 3, 692 (1968)).

15. LAVAL, G. MASCHKE, K.E. and PELLAT, R., Phys. Rev. Letters 24, 22, 1229 (1970).

16. SHAFRANOV, V.D. and YURCHENKO, E.I., Nuclear Fusion 8, 4, 329 (1968).

17. WARE, A.A., Physics Fluids 7, 12, 2006 (1964).

18. JUKES, J.D. (to be published).

19. FORREST, M.J., PEACOCK, N.J., ROBINSON, D.C., SANNIKOV, V.V. and WILCOCK, P.D., Culham Report CLM-R107 (1970).

20. GREENE, J.M., JOHNSON, J.L. and WEIMER, K.E., Physics Fluids 14, 3, 671 (1971).

21. HOSEA, J.C. et al., Paper CN-28/F-7, Proc.4th.Int.Conf. Plasma Physics and Controlled Nuclear Fusion Research, Madison (1971).

APPENDIX A GEOMETRY OF THE FLUX SURFACES

Figure 3 shows the way in which the various geometrical quantities associated with two neighbouring flux surfaces are defined. It is clear that

$$B_r = -\frac{1}{Rr}\left.\frac{\partial\psi}{\partial\theta}\right|_r = B_\chi \sin(\theta+\lambda) \;, \qquad \text{(A.1)}$$

and

$$B_\theta = \frac{1}{R}\left.\frac{\partial\psi}{\partial r}\right|_\theta = B_\chi \cos(\theta+\lambda) \;. \qquad \text{(A.2)}$$

We now require to evaluate $\left.\frac{\partial R}{\partial\psi}\right|_\chi$. We observe that

$$\delta R = \left.\frac{\partial R}{\partial\psi}\right|_\chi \delta\psi = \delta\ell \cos\lambda \;,$$

and further that

$$\delta\psi = RB_\chi\, \delta\ell \;,$$

thus giving

$$\left.\frac{\partial R}{\partial\psi}\right|_\chi = \frac{\cos\lambda}{RB_\chi} \;. \qquad \text{(A.3)}$$

It follows from (A.1) and (A.2) that

$$B_r \sin\theta + B_\theta \cos\theta = B_\chi \cos\lambda \;,$$

and hence

$$\left.\frac{\partial R}{\partial\psi}\right|_\chi = \frac{1}{R^2B^2}\left(\left.\frac{\partial\psi}{\partial r}\right|_\theta \cos\theta - \left.\frac{\partial\psi}{\partial\theta}\right|_r \frac{\sin\theta}{r}\right) \;. \qquad \text{(A.4)}$$

From the figure, $\cos(\theta+\lambda) = r\,\frac{\delta\theta}{\delta S}$,

and thus using (A.2) we obtain

$$\frac{ds}{B} = \frac{r\,d\theta}{\frac{1}{R}\left.\frac{\partial\psi}{\partial r}\right|_\theta 2\pi} \;, \qquad \text{(A.5)}$$

where the 2π has been introduced to ensure that a complete period in χ (or s) corresponds to θ running from 0 to 2π.

APPENDIX B SPECIAL EQUILIBRIA

In this appendix we solve Equation (2.10) for some special models of Tokamak. Making use of the transformations in Equation (2.11) the equilibrium equation becomes approximately

$$\nabla^2\psi - \frac{1}{R_o}\frac{\partial\psi}{\partial R} = Rj_\phi = -\left(II' + R_o{}^2 p' + 2R_o r\cos\theta\, p'\right). \qquad (B.1)$$

