Binding

CLM - P 305

UKAEA RESEARCH GROUP

Preprint

PLASMA DIFFUSION IN TWO DIMENSIONS

J H WILLIAMSON

CULHAM LABORATORY
Abingdon Berkshire

1972

CLM-P 305

PLASMA DIFFUSION IN TWO DIMENSIONS

by

J.H. Williamson

(Submitted for publication in Physics of Fluids)

ABSTRACT

The non-linear differential equations describing the interaction of waves in a two-dimensional guiding centre plasma form the basis for computer simulation. The method is often more efficient than the particle codes now in use and it is readily modified to allow direct computation of ensemble averages. It is shown that the electric field correlation and the particle velocity correlation are not equivalent and so estimates of the diffusion coefficient based on the field correlation are pessimistic. Polarization drifts seem to have little effect on the results until ω_p exceeds ω_c.

UKAEA Research Group,
Culham Laboratory,
Abingdon,
Berks.

April, 1972

I. INTRODUCTION

Plasmas can be simulated in computer experiments by moving charged particles about under the influence of the fields they themselves generate. Often the statistical properties of the fields are more interesting than the details of the particle motion, so it would be nice to be able to dispense with the particles altogether and to describe the plasma by its fields alone. We shall show that this is particularly easy to do for a two-dimensional plasma in which particle motion is described by

$$\underset{\sim}{v} = \frac{\underset{\sim}{E} \times \underset{\sim}{B}}{B^2} \qquad (1)$$

with $\underset{\sim}{E}$ given by Poisson's equation. The non-linear differential equations provide an alternative method of simulating a plasma, which is superior in some important respects to the usual particle codes.

The most important previous work on this system was performed by Taylor and McNamara[1] who showed on dimensional grounds that the diffusion coefficient D, defined by

$$D = B^{-2} \int_0^\infty Q(t)\, dt \ , \qquad (2)$$

where

$$Q(t) = \langle \tfrac{1}{2} \underset{\sim}{E}(t_o) \cdot \underset{\sim}{E}(t + t_o) \rangle \ , \qquad (3)$$

must be proportional to B^{-1}. In order to obtain numerical values they assumed a Gaussian probability functional for the electric field. However, we show here that a simpler assumption can give a better approximation to $Q(t)$.

Sections II to IV develop the basic theory and draw the distinction between the field and particle correlations.

The merits of the computer code and its modification for the direct calculation of ensemble averages are considered in sections V and VI. The results are presented in the next section and finally, in section VIII, we discuss the limit to the theory which occurs at high plasma density.

II THEORY

The two-dimensional plasma is represented by rods with charge $\pm e/\ell$ per unit length moving in the x, y plane. We apply periodic boundary conditions to a square of unit area and so the number of particles in the experiment is equal to n, the mean density. The charge density is

$$\rho(\underset{\sim}{x}) = \frac{e}{\ell} \sum_{j=1}^{n} (\pm 1)_j \, \delta(\underset{\sim}{x} - \underset{\sim}{x}_j)$$

where the index j labels each particle. The spectrum is then

$$\rho_{\underset{\sim}{k}} = \frac{e}{\ell} \sum_j (\pm 1)_j e^{-i\underset{\sim}{k} \cdot \underset{\sim}{x}_j} \tag{4}$$

and its time derivative is

$$\dot{\rho}_{\underset{\sim}{k}} = \frac{e}{\ell} \sum_j -i\underset{\sim}{k} \cdot \underset{\sim}{v}_j \; (\pm 1)_j \; e^{-i\underset{\sim}{k} \cdot \underset{\sim}{x}_j} .$$

The velocity $\underset{\sim}{v}_j$ is given by equation (1) and the electric field is

$$\underset{\sim}{E}(\underset{\sim}{x}) = 4\pi \sum_{\underset{\sim}{k}' \neq 0} \frac{-i\underset{\sim}{k}'}{k'^2} \rho_{\underset{\sim}{k}'} \, e^{i\underset{\sim}{k}' \cdot \underset{\sim}{x}} . \tag{5}$$

Thus

$$\dot{\rho}_{\underset{\sim}{k}} = \frac{4\pi e}{B^2 \ell} \sum_{\underset{\sim}{k}' \neq 0} \frac{i\underset{\sim}{k} \cdot i\underset{\sim}{k}' \times B}{k'^2} \rho_{\underset{\sim}{k}'} \sum_j (\pm 1)_j \, e^{-i\underset{\sim}{k} \cdot \underset{\sim}{x}_j + i\underset{\sim}{k}' \cdot \underset{\sim}{x}_j}$$

$$= \frac{4\pi}{B} \sum_{\underset{\sim}{k}' \neq 0} \frac{\underset{\sim}{k}' \times \underset{\sim}{k}}{k'^2} \rho_{\underset{\sim}{k}'} \rho_{\underset{\sim}{k} - \underset{\sim}{k}'} . \tag{6}$$

