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ABSTRACT

The simple model of a Tokamak studied by Strauss[10] is reinvestigated. Defining poloidal-β to be the ratio of the integrated pressure to the square of the toroidal current it is shown that this quantity is bounded.

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As a result of the encouraging studies made on T3 and ST Tokamaks[1],[2] it is of considerable interest to determine the theoretical upper limit (if any) to the pressure which may be confined. More precisely, we require to evaluate the limit on the poloidal-β, β_I. To be of practical value, this is usually (see for example Greene et al.[3]) defined to be

$$\beta_I = 8\pi I^{-2} \int p dS , \qquad (1)$$

where I is the toroidal component of current and the integral is taken over the minor cross-sectional area of the plasma.

Most studies of MHD equilibria in Tokamaks have been made in terms of the inverse aspect ratio ε, where $\varepsilon \ll 1$. For $\beta_I \sim 1$[3],[4],[5] the analysis can be taken through for an arbitrary pressure distribution. In the work of Shafranov[4], and Ware and Haas[5], it is shown that to $O(\varepsilon^2)$ the flux-surfaces are non-concentric circles. Recently, using the same ordering and considering a model in which a sharp-boundary separates plasma and vacuum, Greene et al.[3] have taken their calculation to $O(\varepsilon^3)$ and shown the flux-surfaces to be elliptically distorted as well as non-concentric.

To investigate systems with $\beta_I \sim \varepsilon^{-1}$ it is necessary to prescribe simple models for the pressure and toroidal current distribution[6],[7]. Laval et al.[6] have studied a diffuse plasma contained in a torus with slightly elliptical cross-section. Jukes and Haas[7] have described both diffuse and sharp-boundary models, the latter having a significantly distorted interface. More recently Haas[8] has demonstrated the existence of a sharp-boundary model with parabolic pressure distribution and of circular cross-section. The common feature to all this work is that the confined pressure is limited by the appearance of a second magnetic axis, the upper limit to β_I being

$$\beta_I = A\,\varepsilon^{-1}, \qquad (2)$$

where A is a number of order one and depends on the precise forms of pressure and current distribution, as well as on the shape of the plasma cross-section[9]. Recently however, defining poloidal-β to be given by $\beta_I \equiv 2p_{ma}/B_p^2$ (p_{ma} and B_p denote the pressure at the magnetic axis and maximum poloidal field respectively), Strauss[10] has shown β_I to be unlimited. Strauss' calculation contains no mention of a second magnetic axis. His approach differs from that of earlier workers, in that, instead of fully prescribing the plasma boundary, he chooses a particular solution for the poloidal-flux ψ and allows the flux-surfaces to take up their natural positions subject to a certain constraint. The latter requires that the plasma cross-section always lies within a square, the boundary surface touching each side of the square once only (see figure 3). This effectively excludes consideration of plasmas whose boundaries are a long way from circular.

In the present note we reconsider the pressure and current forms of Strauss and evaluate β_I as defined by equation (1). We show that for a fully prescribed boundary (circular or non-circular) the poloidal-β attains a limit of the form in equation (2). For a free-boundary subject to the constraint described above, the poloidal-β reaches a similar limit, but this is set by the size of the square - not by the appearance of a second magnetic axis.

It is well-known[11] that for an axisymmetric system the equilibrium equation for the poloidal-flux is

$$R \frac{\partial}{\partial r}\left(\frac{1}{R}\ \frac{\partial\psi}{\partial R}\right) + \frac{\partial^2\psi}{\partial Z^2} = Rj_\phi = -\ FF'(\psi)\ -\ R^2p'(\psi), \quad (3)$$

where (R, ϕ, Z) are the usual cylindrical coordinates (see figure 1). We assume that the plasma occupies a perfectly conducting torus with circular cross section centre O, the major and minor radii being R_o and r_o ($\varepsilon \equiv r_o/R_o$). For $F(\psi)$, $p(\psi)$ we take