It is convenient to introduce the dimensionless quantities

$$\left.\begin{aligned} &\rho \equiv \frac{r}{\kappa R_o}, \quad y \equiv \frac{\psi}{\bar{\psi}}, \quad \zeta \equiv \rho\cos\theta, \\ \text{and } a(y) \equiv -&\left(II' + R_o{}^2 p'\right)\frac{\kappa^2 R_o{}^2}{\bar{\psi}}, \quad b(y) \equiv -\frac{2p'\kappa^3 R_o{}^4}{\bar{\psi}}, \end{aligned}\right\} \qquad (B.2)$$

so that Equation (B.1) becomes

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial y}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 y}{\partial\theta^2} - \kappa\frac{\partial y}{\partial\zeta} = a(y) + b(y)\,\rho\cos\theta, \qquad (B.3)$$

and the toroidal current density can be expressed as

$$j_\phi = \frac{\bar{\psi}}{\kappa^2 R_o{}^2}\cdot\frac{1}{R}\left(a(y) + b(y)\,\rho\cos\theta\right). \qquad (B.4)$$

With a suitable choice of the functions a and b we can investigate two typical equilibria:

(i) quasi-uniform current, where $a(y) \equiv a$ and $b(y) \equiv b$ are both constants, (but because of the ρ, θ dependence, when b is finite j_ϕ is neither uniform nor constant on a flux surface), and

(ii) parabolic current, where $a(y) \equiv (1 - y)a$ and

$b(y) \equiv (1 - y)b$ and a, b are constants ('parabolic' refers to the approximate current distribution over ρ).

We now solve Equation (B.3) for these models and this requires that the boundary conditions on y must be satisfied. The latter may be imposed experimentally by either (a) a highly conducting closed wall of an arbitrary shape coinciding with a flux surface y = constant, or (b) an external vacuum magnetic field established by a suitable array of conductors.

(i) <u>Quasi-Uniform Current</u>

Expanding y such that

$$y = y^{(o)} + \kappa y^{(1)} + \dots , \qquad \text{(B.5)}$$

we may solve Equation (B.3) by a simple iterative procedure, and obtain

$$\nabla^2 y^{(o)} = a + b \, \rho \, \cos\theta \quad \text{and} \quad \nabla^2 y^{(1)} = \frac{\partial y^{(o)}}{\partial \zeta} \; . \qquad \text{(B.6)}$$

A solution which possesses an elliptic magnetic axis at the origin and symmetry about the plane Z = 0 can be expressed as

$$y = \Omega^2 \rho^2 + \rho^3 \, C_1 \, \cos\theta \; , \qquad \text{(B.7)}$$

where

$$4\,\Omega^2 = a \; , \quad C_1 = \frac{b}{8} + \frac{\kappa \Omega^2}{4} \; . \qquad \text{(B.8)}$$

It is assumed that the range of validity extends to $0 \leqslant \rho \lesssim 1$ and $0 \leqslant y \leqslant 1$, the upper limit representing the plasma boundary where $p = 0$. The constant C_1 is, of course, of order κ.

Since the confined pressure is directly

proportional to b it is of interest to ascertain the maximum possible value of the latter quantity. It is determined by the condition that there be only one (elliptic) magnetic axis within the plasma boundary and that the maximum dimension of the plasma minor cross-section be two units. The following inequalities must then apply

$$\Omega^2 \leqslant \frac{27}{16} \ , \quad C_1 \leqslant \frac{27}{32} \ , \quad b \simeq 8\, C_1 \leqslant \frac{27}{4} \ .$$

We now relate b to the experimentally significant parameters. In the present model $p = p_o(1 - y)$, and at the magnetic axis $\beta = \beta_o \equiv \dfrac{2p_o}{B_o^{\,2}}$, where p_o and B_o are the pressure and magnetic field respectively at the magnetic axis. It follows that

$$\beta_o = \frac{b\ \bar{\psi}^2}{\kappa^3 R_o^{\,4}\, B_o^{\,2}} \ , \tag{B.9}$$

and $\bar{\psi}$, the total enclosed flux, can be expressed in terms of the reciprocal rotational transform over a flux surface $q(y)$, where

$$q \equiv I \oint \frac{J d\chi}{R^2} = \frac{I}{2\pi} \int_0^{2\pi} \frac{d\theta}{\frac{R}{r} \left.\frac{\partial \psi}{\partial r}\right|_\theta} \ . \tag{B.10}$$

Therefore

$$\left.\begin{aligned} q_o &= \frac{\kappa^2 R_o^{\,2} B_o}{2\, \bar{\psi}\, \Omega^2} \ , \qquad \beta_o = \frac{\kappa b}{4\, \Omega^4\, q_o^{\,2}} \ , \\ &\text{and} \\ C_1 &= \frac{\kappa\, \Omega^2}{4} + \frac{\beta_o\, \Omega^4\, q_o^{\,2}}{2\kappa} \end{aligned}\right\} \tag{B.11}$$

From the preceding results it follows that the maximum value of β_o is $\frac{16\kappa}{27\, q_o^2}$.