The scalar product with $\underset{\sim}{B}$ has been dropped here because the vector $\underset{\sim}{k}' \times \underset{\sim}{k}$ is necessarily parallel to $\underset{\sim}{B}$. It is convenient to rewrite this equation in a symmetric form

$$\dot{\rho}_{\underset{\sim}{k}} = \sum_{\underset{\sim}{k}_1, \underset{\sim}{k}_2} W(\underset{\sim}{k}_1, \underset{\sim}{k}_2)\ \delta_{\underset{\sim}{k} - \underset{\sim}{k}_1 - \underset{\sim}{k}_2}\ \rho_{\underset{\sim}{k}_1}\ \rho_{\underset{\sim}{k}_2}, \qquad (7)$$

where

$$W(\underset{\sim}{k}_1, \underset{\sim}{k}_2) = \frac{2\pi}{B}\ \underset{\sim}{k}_1 \times \underset{\sim}{k}_2 \left(\frac{1}{k_1^{\,2}} - \frac{1}{k_2^{\,2}} \right). \qquad (8)$$

This equation[2] subsumes all the information about the electric field inherent in the original guiding centre equations of motion and so it can be used as the basis of simulation.

III CONSERVATION LAWS AND EQUILIBRIA

Obviously the total number of particles must be conserved under the operation of equation (7) and this is true because $\dot{\rho}_o$ is zero. We must also check that there is no tendency for particles to decompose while the plasma evolves. A necessary but not sufficient diagnostic is $P = \int \rho^2(\underset{\sim}{x})\, d\underset{\sim}{x}$ which would fall if particles were to split up into fragments. By the convolution theorem $\rho^2(\underset{\sim}{x})$ is the Fourier transform of $\int \rho(\underset{\sim}{k}) \rho(\underset{\sim}{k} - \underset{\sim}{k}')\, d\underset{\sim}{k}'$ and here we are interested in the $\underset{\sim}{k} = 0$ transform of $\rho^2(\underset{\sim}{x})$. Thus

$$P = \Sigma_{\underset{\sim}{k}}\ \rho_{\underset{\sim}{k}}\ \rho_{-\underset{\sim}{k}} \qquad (9)$$

and

$$\dot{P} = \frac{4\pi}{B} \sum_{\underset{\sim}{k}\underset{\sim}{k}'} \frac{\underset{\sim}{k}' \times \underset{\sim}{k}}{k'^2}\ \rho_{\underset{\sim}{k}'}\ \rho_{\underset{\sim}{k} - \underset{\sim}{k}'}\ \rho_{-\underset{\sim}{k}}$$

$$= 0$$

because each term $\underset{\sim}{k}, \underset{\sim}{k}'$ is cancelled by the term $\underset{\sim}{k}' - \underset{\sim}{k}, \underset{\sim}{k}'$.

The total energy in the electrostatic field is

$$\mathcal{E} = \frac{1}{8\pi} \sum_{\underset{\sim}{k}} |E_{\underset{\sim}{k}}|^2 = 2\pi \sum_{\underset{\sim}{k}} \frac{1}{k^2} \rho_{\underset{\sim}{k}} \rho_{-\underset{\sim}{k}} . \qquad (10)$$

The rate of change of $\mathcal{E}$ is

$$\dot{\mathcal{E}} = 4\pi \sum \frac{1}{k^2} \dot{\rho}_{\underset{\sim}{k}} \rho_{-\underset{\sim}{k}}$$

$$= 4\pi \frac{4\pi}{B} \sum_{\underset{\sim}{k}\underset{\sim}{k}'} \frac{\underset{\sim}{k}' \times \underset{\sim}{k}}{k^2 k'^2} \rho_{\underset{\sim}{k}'} \rho_{\underset{\sim}{k} - \underset{\sim}{k}'} \rho_{-\underset{\sim}{k}}$$

$$= 0$$

because the term $\underset{\sim}{k}, \underset{\sim}{k}'$ is cancelled by $-\underset{\sim}{k}', -\underset{\sim}{k}$.

The quantities P and $\mathcal{E}$ are conserved in systems with an infinite number of Fourier modes, M. When M is limited, P is conserved only if we truncate the series so that $\rho_{\underset{\sim}{k}}$, $\rho_{\underset{\sim}{k}'}$ and $\rho_{\underset{\sim}{k} - \underset{\sim}{k}'}$ in equation (6) all contain the same M terms. This choice, which has been adopted in this paper, also ensures that $\mathcal{E}$ is conserved although this would still be true if the restriction on $\rho_{\underset{\sim}{k} - \underset{\sim}{k}'}$ were lifted.