$$F(\psi) = (C + 2d\ R_o^2\, p(\psi))^{\frac{1}{2}} \text{ and } p = \frac{\alpha}{2R_o^4\,\varepsilon^2}\,(\psi^2 - \psi_B^2), \quad (4)$$

the latter ensuring that the pressure vanish at the boundary $\psi = \psi_B$. Because of the form of F the parameter C does not enter equation (3). Thus the dimensionless quantities α and d are free parameters, since it will always be possible to choose C such that F, and hence the toroidal field B_ϕ, are real. Transforming equation (3) to local-polar coordinates (r, θ)[8],[11] based on the point O (see figure 1), we can obtain ψ as an expansion in ε. Since the procedure follows that given in an earlier publication[8] we shall only give the results.

Taking $\alpha \sim 1$ and $d + 1 \sim \varepsilon$, we find that

$$\psi = \psi_o + \frac{\alpha}{4}\,\psi_o(d + 1 + \varepsilon r\cos\theta)(1 - r^2), \quad (5)$$

where $\psi_o = \psi_B$, and r is now dimensionless (r = 1 defines the minor radius of the torus). To leading-order the toroidal current density is given by

$$j_\phi = - \frac{\alpha}{R_o r_o^2} (d + 1 + 2\varepsilon r \cos\theta)\psi_o , \qquad (6)$$

and we note that the constant j_ϕ contours are planes perpendicular to the R-axis and parallel to the axis of symmetry. Writing equation (1) in polar coordinates we have

$$\beta_I = 8\pi r_o^{-2} \int_0^{2\pi}\int_0^1 p(r,\theta) r dr d\theta \left[\int_0^{2\pi}\int_0^1 j_\phi(r,\theta) r dr d\theta \right]^{-2} , (7)$$

which using the above equations gives

$$\beta_I = (d + 1)^{-1} . \qquad (8)$$

The quantity α can be written as

$$\alpha = \frac{\beta_I}{\pi} \frac{I}{I_c} , \qquad (9)$$

where I_c is a characteristic current, $I_c = \psi_B R_o^{-1}$. Thus through equations (8) and (9) we can replace the parameters α and d by the physical quantities β_I and I. Equations (5) and (6) can now be expressed as

$$\psi = \psi_o \left[1 + \frac{I}{4\pi I_c} (1 + \varepsilon\beta_I r \cos\theta) (1 - r^2) \right] \qquad (10)$$

and

$$j_\phi = \frac{I}{\pi r_o^2} (1 + 2\varepsilon\beta_I r \cos\theta) . \qquad (11)$$

From equation (11) we observe that for $\beta_I = 0.5 \varepsilon^{-1}$ the toroidal current density is zero at the innermost point of the torus, A say (see figure 1). Increasing β_I above this value leads to a region of reversed current spreading across the plasma. For $\beta_I < \varepsilon^{-1}$ it can be shown that there is one (outward) magnetic axis corresponding to a pressure maximum displaced a distance

$$\Delta = (3\varepsilon \beta_I)^{-1} \{ - 1 + (1 + 3\varepsilon^2\beta_I^2)^{\frac{1}{2}} \} , \qquad (12)$$

from the centre of the tube. Alternatively, equation (12) can be written as

$$\beta_I = 2\Delta(1 - 3\Delta^2)^{-1}\varepsilon^{-1}. \qquad (13)$$

The upper limit $\beta_{Icrit} = \varepsilon^{-1}$, is achieved for $\Delta = \frac{1}{3}$. Taking β_I above this value leads to the appearance of a second magnetic axis[12]. The above analysis is inapplicable to low pressures ($\beta_I \sim 1$) since we have neglected higher-order terms.