If the conducting wall is circular it is possible to find an approximate analytic solution for y and q(y). This shows that the magnetic axis is displaced a distance Δ outwards from the centre given by

$$\frac{\Delta}{\kappa R_o} \simeq \frac{\kappa}{8} + \frac{\beta_o\, q_o^2}{4\kappa} , \qquad (B.12)$$

and q(y) is given by

$$q(y) \simeq q_o \left(1 - \left(\frac{3C_1}{2\Omega^3}\right)^2 y \right)^{-1/2} , \qquad (B.13)$$

where terms $\sim \kappa/C_1$ are assumed small. When y is small, on the other hand,

$$q(y) \simeq q_o \left(1 + \left(\frac{9\, C_1^2}{4\Omega^4} + \frac{2\kappa\, C_1}{\Omega^2} + \kappa^2 \right) \frac{y}{2\Omega^2} \right) . \qquad (B.14)$$

The above formulae cease to be valid when $8C_1 \simeq b \to \frac{27}{4}$, at which value $\Delta/\kappa R_o = 1/3$ and $q(y) \to \infty$ on $y = 1$.

(ii) <u>Parabolic Current</u>

This case is of more practical interest than the previous one, since it is natural for the toroidal current density to fall to zero along with the plasma pressure at the plasma boundary. We shall again assume that this boundary coincides with a rigid circular conducting wall. We suppose Rj_ϕ and p' are proportional to $1 - \psi/\bar{\psi}$, then $p = p_o (1 - y)^2$ where

$$p_o = \frac{b\, \bar{\psi}^2}{4\kappa^3 R_o^4} \quad \text{and} \quad \beta_o = \frac{b\, \bar{\psi}^2}{2\kappa^3 R_o^4\, B_o^2} . \qquad (B.15)$$

The equation to be solved is more complicated than previously. Omitting terms $\sim 0(\kappa)$ it is

$$\nabla^2 y = (1 - y)(a + b\rho \cos\theta) \quad . \tag{B.16}$$

With a change of variable to $t \equiv 1 - y$ Equation (B.16) can be put in a homogeneous form for which a suitable variational principle exists for determining a and b. Thus we have

$$\int (a + b\rho \cos\theta) t^2 dS = \int (\nabla t)^2 dS \quad ,$$

where

$$\delta \int (a + b\rho \cos\theta) t^2 dS \simeq 0(\delta^2) \quad ,$$

and $\int dS$ denotes the surface integral. With the use of a simple one-parameter trial function it can be shown that the maximum value of β_o is $\frac{0.4\kappa}{q_o^2}$.

Summarising then, sufficiently close to the magnetic axis, both models have flux surfaces of the form

$$y = \Omega^2 \rho^2 + \rho^3 C_1 \cos\theta \quad , \tag{B.17}$$

with

$$C_1 = \frac{\kappa \Omega^2}{4} + \frac{\Omega^4 q_o^2 \beta_o (5 - \tau)}{2\kappa} \quad , \tag{B.18}$$

where $\tau = 4$ or 3 according as the current distribution is quasi-uniform or parabolic. Similarly the pressure gradient can be expressed as

$$\frac{dp}{dy} = - \frac{1}{2} (5 - \tau)\beta_o B_o^2 \; . \tag{B.19}$$

If we consider the particular case $\Omega = 1$ then it is just as convenient to express Equations (B. 17) to (B.19) in terms of the 'poloidal-β', β_I, which is defined maximum value of β_o is $\frac{16\kappa}{27\, q_o^2}$.