An alternative way of measuring the energy in the system is to add up the potential energy of each particle. This is

$$\mathcal{E}_\Phi = \int \rho(\underset{\sim}{x}) \Phi(\underset{\sim}{x}) \, d\underset{\sim}{x} = 4\pi \sum \rho_{-\underset{\sim}{k}} \frac{\rho_{\underset{\sim}{k}}}{k^2} = 2\mathcal{E} . \qquad (11)$$

There is, however, no such thing as kinetic energy in the system. We can also calculate the overall electric current flowing in the plasma. This is proportional to

$$\underset{\sim}{J} = \int \underset{\sim}{E}(\underset{\sim}{x}) \rho(\underset{\sim}{x}) \, d\underset{\sim}{x} = \sum \underset{\sim}{E}_{\underset{\sim}{k}} \rho_{-\underset{\sim}{k}} = 4\pi \sum \frac{i\underset{\sim}{k}}{k^2} |\rho_{\underset{\sim}{k}}|^2 \qquad (12)$$

and hence it is identically zero because of the cancellation

beteeen each term $\underset{\sim}{k}$ and its complex conjugate.

The system will be in statistical equilibrium if the probability distribution $\wp(\{\rho_{\underset{\sim}{k}}(t)\})$ is independent of t. The conservation of P and $\mathcal{E}$ immediately shows that $\wp$ is zero if $\wp$ is any function of the form

$$\wp = \wp\left(\sum_{\underset{\sim}{k}} \frac{|\rho_{\underset{\sim}{k}}|^2}{\langle |\rho_{\underset{\sim}{k}}|^2 \rangle}\right) \tag{13}$$

with a thermal spectrum, i.e.,

$$\langle |\rho_{\underset{\sim}{k}}|^2 \rangle \propto \frac{k^2}{k^2 + A} \quad . \tag{14}$$

Of these functions, the most plausible is the Gaussian distribution; this is the only one which can be factorised into independent probability distributions for each of the $\rho_{\underset{\sim}{k}}$. We are thus using a canonical ensemble; a microcanonical ensemble, in which P and $\mathcal{E}$ have specified values but the $|\rho_{\underset{\sim}{k}}|^2$ are correlated, would be much harder to set up. The constant A in equation (14) can have any positive value and even, when $\underset{\sim}{k}$ is discrete as here, a small negative one.

In addition the system will support an infinity of exact equilibria. Here the fields do not vary with time although the particles continue to drift about. However, all these equilibria are unstable because almost any small perturbation will grow indefinitely, so their effect on the correlation function Q(t) is negligible.

IV PARTICULATE DIFFUSION

Although it was stated in the introduction that the correlation properties of the electric field are more

interesting than the details of the particle motion, yet in the final resort it is the particle diffusion that matters. Luckily the wave code can be extended to cover this in an efficient manner. (Particle codes in practice tend to compute the diffusion of only a subset of the particles and the periodic boundary conditions may cause complications.) The particulate analogue of

$$Q(t) = 8\pi^2 \sum_{\underset{\sim}{k}} k^{-2} \rho_{\underset{\sim}{k}}(t)\ \rho_{\underset{\sim}{k}}(o) \qquad (15)$$

is

$$C(t) = \frac{B^2}{n} \sum_{j=1}^{n} \tfrac{1}{2}\, \underset{\sim}{v}_j(o) \cdot \underset{\sim}{v}_j(t) \ . \qquad (16)$$

Substituting for $\underset{\sim}{v}_j(t)$ as in section II, we obtain

$$C(t) = 8\pi^2 \sum_{\underset{\sim}{k}} \frac{i\underset{\sim}{k} \cdot \underset{\sim}{\zeta}_{\underset{\sim}{k}}(t)}{k^2}\ \rho_{-\underset{\sim}{k}}(t) \ , \qquad (17)$$

where

$$\underset{\sim}{\zeta}_{\underset{\sim}{k}}(t) = \frac{1}{4\pi n} \sum_{j} \underset{\sim}{v}_j(o) \times \underset{\sim}{B}\ e^{-i\underset{\sim}{k} \cdot \underset{\sim}{x}_j} \ . \qquad (18)$$

The evolution of $\underset{\sim}{\zeta}_{\underset{\sim}{k}}$ is given by

$$\dot{\underset{\sim}{\zeta}}_{\underset{\sim}{k}} = \frac{4\pi}{B} \sum_{\underset{\sim}{k}' \neq 0} \frac{\underset{\sim}{k}' \times \underset{\sim}{k}}{k'^2}\ \rho_{\underset{\sim}{k}'}\ \underset{\sim}{\zeta}_{\underset{\sim}{k} - \underset{\sim}{k}'} \ , \qquad (19)$$

with the initial conditions

$$\underset{\sim}{\zeta}_{\underset{\sim}{k}}(o) = \frac{-i\underset{\sim}{k}}{k^2}\ \rho_{\underset{\sim}{k}} \ . \qquad (20)$$