We now consider the problem of a plasma with a non-circular boundary. Taking the forms of equation (4) we suppose the plasma to have an 'egg-shaped' boundary given by the formula

$$y^2 = (1 - x^2)(1 + \delta^2 - 2\delta x)^{-1}, \qquad (14)$$

and illustrated in figure 2. The plasma lies inside a square ($-1 < x < +1$, $-1 < y < +1$) - the boundary touching each side of the square once only. Solving equation (3) numerically for a <u>given</u> δ and different values of α and d, we find as before, that a limit on β_I is reached due to the appearance of a second magnetic axis (see figure 2). Even in the presence of a significant distortion of the boundary, $\delta = 0.75$ say, it is found that the β_I limit occurs for α and d such that $\alpha \sim 1$ and $d + 1 \sim \varepsilon$. Thus for a given ε the magnitude of the β_I limit is close to that for a circular boundary. Details of these calculations will be presented in a further paper.

We now turn to the principal point of this communication, the evaluation of β_I (as defined in equation (1)) for Strauss' problem. The disposition of the boundary and its constraint have been described earlier. Introducing the dimensionless coordinates x, y (see figure 3) and making the same approximations as Strauss, equation (3) can be written as

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + b^3(\lambda + x)\psi = 0 , \qquad (15)$$

where b and λ are defined through

$$b^3 \equiv 2\alpha\varepsilon \quad \text{and} \quad \lambda \equiv (d+1)(2\varepsilon)^{-1} . \qquad (16)$$

A particular solution of equation (15) is given by

$$\psi = U(\rho) \cos ky , \qquad (17)$$

where $U(\rho) = Ai(-\rho)$ (see figure 1 of Strauss), ρ being defined by

$$\rho = b(x + \lambda) - k^2 b^{-2} . \qquad (18)$$

If ρ_2 and ρ_1 signify the boundary points corresponding to $x = +1$ and $x = -1$ respectively, then

$$U(\rho_2) = U(\rho_1) = U_{max} \cos k = U_{min} \equiv \psi_B . \qquad (19)$$

If we specify ψ_B then the equilibrium and shape of the boundary are fully determined. For we can immediately evaluate k, ρ_2 and ρ_1 from equation (19), and since $2b = \rho_2 - \rho_1$, the quantity b is calculable. Finally, λ can be found from equation (18). Thus as we reduce ψ_B from U_{max} (figure 1 of Strauss) the magnetic axis (see figure 3) moves outward approaching $x = +1$ as $b \to \infty$, the boundary taking up the appropriate shape. With his definition, Strauss shows β_I to increase indefinitely as this limit is approached.

We now evaluate β_I as defined in equation (1). The toroidal current I is evaluated from

$$j_\phi = \frac{1}{R_o(\varepsilon R_o)^2}\left[\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right] . \qquad (20)$$

Using equations (17) and (18), we obtain

$$I = \frac{1}{bR_o} \iint \left(b^2 \frac{d^2U}{d\rho^2} - k^2 U \right) \cos ky \; dy \; d\rho , \qquad (21)$$

which, using the Airy equation and noting that the equation of the boundary is

$$U(\rho_1) = U(\rho) \cos ky \; , \qquad (22)$$

becomes

$$I = - \frac{2}{kbR_o} \int_{\rho_1}^{\rho_2} (b^2\rho + k^2)\left(U^2(\rho) - U^2(\rho_1)\right)^{\frac{1}{2}} d\rho \; . \qquad (23)$$

Integrating by parts this can be cast into the form

$$I = - \frac{2}{kbR_o} \left\{ k^2 \int_{\rho_1}^{\rho_2} \left(U^2(\rho) - U^2(\rho_1)\right)^{\frac{1}{2}} d\rho + b^2 U^2(\rho_1) \int_{\rho_1}^{\rho_2} \frac{\left(\frac{dU}{d\rho}\right)^2 d\rho}{U^2(\rho)\left(U^2(\rho) - U^2(\rho_1)\right)^{\frac{1}{2}}} \right\} , \qquad (24)$$