If the conducting wall is circular it is possible to find an approximate analytic solution for y and $q(y)$. This shows that the magnetic axis is displaced a distance Δ outwards from the centre given by

$$\frac{\Delta}{\kappa R_o} \simeq \frac{\kappa}{8} + \frac{\beta_o\, q_o^2}{4\kappa} , \tag{B.12}$$

and $q(y)$ is given by

$$q(y) \simeq q_o \left(1 - \left(\frac{3C_1}{2\Omega^3}\right)^2 y\right)^{-1/2} , \tag{B.13}$$

where terms $\sim \kappa/C_1$ are assumed small. When y is small, on the other hand,

$$q(y) \simeq q_o \left(1 + \left(\frac{9\, C_1^2}{4\Omega^4} + \frac{2\kappa\, C_1}{\Omega^2} + \kappa^2\right) \frac{y}{2\Omega^2}\right) . \tag{B.14}$$

The above formulae cease to be valid when $8C_1 \simeq b \to \frac{27}{4}$, at which value $\Delta/\kappa R_o = 1/3$ and $q(y) \to \infty$ on $y = 1$.

(ii) Parabolic Current

This case is of more practical interest than the previous one, since it is natural for the toroidal current density to fall to zero along with the plasma pressure at the plasma boundary. We shall again assume that this boundary coincides with a rigid circular conducting wall. We suppose Rj_ϕ and p' are proportional to $1 - \psi/\bar{\psi}$, then $p = p_o\,(1 - y)^2$ where

$$p_o = \frac{b\, \bar{\psi}^2}{4\kappa^3 R_o^4} \quad \text{and} \quad \beta_o = \frac{b\, \bar{\psi}^2}{2\kappa^3 R_o^4\, B_o^2} . \tag{B.15}$$

The equation to be solved is more complicated than previously. Omitting terms $\sim O(\kappa)$ it is

$$\nabla^2 y = (1 - y)(a + b\rho \cos\theta) . \qquad (B.16)$$

With a change of variable to $t \equiv 1 - y$ Equation (B.16) can be put in a homogeneous form for which a suitable variational principle exists for determining a and b. Thus we have

$$\int (a + b\rho \cos\theta) t^2 dS = \int (\nabla t)^2 dS ,$$

where

$$\delta \int (a + b\rho \cos\theta) t^2 dS \simeq O(\delta^2) ,$$

and $\int dS$ denotes the surface integral. With the use of a simple one-parameter trial function it can be shown that the maximum value of β_o is $\frac{0.4\kappa}{q_o^2}$.

Summarising then, sufficiently close to the magnetic axis, both models have flux surfaces of the form

$$y = \Omega^2 \rho^2 + \rho^3 C_1 \cos\theta , \qquad (B.17)$$

with

$$C_1 = \frac{\kappa \Omega^2}{4} + \frac{\Omega^4 q_o^2 \beta_o (5 - \tau)}{2\kappa} , \qquad (B.18)$$

where $\tau = 4$ or 3 according as the current distribution is quasi-uniform or parabolic. Similarly the pressure gradient can be expressed as

$$\frac{dp}{dy} = - \frac{1}{2} (5 - \tau) \beta_o B_o^2 . \qquad (B.19)$$

If we consider the particular case $\Omega = 1$ then it is just as convenient to express Equations (B. 17) to (B.19) in terms of the 'poloidal-β', β_I, which is defined by[20]

$$\beta_I \equiv \frac{8\pi}{I^2} \int_0^{\kappa R_o} \int_0^{2\pi} r p(r, \theta) dr \, d\theta \ , \qquad (B.20)$$

where I is the toroidal current. Thus we obtain

$$y = \rho^2 + \rho^3 C_1 \cos\theta \ , \qquad (B.21)$$

with

$$C_1 = \frac{\kappa}{4} (1 + \tau\beta_I) \ , \qquad (B.22)$$

where τ is defined as above. The pressure gradient can be expressed in the form

$$\frac{dp}{dy} = - \frac{\tau \bar{\psi}^2}{\kappa^2 R_o^{\,4}} \beta_I \ . \qquad (B.23)$$