There is an alternative way of computing the diffusion of the particles - in terms of the mean square displacement. The equations are:

$$H(t) = \frac{B^2}{n} \sum_j \tfrac{1}{4} \Delta x_j^2 \ , \qquad (21)$$

$$\frac{dH}{dt} = 8\pi^2 \sum_{\underset{\sim}{k}} \frac{i\underset{\sim}{k} \cdot \underset{\sim}{\chi}(t)}{k^2} \rho_{-\underset{\sim}{k}}(t) \ , \qquad (22)$$

$$\underset{\sim}{\chi}_{\underset{\sim}{k}} = \frac{1}{4\pi n} \sum \Delta \underset{\sim}{x}_j \times \underset{\sim}{B}\, e^{-i\underset{\sim}{k} \cdot \underset{\sim}{x}_j} \qquad (23)$$

$$\dot{\underset{\sim}{\chi}}_{\underset{\sim}{k}} = \frac{4\pi}{B} \sum_{\underset{\sim}{k} \neq 0} \frac{\underset{\sim}{k}' \times \underset{\sim}{k}}{k'^2} \rho_{\underset{\sim}{k}'} \chi_{\underset{\sim}{k} - \underset{\sim}{k}'} - \frac{i\underset{\sim}{k}}{k^2} \rho_{\underset{\sim}{k}} \qquad (24)$$

$$\underset{\sim}{\chi}_{\underset{\sim}{k}}(o) = 0 \ . \qquad (25)$$

The results given in section VII demonstrate that the distinction between Q(t) and C(t) is a real one; it had been earlier assumed that Q(t) adequately described the diffusion of particles.

V COMPARISON WITH PARTICLE CODES

As mentioned above we truncate the Fourier series so that $\rho_{\underset{\sim}{k}'}$ and $\rho_{\underset{\sim}{k} - \underset{\sim}{k}'}$ in equation (6) both contain the same M terms. In particle codes, on the other hand, only a finite number of Fourier modes are retained in the electric field whereas the equations of motion use the exact particle positions. This can be reproduced in equation (6) by limiting $\underset{\sim}{k}'$ to its M smallest values while not restricting the coupling vector $\underset{\sim}{k} - \underset{\sim}{k}'$. In one sense this is closer to reality, but on the other hand the terms in $\underset{\sim}{k}'$, $\underset{\sim}{k} - \underset{\sim}{k}'$ and $\underset{\sim}{k} - \underset{\sim}{k}'$, $\underset{\sim}{k}'$ always tend to cancel each other and omitting one of them leads to an unnatural inflation in $\dot{\rho}_{\underset{\sim}{k}}$. In running the same simulation experiment

with the two different types of code, this possible source of discrepancy should be borne in mind.

We now consider whether this alternative method of plasma simulation is as efficient as the usual one. About $1.37\ M^2 + M$ additions are needed to advance equation (7) by one time step when M Fourier modes are retained in the simulation. A particle code computing the field from the exact positions of the particles would be prohibitively slow and so approximate positions are used although this introduces errors in the form of grid noise[3] (a dependence of the force between two particles on their absolute, not just relative, positions). Hockney[4] has compared the various forms of fast Fourier transformation techniques. The quickest of these with nearest grid point weighting would need $1.25\ M \log_2 M + 8M + 3n$ operations per time step when there are n particles in the experiment. The wave code is therefore superior up to $M = 155$ for the case of $n = 10^4$. However this comparison is unduly pessimistic because particle codes would never use such a small M throughout the computation because of the severe grid noise. For example in reference 1 the particles are assigned to a 64×64 mesh and then the electric field is smoothed by truncating all except the lowest M modes; in this case the wave code wins out up to $M = 300$.

The representation of the system by a finite number of Fourier modes means that there is a limit to the number of particles that can be usefully employed in an experiment. In addition particle codes have an explicit restriction on

n but the wave code does not suffer in this way. Equation (7) describes plasmas equally well for any value of n because all the $\rho_{\underset{\sim}{k}}$ which appear in it are $O(n^{\frac{1}{2}} e/\ell)$; hence a change in n will merely alter the time scale. The term ρ_o is equal to ne/ℓ but it is not present in the equation and $\dot{\rho}_o$ is identically zero. The value of n has an indirect effect on the behaviour of the simulated plasma through its influence on the initial conditions. In a plasma with small n, the several $\rho_{\underset{\sim}{k}}$ will have statistical correlations but we can simulate a plasma with infinite n merely by eschewing such correlations. Additional advantages of the new method are: the coding is much simpler as no Fourier transformations are required allowing a much more flexible program than the usual monolithic particle codes; any desired initial spectrum can be set up without trouble[1, 5]; there is no grid noise; and Runge-Kutta or similar methods of integration are readily included reducing $\Delta\mathcal{E}/\mathcal{E}$ to e.g. 10^{-7} per time step compared with 2×10^{-3} using crude integration with the same number of subroutine calls.

The lack of worry about whether n is large enough or whether grid noise or an excessive Δt is disturbing the system allows the user to concentrate on the one limitation the wave code has - a small M. If M has to be large, then a particle code is essential. However, much useful physical insight can be gained from the study of systems with very small M, systems that would be swamped by grid noise if a particle code were used.

