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NON-LINEAR HOMOGENEOUS FREDHOLM EQUATIONS OF THE SECOND KIND

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NON-LINEAR HOMOGENEOUS FREDHOLM EQUATIONS OF THE SECOND KIND

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ABSTRACT

It is shown how the solution of a non-linear homogeneous Fredholm equation of the second kind may be reduced to the solution of a set of simultaneous non-linear algebraic equations. The kernel of the integral equation is degenerate.

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We show how the solution of a certain class of non-linear Fredholm equations may be reduced to the problem of solving a set of simultaneous non-linear algebraic equations. Consider the equation

$$F(x) = \lambda \int_a^b K(x,y)[F(y)]^n \, dy \qquad (1)$$

where $K(x,y)$ is a known kernel and n is a positive integer. If $n = 1$ the equation is linear and has solutions only for specific values of λ. If $n > 1$ the equation is non-linear and λ can be set equal to unity without loss of generality (Davis 1962). Davis (1962) has also shown how equation (1) may be solved if the kernel has the degenerate form

$$K(x,y) = \sum_{i=1}^{R} X_i(x) Y_i(y) \qquad (2)$$

where the functions $X_1(x)$, $X_2(x)$, ... are assumed to form a linearly independent set. We extend his solution to cover the case of the X_i being linearly dependent.

Set $\lambda = 1$ and replace $K(x,y)$ by equation (2). Thus equation (1) becomes

$$F(x) = \sum_{i=1}^{R} X_i(x) F_i \qquad (3)$$

where

$$F_i = \int_a^b Y_i(y)[F(y)]^n \, dy \quad .$$

We now find a set of simultaneous non-linear algebraic equations for the F_i. Multiply equation (3) by $[F(x)]^{n-1} Y_j(x)$ and integrate with respect to x. Then

$$F_j = \sum_{i=1}^{R} c_{ij} F_i \qquad (4)$$

where

$$c_{ij} = \int_a^b X_i(x) Y_j(x)[F(x)]^{n-1} \, dx \quad .$$

To find the c_{ij} multiply equation (3) by $[F(x)]^{n-2} X_k(x) Y_j(x)$ and integrate with respect to x. Thus

$$c_{kj} = \sum_{i=1}^{R} d_{ikj} F_i \qquad (5)$$

where

$$d_{ikj} = \int_a^b X_i(x)\, X_k(x)\, Y_j(x) [F(x)]^{n-2}\, dx \quad .$$

Clearly the d_{ikj} may be found in terms of integrals over the product of three X functions, a Y function and F^{n-3}. If n is a positive integer we will eventually be able to find the coefficients in terms of integrals involving only the X and Y functions and the unknown F_i . Thus the set of equations (d) becomes a set of R simultaneous non-linear algebraic equations (of degree n) for the F_i . For example, if n = 3 then

$$F_j = \sum_i \sum_m \sum_k e_{kmij}\, F_k\, F_m\, F_i \tag{6}$$

where

$$e_{kmij} = \int_a^b X_k(x)\, X_m(x)\, X_i(x)\, Y_j(x)\, dx \quad .$$

The calculation of the e_{kmij} is greatly facilitated by the fact that e_{kmij} is completely symmetric in the indices k, m and i. Even so the application of this method is likely to be difficult for two reasons. Firstly, the calculation of the coefficients of the algebraic equations involves the calculation of a considerable number of integrals. In principle, however, this problem presents no real difficulty; a computer program to evaluate the integrals may be easily devised. Secondly, and much more seriously, is the problem of solving the algebraic equations once the coefficients are known. Neither the theory nor the numerical methods of solving simultaneous non-linear algebraic equations are well-developed. The book Numerical Methods for Non-linear Algebraic Equations (Rabinowitz 1970) is a useful source of references and also contains a FORTRAN subroutine (Powell 1970) for solving systems of non-linear algebraic equations.

It may be argued that direct numerical solution of equation (1) is better than the method presented here. This, however, is not always possible. One method of solving equation (1) directly is the iterative technique

$$F_1 = \int_a^b K(x,y)\, [F_o(y)]^n\, dy$$

$$F_2 = \int_a^b K(x,y)\, [F_1(y)]^n\, dy$$

$$\vdots$$

where F_o is some initial guess. A necessary condition for the convergence of such a procedure is $M(b-a) < 1$ where M is such that $K(x,y)F^n$ satisfies the Lipschitz condition in $[a,b]$,

i.e.

$$| K(x,y)F^n - K(x,y)F'^n | < M | F - F' |$$

(see Davis 1962). Obviously if this condition is not satisfied an alternative method of solution must be sought.

Finally it should be noted that the method described in this paper is easily extended to problems of the type

$$F(x) = \int_a^b K(x,y) \left\{ c_o + c_1 F(y) + c_2 [F(y)]^2 + \ldots \right\} dy$$

where $K(x,y)$ has the form of equation (2).

References

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