We further note that β_I can be related to β through the expression

$$\beta_I = \frac{2q^2}{\kappa^2} \left(\frac{5}{\tau} - 1\right) \beta \ . \qquad (B.24)$$

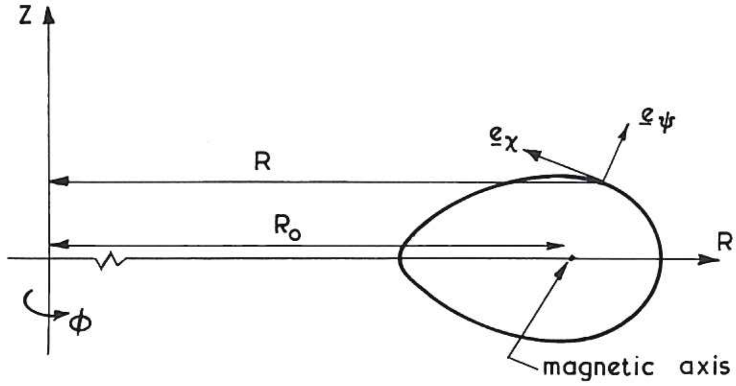

Fig.1 Geometry of toroidal flux surface in (R, ϕ, Z) and (ψ, ϕ, χ) coordinates.

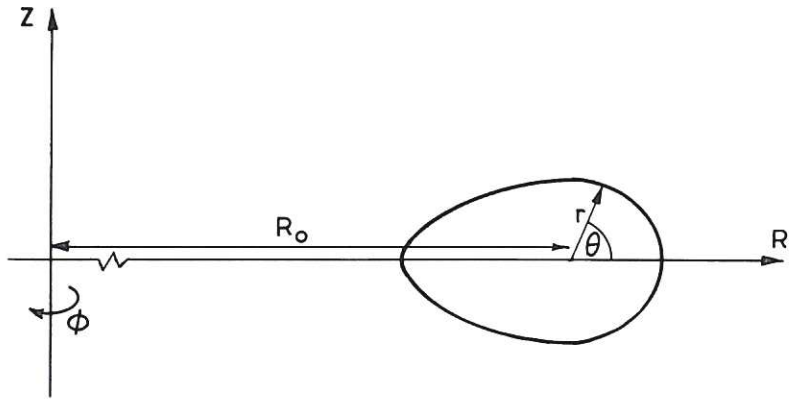

Fig.2 Geometry of toroidal flux surface in local polar coordinates (r, θ, ϕ).

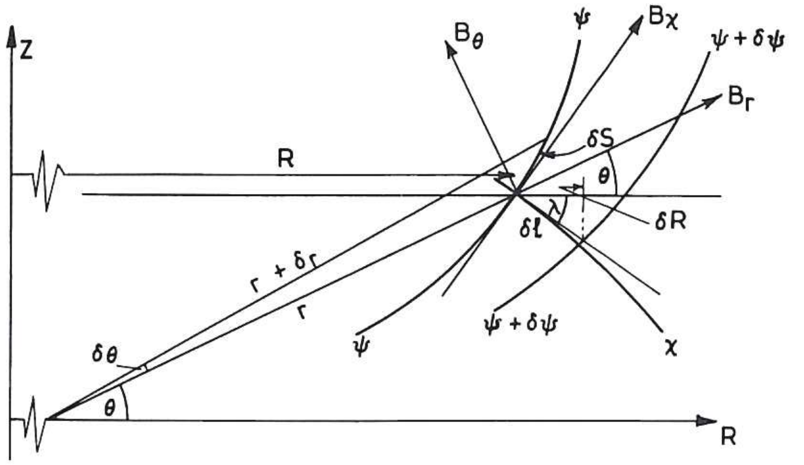

Fig.3 Geometry of two neighbouring flux-surfaces.