VI THE CORRELATION FUNCTION AS A POWER SERIES

A useful supplementary technique is to compute the correlation function as a power series in t,

$$Q(t) = 8\pi^2 \Sigma_p \, q_p t^p \, . \tag{26}$$

The coefficients can be found by iterating equation (7), e.g.

$$q_2 = \tfrac{1}{2} \left\langle \sum_{\underset{\sim}{k}} k^{-2} \, \ddot{\rho}_{\underset{\sim}{k}}(o) \, \rho_{-\underset{\sim}{k}}(o) \right\rangle$$

$$= \sum k^{-2} \, W(\underset{\sim}{k}, \underset{\sim}{\ell}) \, W(\underset{\sim}{k}+\underset{\sim}{\ell}, -\underset{\sim}{\ell}) \langle |\rho_{\underset{\sim}{k}}|^2 \rangle \langle |\rho_{\underset{\sim}{\ell}}|^2 \rangle \tag{27}$$

Here we have used the fact that, in equilibrium, the ensemble average is zero unless all the ρ occur as complex conjugate pairs. There were only two (equal) terms instead of the six which might have been expected because the property

$$W(\underset{\sim}{k}, \underset{\sim}{k}) = W(\underset{\sim}{k}, \underset{\sim}{0}) = 0 \tag{28}$$

eliminates many possible contenders.

A computer program based on graph theory[6] was written to compute and print the coefficients in FORTRAN in a form like equation (27). The resulting program was then used to evaluate the coefficients numerically for the chosen spectrum $\langle |\rho_{\underset{\sim}{k}}|^2 \rangle$. The procedure was checked by summing the series to compare with direct simulation runs. In addition a simulation experiment was set up to evaluate $d^p\rho_{\underset{\sim}{k}}/dt^p$ and hence q_p, instead of $\rho_{\underset{\sim}{k}}(t)$ and hence $Q(t)$. The results, in Table I, demonstrate the advantage of being able to compute the ensemble average directly instead of having to rely on simulation experiments with their large

statistical fluctuations.

The number of terms to be evaluated increases somewhat faster than $M^{\frac{1}{2}p+1}$, so it is necessary to resort to Monte Carlo evaluation of the summations when p is large. However, this immediately removes the restriction that M must be finite and so for instance we can allow a continuous spectrum of $\underset{\sim}{k}$ values. In this form the wave code provides an intrinsically noiseless though inefficient simulation of a plasma containing a infinite number of particles whose law of force has an infinite number of Fourier modes.

VII MODELS AND RESULTS

Figure 1 shows the ensemble average $Q(t)$ for a series of 13 runs of the code based on equation (7). The system contains 121 Fourier modes and the spectrum was appropriate for randomly placed particles ($A = 0$ in equation (14)). The correlation function decays monotonically but the statistical accuracy is rather poor. It would therefore be useful to find a model giving $Q(t)$ directly.

Taylor and McNamara[1] did this by assuming that the probability functional of $\underset{\sim}{E}(t)$ is Gaussian. They obtained the result that $Q(t)$ is given by

$$B^2\ddot{R} = Q(t) = \sum_{k} \tfrac{1}{2} \langle |E_{\underset{\sim}{k}}|^2 \rangle \, e^{-k^2 R(t)} \qquad (29)$$

but, as figure 1 shows, this Q falls off too quickly. In part this is to be expected as the authors implicitly adopt the unsymmetric truncation of Fourier modes described in Section V. However putting that truncation into equation (6)

accounts for only 42% of the discrepancy in q_2.

Another way of getting a model for $Q(t)$ is based on the cumulant discard approximation discussed by Betchov[7]. By iterating equation (7) we obtain

$$\ddot{S}_{\underset{\sim}{k}}(t) = - \sum_{\underset{\sim}{\ell}} W(\underset{\sim}{k},\underset{\sim}{\ell})\ W(\underset{\sim}{k}+\underset{\sim}{\ell},\ \underset{\sim}{\ell})\ |\rho_{\underset{\sim}{\ell}}(t)|^2\ S_{\underset{\sim}{k}}(t), \quad (30)$$

where

$$S_{\underset{\sim}{k}}(t) = \rho_{\underset{\sim}{k}}(t)\ \rho_{-\underset{\sim}{k}}(o) \quad (31)$$

and we have neglected all terms with four distinct ρ on the right hand side. This is valid at $t = 0$ and, if we are lucky, the fourth-order cumulants will grow slowly enough for it to remain a reasonable approximation. If $|\rho_{\underset{\sim}{\ell}}(t)|^2$ fluctuates rapidly compared with $S_{\underset{\sim}{k}}(t)$, it can be replaced by its time average assumed here to be $\langle |\rho_{\underset{\sim}{\ell}}|^2 \rangle$. Then the correlation function is[8]

$$Q(t) = 8\pi^2\ \Sigma_{\underset{\sim}{k}}\ k^{-2}\ \langle |\rho_{\underset{\sim}{k}}|^2 \rangle \cos\ \Omega_{\underset{\sim}{k}}\ t\ , \quad (32)$$

where

$$\Omega^2_{\underset{\sim}{k}} = \Sigma_{\underset{\sim}{\ell}}\ W(\underset{\sim}{k},\ \underset{\sim}{\ell})\ W(\underset{\sim}{k}+\underset{\sim}{\ell},\ \underset{\sim}{\ell}) \langle |\rho_{\underset{\sim}{\ell}}|^2 \rangle\ . \quad (33)$$