showing that for any choice of ψ_B (i.e. setting of ρ_1, ρ_2), I is always negative and hence never passes through zero. After some straightforward algebra we obtain

$$\beta_I = L(b)\, \varepsilon^{-1} \qquad (25)$$

where

$$L(b) = \frac{\pi k \int_{\rho_1}^{\rho_2} \left[\left(\frac{1}{2} U^2(\rho) - U^2(\rho_1)\right) \cos^{-1}\left(\frac{U(\rho_1)}{U(\rho)}\right) + \frac{1}{2} U(\rho_1)\left(U^2(\rho) - U^2(\rho_1)\right)^{\frac{1}{2}} \right] d\rho}{\left[\int_{\rho_1}^{\rho_2} \left(\rho + \frac{k^2}{b^2}\right)\left(U^2(\rho) - U^2(\rho_1)\right)^{\frac{1}{2}} d\rho \right]^2} \; . \qquad (26)$$

In the limit $b \to \infty$

$$L(b) \to \frac{\pi^3}{8} \left(\frac{dU}{d\rho}\right)^{-2}_{\rho = 2.34} \int_{-\infty}^{2.34} U^2(\rho)\, d\rho = 3.86 \; . \qquad (27)$$

Numerical evaluation of L(b) shows this quantity to approach 3.86 asymptotically from below (see figure 4). As before we expect the curve to be incorrect for sufficiently small b since we have neglected higher-order terms. Finally we note that b is related to the displacement of the magnetic axis through

$$\Delta = 1 - (\rho_2 - \rho_o)\, b^{-1} \; , \qquad (28)$$

$\rho = \rho_o$ corresponding to U_{max}. As $b \to \infty$, $\rho_2 \to 2.34$ and $\Delta \to 1$. Thus the upper limit to β_I is set by the finite size of the square.

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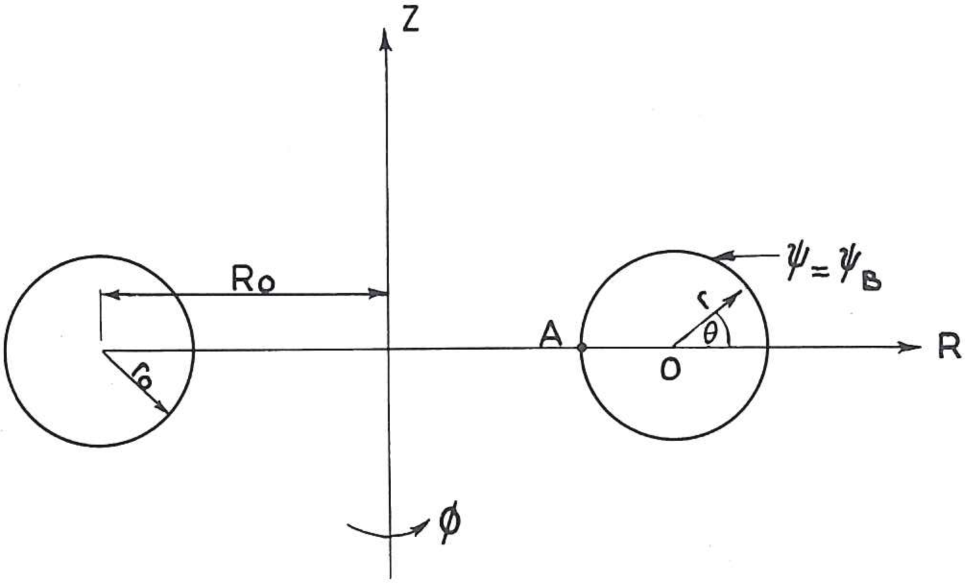

Fig.1 Coordinate systems.