The next model assumes conversely that $|\rho_{\underset{\sim}{\ell}}(t)|^2$ varies much more slowly than $S_{\underset{\sim}{k}}(t)$ in which case we can substitute its initial value $|\rho_{\underset{\sim}{\ell}}(o)|^2$ giving eventually

$$Q(t) = 8\pi^2\ \Sigma_{\underset{\sim}{k}}\ k^{-2} \langle |\rho_{\underset{\sim}{k}}|^2 \rangle \cos(\Omega_{\underset{\sim}{k}}^2 - \sigma_{\underset{\sim}{k}}^2)^{\frac{1}{2}} t\ \exp(-\tfrac{1}{2}\sigma_{\underset{\sim}{k}}^2\ t^2), \quad (34)$$

where

$$\sigma_{\underset{\sim}{k}}^2 = \Sigma_{\underset{\sim}{\ell}}\ [W(\underset{\sim}{k},\underset{\sim}{\ell})\ W(\underset{\sim}{k}+\underset{\sim}{\ell},\ \underset{\sim}{\ell}) \langle |\rho_{\underset{\sim}{\ell}}|^2 \rangle\ ]^2\ . \quad (35)$$

A further model can be constructed by exploiting the

fact that $Q(t)$ is a stationary function. Thus $\ddot{S}_{\underset{\sim}{k}}(t)$ can be written as $-\dot{\rho}_{\underset{\sim}{k}}(t)\,\dot{\rho}_{-\underset{\sim}{k}}(o)$ and, again neglecting terms with four distinct ρ, we obtain

$$S_{\underset{\sim}{k}}(t) = -\sum_{\underset{\sim}{\ell}} W^2(\underset{\sim}{k},\underset{\sim}{\ell})\; S_{\underset{\sim}{\ell}}(t)\; S_{\underset{\sim}{k}-\underset{\sim}{\ell}}(t)\ . \qquad (36)$$

This equation has been solved numerically using a modification of the wave code. Figure 1 shows that these three models give the correct initial curvature but they all display oscillations which are not seen in the simulation runs.

A more detailed study of the correlations between individual modes showed no coherent behaviour among the oscillations which followed the decay of the initial correlation. This suggested the model

$$Q(t) = 8\pi^2 \sum k^{-2} \langle |\rho_k|^2 \rangle \exp(-\tfrac{1}{2}\Omega_k^2\, t^2) \qquad (37)$$

which fits the data in figure 1 almost too well (χ^2 per degree of freedom = 0.147).

Another way of judging the models is to compare their power series expansions with the exact result and the simulation experiments in table I. The first experiment consisted of 185 runs with the conventional definition for the coefficients, e.g. $\langle \ddot{\underset{\sim}{E}} \cdot \underset{\sim}{E} \rangle$ and the second was made up of 100 runs using the symmetrical form e.g. $\langle -\dot{\underset{\sim}{E}} \cdot \dot{\underset{\sim}{E}} \rangle$. The model based on equation (37) is clearly consistent with the exact result and the experiments, but the other models are way out. A respectable pedigree for equation (37) could be dreamed up if desired; its Gaussian autocorrelation function for each $\rho_{\underset{\sim}{k}}(t)$ is at least as plausible as the Gaussian functional

for $\underset{\sim}{E}$ adopted in reference 1.

Having established then that equation (37) describes $Q(t)$ very well, we can integrate it to obtain the diffusion coefficient D. We do this for a thermal spectrum with continuous $\underset{\sim}{k}$,

$$\langle |\rho(\underset{\sim}{k})|^2 \rangle = \frac{\kappa T}{4\pi \ell} \frac{k^2}{1 + k^2 \lambda^2} \tag{38}$$

where the Debye length is given by $\lambda^2 = \kappa T \ell / 4\pi n e^2$. In evaluating $\Omega^2(\underset{\sim}{k})$, it turns out that modes with wave number greater than k give no contribution to the integral. The result is

$$\Omega^2(\underset{\sim}{k}) = \frac{\kappa T}{8 \ell B^2 \lambda^4} \left\{ \frac{(1 + k^2\lambda^2)^2}{k^2\lambda^2} \ell n(1 + k^2\lambda^2) - 1 - {}^3\!/_2\, k^2\lambda^2 \right\}. \tag{39}$$

The diffusion coefficient is

$$D = \frac{8^{\frac{1}{2}} \kappa T}{\pi^{\frac{1}{2}} \ell B^2} \int_{4\pi^2/L^2}^{\infty} \frac{dk^2}{1 + k^2\lambda^2} \Omega^{-1}$$

$$= \frac{\kappa T}{eBN_D^{\frac{1}{2}}} \frac{3^{\frac{1}{2}} 8}{\pi} \ell n(L/2\pi\lambda), \tag{40}$$

where $N_D = n\lambda^2$ and L is the size of the system, put in explicitly here for clarity. Because this was based on equation (37), which appears to fit the observed correlation more accurately than the expression derived by Taylor and McNamara, this is presumably a more accurate estimate of D than their result,

$$D = \frac{\kappa T}{eBN_D^{\frac{1}{2}}} \left[\frac{1}{2\pi} \ell n(L/2\pi\lambda) \right]^{\frac{1}{2}}. \tag{41}$$