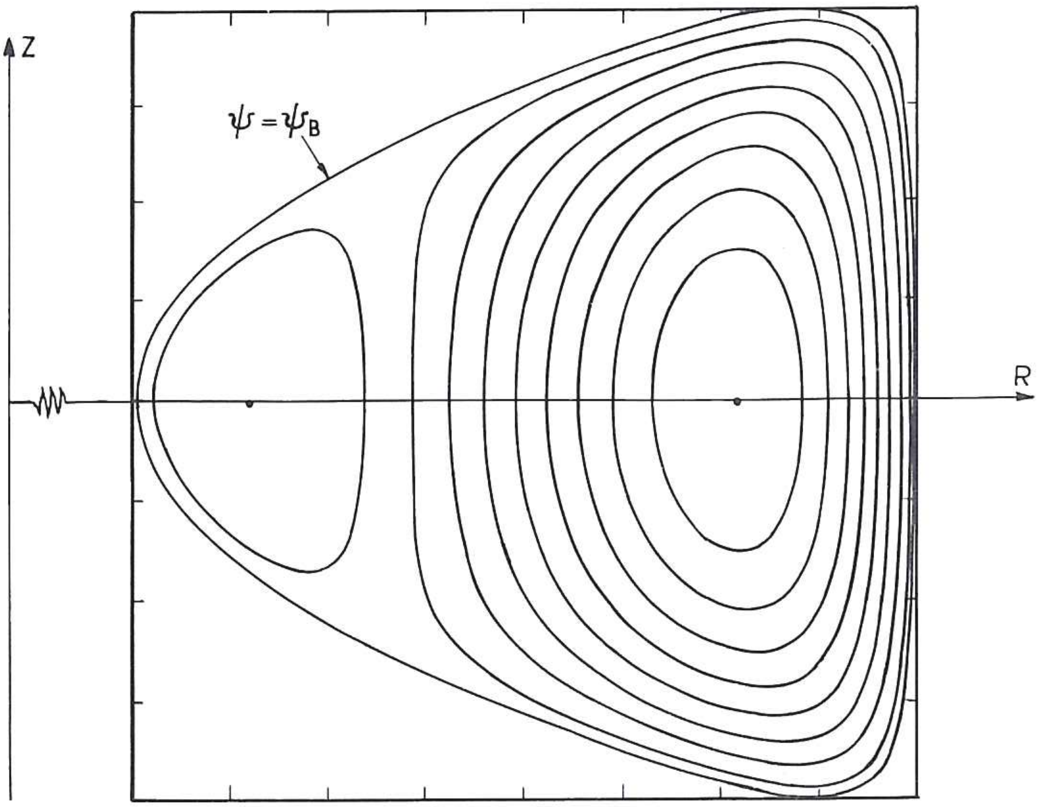

Fig.2 Flux surfaces for plasma with an 'egg-shaped' cross-section (δ = 7.75, α = 5.0, d = - 0.95).

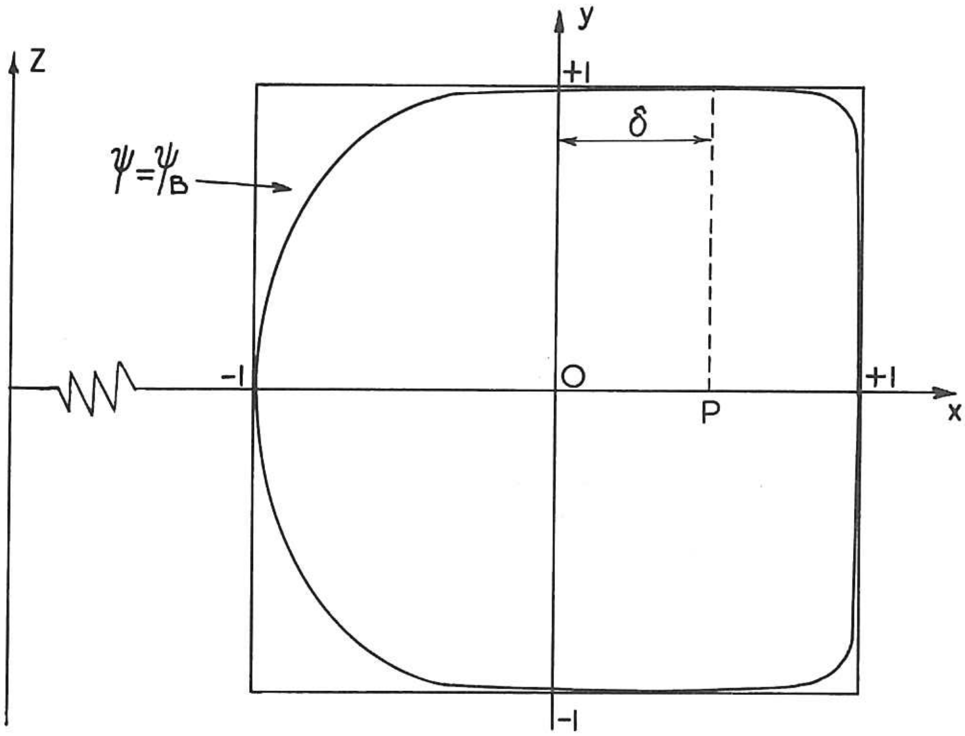

Fig.3 Rectangular coordinate system (x, y) and plasma boundary for b = 2.72. P denotes the position of the magnetic axis.

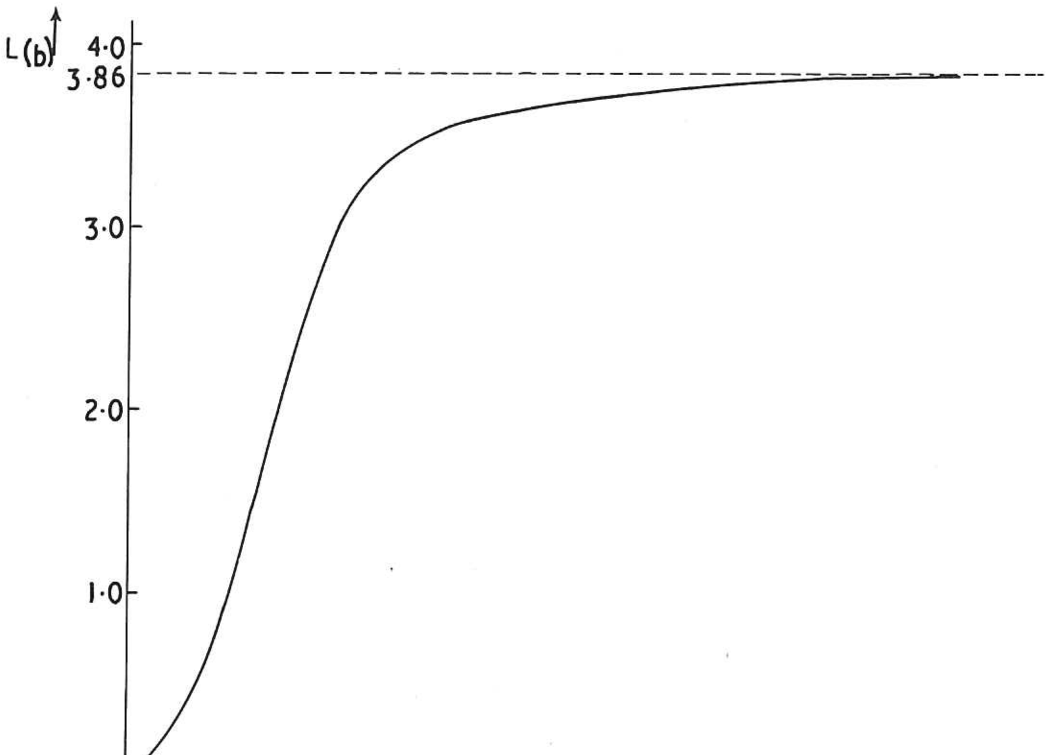

Fig.4 Plot of L(b).