Before jumping to the conclusion that equation (40)

is the end of the story we must study the particulate diffusion. Figures 2 and 3 compare the correlation function etc. for the field and particles. They are plotted for a single run to emphasise that Q(t) and C(t) are not the same thing. The difference is not an artifact caused by a finite number of particles; C follows the trajectory of each particle whereas Q does not distinguish one particle from another nor indeed a particle from an empty space. The conclusion from an ensemble of runs like this showed that C initially falls off more rapidly than Q to reach an average value of zero, so the particles diffuse more slowly than predicted by equation (40). However, a suitable model for C(t) has not yet been found because now the modes above k contribute to $\Omega^2(k)$ and give an infinite result. This leaves open the possibility that D should be finite when $L \to \infty$.

VIII POLARISATION DRIFTS

The drift velocity is only given by equation (1) when the electric field is static but the whole point of this study is to investigate the effect of time varying fields. A better approximation to the motion of the guiding centre is obtained by including the polarisation drift[9]

$$\underset{\sim}{v} = \frac{\underset{\sim}{E} \times \underset{\sim}{B}}{B^2} + \frac{\dot{\underset{\sim}{E}}}{B\omega_c} \quad . \tag{42}$$

In order to use the wave code here, we have to neglect the advective part of $\dot{\underset{\sim}{E}}$, leading to an implicit equation for $\dot{\rho}$,

$$\dot{\rho}_{\underset{\sim}{k}} = \frac{4\pi}{B} \sum \left(\frac{\underset{\sim}{k}' \times \underset{\sim}{k}}{k'^2} \rho_{\underset{\sim}{k}'} - \frac{\underset{\sim}{k}' \cdot \underset{\sim}{k}}{k'^2 \omega_c} \dot{\rho}_{\underset{\sim}{k}'} \right) \rho_{\underset{\sim}{k} - \underset{\sim}{k}'} \quad , \tag{43}$$

which can be solved by iteration. (Particle codes need not approximate $\dot{\underset{\sim}{E}}$ in this way and are in general to be preferred when polarisation drifts are present). We find that q_2 is unchanged to $O(\varepsilon)$ where the dimensionless expansion parameter is

$$\varepsilon = n^{-\frac{1}{2}} \omega_p^2 \ \omega_c^{-2} \tag{44}$$

(n is the number of particles in the experiment). To obtain q_2 to the next order we must add yet another term to the $\underset{\sim}{E} \times \underset{\sim}{B}$ drift, viz. $- \underset{\sim}{E} \times \underset{\sim}{B} \ \omega_c^{-2} \ B^{-2}$. With this correction a series of 50 runs of the system in Table I have given the result

$$q_2 = - 0.876 - (105 \pm 12) \ \varepsilon^2 . \tag{45}$$

Thus we are justified in ignoring the polarisation drift until ω_p approaches $0.30 \ n^{\frac{1}{4}} \omega_c$.

For a thermal spectrum the appropriate limit is of the order $N_D^{\frac{1}{4}} \omega_c$, and above this the effect of the polarisation drift is to reduce D. This provides an alternative explanation to the one adduced by Dawson et al.[10] to explain the relatively constant value of D for $\omega_p \gtrsim \omega_c$ which they found in their simulation experiment.

IX CONCLUSION

This study of the two-dimensional guiding centre plasma has demonstrated the merits of simulation using a wave code. The motion of the plasma is constrained by two important invariants and any thermal spectrum constitutes an equilibrium state of the system. The correlation function of the electric field is described accurately by a simple model,

but the particles do not diffuse as rapidly as this would suggest. Some doubt is therefore cast on the previously held view that D depends weakly on the size of the system. The results of Dawson et al. can be qualitatively explained by the onset of polarisation drifts when ω_p approaches ω_c.

ACKNOWLEDGMENTS

I am grateful to J.B. Taylor, I. Cook and J.P. Christiansen for their useful suggestions and critical appraisal of this work.

REFERENCES

1 J.B. Taylor and B. McNamara, Phys. Fluids 14, 1429 (1971).

2 J.B. Taylor and W.B. Thompson, to be published.

3 A.B. Langdon, J. Comp. Phys. 6, 247 (1970).

4 R.W. Hockney, in Methods of Computational Physics, Vol. 9, p. 136,[Academic Press, 1970].

5 J.H. Williamson, J. Comp. Phys. 8, 258 (1971).

6 F. Harary, Graph Theory, [Addison-Wesley, 1969].

7 R. Betchov, in Dynamics of Fluids and Plasmas, ed. S.I. Pai, p. 215, [Academic Press, 1966].

8 Taylor and Thompson (Ref. 2) obtained this equation using a somewhat different rationale and indeed claim it to be exact when A is zero as for instance in fig.1.

9 S. Chandrasekhar, Plasma Physics, [Univ. of Chicago Press, 1960].

10 J.M. Dawson, H. Okuda and R.N. Carlile, Phys. Rev. Letters, 27, 491 (1971).

TABLE I

The coefficients q_p in the expansion $Q(t) = 8\pi^2 \sum_p q_p t^p$ evaluated for the system of Fig. 1

p	0	2	4	6	8	10	12
exact theory	1.000	- .876	.962				
simulation	.993	- .857	.929	- .745			
± stand. dev.	.026	.043	.089	.105			
simulation	1.029	- .926	1.052	- .919	.637	- .364	.178
± stand. dev.	.024	.044	.108	.162	.166	.124	.072
model: Ref. 1	1.000	- 2.14	6.17	- 17.3	45.4		
model: Eq. 32	1.000	- .876	.326	- .059	.006	0	0
model: Eq. 34	1.000	- .876	.372	- .087	.013	- .001	0
model: Eq. 37	1.000	- .876	.977	- .883	.648	- .397	.209

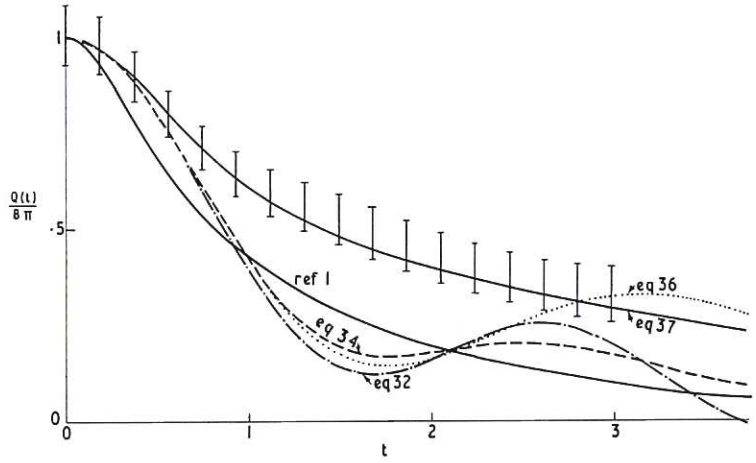

Fig.1 The correlation function $Q(t)$ for the spectrum $\langle |E_k|^2 \rangle \propto k^{-2}$, normalised so that q_o is unity, in a system with $M = 121$ modes. The vertical bars denote the standard deviations of the simulation runs and the curves represent various models for $Q(t)$.

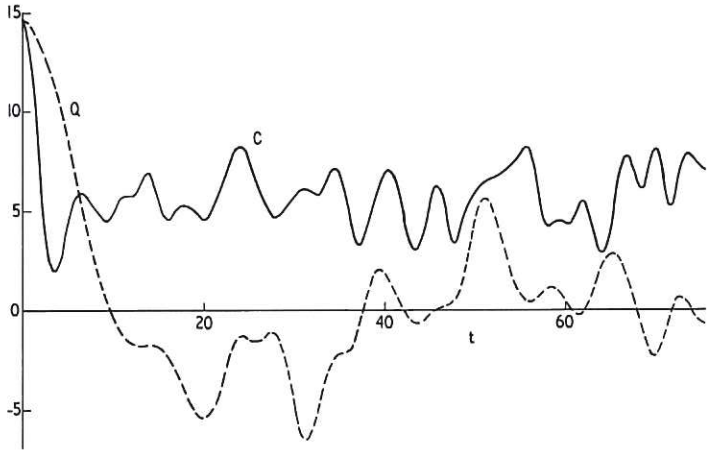

Fig.2 The velocity correlation functions, $Q(t)$ for the electric field and $C(t)$ for the particles, in a single run with $M = 25$.

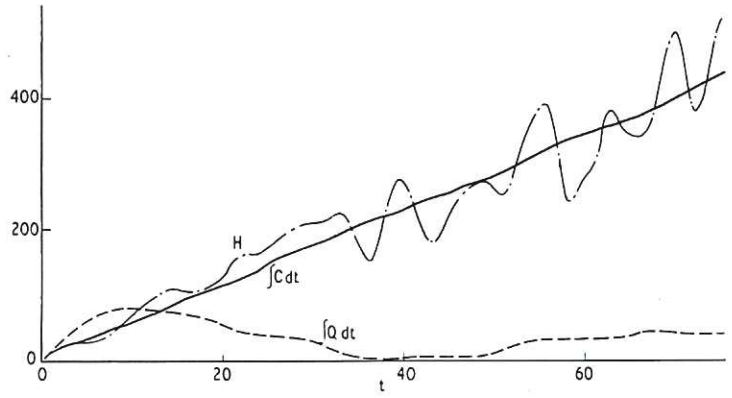

Fig.3 The running diffusion coefficient for the run in Fig.2. The integrals of Q and C are compared with $H(t)$ which is the derivative of the mean square displacement of the particles.